A General Theory of Optimal Capital Accumulation under Price Uncertainty and Costly Reversibility

Luis H. R. Alvarez E.
Turku School of Economics and Business Administration,
RUESG and HECER

Discussion Paper No. 104
April 2006
ISSN 1795-0562
A General Theory of Optimal Capital Accumulation under Price Uncertainty and Costly Reversibility*

Abstract

We consider the optimal capital accumulation policy of a neoclassical firm operating in the presence of decreasing returns to scale, price uncertainty, and costly reversibility of investment. We characterize the optimal accumulation policy and derive the value of the firm by focusing on the marginal investment decision and solving the associated optimal timing problem characterizing the option value of the associated opportunity to either disinvest or acquire a marginal unit of capacity. We also characterize the excess returns associated with the optimal policies and demonstrate that hysteresis prevails within this broad class of accumulation problems as well.

JEL Classification: G31, E22, D92.

Keywords: Price uncertainty, capital accumulation, costly reversibility, expansion options, disinvestment opportunity, hysteresis.

Luis H. R. Alvarez E.

Department of Economics
Turku School of Economics and Business Administration
Rehtorinpellonkatu 3
FI-20500 Turku
FINLAND

e-mail: luis.alvarez@tukkk.fi

* Financial support from the Foundation for the Promotion of the Actuarial Profession, the Finnish Insurance Society, and the Research Unit of Economic Structures and Growth (RUESG) at the University of Helsinki is gratefully acknowledged.
1 Introduction

One of the basic lessons of the literature considering optimal capital accumulation is that irreversibility results into a accumulation rule which differs from the standard myopic investment rule requiring that productive stock should be maintained at a level where its marginal revenue product coincides with its marginal user cost (cf. Arrow (1968), Nickell (1974a,b), Baldwin (1982), Bernanke (1983), Pindyck (1988), Pindyck (1991), Bertola and Caballero (1994), Dixit (1995), Caballero and Leahy (1996), Bertola (1998), and Alvarez (2006a, 2006b); for extensive textbook treatment of this subject, see Nickell (1978) and Dixit and Pindyck (1994)). The main reason for this observation is naturally the inability to instantaneously adjust the productive capacity to a desired optimal level should the market conditions change towards a unfavorable direction afterwards. As intuitively is clear, the presence of uncertainty typically pronounces this effect and increases the required exercise premia associated with the irreversible decision by increasing the option value of waiting.

Even though the literature on irreversible capital accumulation in the presence of uncertainty is extensive, the impact of partial reversibility of investment on the optimal capital accumulation policies has typically been overlooked. In this respect, the papers by Abel and Eberly (1996) and Abel, Dixit, Eberly, and Pindyck (1996) constitute the pioneering studies on this topic. Abel and Eberly (1996) consider optimal investment in the presence of uncertainty and costly reversibility. By assuming that the short run profit flow of the firm is homogenous of degree one as a function of the underlying price and the installed capital stock and by modeling the underlying driving stochastic factor dynamics as a geometric Brownian motion they demonstrated that the optimal investment policy is characterizable as a rule requiring that the marginal revenue product of capital should be maintained between two separate optimal thresholds at all times. Abel, Dixit, Eberly, and Pindyck (1996) in turn consider a discrete two period investment model where both expanding the current productive capital stock and disinvestment is costly. They demonstrate how the optimal investment policy can be characterized in terms of put and call options and show that the the relative sizes of
these options determine the net effect of expandability and reversibility on investment.

In this paper we analyze how price uncertainty and costly reversibility affects the optimal accumulation policy of a competitive firm operating in the presence of decreasing returns to scale. We extend previous studies in two ways by assuming that the underlying price dynamics follows a one-dimensional but otherwise general linear diffusion process and that the production technology is characterized by a differentiable but otherwise general production function subject to decreasing returns to scale. In this way our analysis covers a broad class of descriptions both for the production function and for the underlying stochastic price dynamics which, within our approach, includes most typically applied mean reverting models as well. This is advantageous since it admits the analysis of the general properties of optimal accumulation policies within this relatively general class of investment problems and characterizes those circumstances under which the results obtained by relying on simple models based on exponential prices remain qualitatively valid.

Instead of tackling the stochastic capital accumulation problem directly, we follow the seminal studies by Pindyck (1988) and Bertola (1998) and focus on the decision to acquire or sell a marginal unit of capacity. In this way the original accumulation problem is transformed into a simpler timing problem characterizing the marginal value of capital as the value of a single discrete investment opportunity which depends on the prevailing productive capacity but is independent of the subsequent decisions to either invest or disinvest. We state a decomposition of the value of a marginal unit of capacity into its option components and for the sake of comparison analyze separately the two associated optimal accumulation problems subject to either irreversible investment or disinvestment. We characterize the marginal value of capital explicitly and demonstrate that the optimal accumulation policy can be characterized in terms of two boundaries at which the productive stock is optimally adjusted. According to the optimal accumulation rule a further unit of capacity should be acquired whenever the expected cumulative present value of the revenue product it generates exceeds a critical threshold at which the value of a marginal unit of capacity coincides with its
acquisition cost. Analogously, along the optimal accumulation path disinvestment is optimal as soon as the expected cumulative present value of the revenue product of capital falls below a critical threshold at which the marginal value of capital coincides with its selling price. Since each unit of capacity decreases the marginal return it generates we found that the optimal accumulation rule can also be interpreted as a requirement that incremental adjustments to capacity are made each time the underlying price hits one of the optimal monotonically increasing capital dependent exercise boundaries. Whenever the underlying price is between these two critical boundaries the productive capacity is maintained unchanged and the firm continues production with its existing stock. Consequently, the optimal accumulation path consists typically of potentially long periods of time where the firm operates with the prevailing capacity after which the firm enters into regimes of either very intense investing or disinvesting. The duration of these periods naturally depends on the expected growth rate of the underlying price as well as on its volatility. An interesting property of the optimal incremental accumulation policy is that it is path dependent and, therefore, that the future optimal capacities are not only sensitive with respect to changes in the initial stock and price, they are also profoundly affected by the evolution of the underlying price dynamics and especially its historical extreme values (cf. Dixit (1992)). We also characterize the value of the optimal accumulation policy and prove that it can be expressed in terms of the values of the three available operational options of the firm. Namely, in terms of the expected cumulative present value of the revenue product generated by current capacity, the option value of the future disinvestment opportunities, and the value of the options to expand productive capacity later in the future. As intuitively is clear, investment is optimal at the capacity where the future disinvestment options become valueless and, in turn, disinvestment is optimal at the stock where the option value of the future opportunities to expand productive capacity vanishes.

When investment is completely reversible, the optimal myopic accumulation rule dictates that the firm should maintain its productive stock at a level where the expected cumulative present value of its marginal revenue product coincides with its acquisition
The inability to sell excess capital at its acquisition price naturally results in a situation where this principle is no longer valid since a forward-looking firm has to take into account that the acquisition costs of a marginal unit are partly sunk and cannot be recovered by selling the marginal unit should the investment decision turn out to be poor afterwards. Our analysis supports this view and demonstrates that at the optimal investment boundary the revenue product of a marginal unit of capacity exceeds the interest on the marginal acquisition cost (i.e. the marginal user cost of investing). Similarly, our analysis shows that at the optimal disinvestment boundary the interest on the selling price of a marginal unit of capacity exceeds the revenue product it would generate if it would not be sold. Therefore, our findings extend the observations made originally by Dixit (1989) and Dixit (1992) and demonstrate that hysteresis prevails within our general setting as well. It is, however, worth emphasizing that costly reversibility is shown to reduce the magnitude of hysteresis in comparison with the cases where either investment or disinvestment are irreversible. Thus, even while the optimal policy differs from the one which is followed in case investment is perfectly reversible, it does not coincide with the optimal policy in the presence of irreversibility. Especially, the optimal exercise boundaries are located between the optimal boundaries of the two above mentioned extreme cases. This observation is of interest since it supports the economically sensible argument that increased policy flexibility has a positive impact on the optimal accumulation policy and its value (cf. Alvarez and Virtanen (2006)).

It is worth pointing out that our main findings have important implications to other related economically relevant stochastic control problems as well. First, given the close connection of the considered accumulation problem to optimal switching problems encountered in studies considering sequential entry and exit decisions (cf. Dixit (1989)) and in studies considering the optimal operation of a mine (cf. Brennan and Schwartz (1985)) we observe that our results on the existence and uniqueness of a pair of optimal boundaries at which the firm either invests or disinvests extends to those modeling frameworks as well (by adjusting the cash flow appropriately). Similarly, studies considering rational hiring and firing policies of competitive firms are based on a completely
analogous singular stochastic control approach as this study is and, therefore, our find-
ings are applicable within that framework as well (cf. Bentolila and Bertola (1990) and
Shepp and Shiryaev (1996)).

The contents of this study are as follows. In section two we present the consid-
ered optimal accumulation problem and the underlying stochastic dynamics. In section
three we analyze the two associated optimal accumulation problems subject to either
irreversible investment or disinvestment. Our main findings on the general capital ac-
cumulation problem subject to costly reversibility are then stated in section four. Our
general findings are the explicitly illustrated in section five for two different underlying
price dynamics. Some concluding comments and potential extensions are then stated
in section six.

2 Irreversible Capital Accumulation

Consider a competitive value maximizing firm facing a stochastically fluctuating price
evolving on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ according to the
stochastic dynamics characterized by the stochastic differential equation

$$dp_t = \mu(p_t)dt + \sigma(p_t)dW_t, \quad p_0 = p \in \mathbb{R}_+ \quad (2.1)$$

where $W_t$ is standard Brownian motion and both the drift coefficient $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}$ and
the volatility coefficient $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are assumed to be continuously differentiable. As
usually, we denote as

$$\mathcal{A} = \frac{1}{2} \sigma^2(p) \frac{\partial^2}{\partial p^2} + \mu(p) \frac{\partial}{\partial p} \quad (2.2)$$

the differential operator associated with the price dynamics $p_t$. For simplicity, we will
assume that the boundaries of the state space $(0, \infty)$ of the price process $p_t$ are either
natural. Hence, even though the price dynamics may tend toward its boundary it is
never expected to attain it in finite time (for a comprehensive characterization of the

The considered firm is assumed to produce a single homogenous output $F(k)$ by
using a single homogenous and non-depreciating productive input $k$, which is called
capital. As usually, we assume that the function $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuously differentiable, monotonically increasing, strictly concave, and satisfies the Inada-conditions $F(0) = 0$, $\lim_{k \downarrow 0} F'(k) = 0$ and $\lim_{k \to \infty} F'(k) = \infty$. In line with standard studies of irreversible capital accumulation, we assume that acquiring new capital is costly and that the unit cost of increased capacity is an exogenously determined constant $q_+ > 0$.

In order to analyze costly reversibility, we follow the approach developed in Abel and Eberly (1996) and assume that capital stock can be sold at a constant price $q_-$ satisfying the assumption $0 < q_- < q_+$. Thus, even though capital has resale value it is below its acquisition cost making part of the acquisition costs sunk. Given these assumptions, we now plan to analyze the optimal capital accumulation problem (cf. Abel and Eberly (1996))

$$V(k, p) = \sup_{k \in \Lambda} \mathbb{E} \int_0^\infty e^{-rs} (p_s F(k_s) ds - q_+ dk^+_s + q_- dk^-_s),$$

(2.3)

where $r > 0$ is an exogenously given discount rate, $k^+_t$ denotes the cumulative investments up to time $t$, and $k^-_t$ denotes the cumulative disinvestments up to time $t$. Thus, at any date $t$ the operational capital stock is $k_t = k + k^+_t - k^-_t$. As usually, we call a policy admissible if it is non-negative, non-decreasing, right-continuous, and $\{\mathcal{F}_t\}$-adapted, and denote the set of admissible accumulation policies as $\Lambda$. In order to guarantee the finiteness of the objective functional (2.3) we assume that the expected cumulative present value of the maximal short run profit flow $\pi(p) = \sup_{k \in \mathbb{R}_+} [pF(k) - rq_+ k]$ is bounded.

As usually in neoclassical investment theory, the expected cumulative present value of the future marginal revenue products of capital is a central factor affecting the determination of the optimal policy. In the present case, this factor can be expressed as

$$M(k, p) = \mathbb{E}_p \int_0^\infty e^{-rs} p_s F'(k) ds = G(p) F'(k),$$

(2.4)

where

$$G(p) = \mathbb{E} \int_0^\infty e^{-rs} p_s ds$$

(2.5)
denotes the expected cumulative present value of the flow $p$ from the present up to an arbitrary distant future. It is well-known that if this value exists it can be re-expressed as

$$G(p) = B^{-1} \varphi(p) \int_0^p \psi(y) y m'(y) dy + B^{-1} \psi(p) \int_p^\infty \varphi(y) y m'(y) dy,$$  \hspace{1cm} (2.6)

where $B$ denotes the constant Wronskian of the fundamental solutions $\psi(p)$ and $\varphi(p)$ of the ordinary second order differential equation $(Au)(p) = r u(p)$, $m'(p) = 2/(\sigma^2(p) S'(p))$ denotes the density of the speed measure and

$$S'(p) = \exp \left( - \int \frac{2 \mu(p) dp}{\sigma^2(p)} \right)$$

denotes the density of the scale function of the price process $p_t$ (for a complete characterization of the fundamental solutions, see Borodin and Salminen (2002), pp. 18–19).

There are various approaches for analyzing the optimal accumulation problem (2.3). We follow the neoclassical tradition and focus on the decision to acquire or sell a marginal unit of stock and, therefore, concentrate on the marginal value of capital along an optimal accumulation path (cf. Pindyck (1988), Bertola (1998), and Alvarez (2006a,b)). The major advantage of this approach is that it reduces the original sequential incremental accumulation problem into an associated simpler optimal timing problem where the only endogenous variable is the exact timing at which the capital stock should be optimally adjusted for the next time. Moreover, given the close connection of the marginal value of the stock with Tobin’s marginal $q$, it provides valuable information on this classical capital theoretic concept (cf. Hayashi (1982); for excellent and extensive surveys of this classical subject see Abel (1990) and Caballero (1999)).

As was established in Abel and Eberly (1996), in the case where the underlying price follows a geometric Brownian motion, the optimal capital accumulation policy can be characterized in terms of two boundaries: one at which investment is optimal and one at which disinvestment will occur. In line with their findings, we now introduce the two separate but otherwise arbitrary boundaries $p_l < p_u$ and consider for any $p \in (p_l, p_u)$ the value of a marginal unit of capacity defined as

$$J(k, p) = E_p \left[ \int_0^{\tau(p_l, p_u)} e^{-r s} p_s F'(k) ds + e^{-r \tau(p_l, p_u)} C(p_{\tau(p_l, p_u)}) \right],$$  \hspace{1cm} (2.7)
where $\tau_{(p_l,p_u)} = \inf\{t \geq 0: p_t \notin (p_l,p_u)\}$ denotes the first exit time of the underlying price process from the continuation set $(p_l,p_u)$ where the operating capacity is left unchanged, and

$$C(p) = \begin{cases} 
q_+ & p \geq p_u \\
0 & p_l < p < p_u \\
q_- & p \leq p_l.
\end{cases}$$

The functional (2.7) measures the expected cumulative present value of the marginal revenue product of the current capital stock from the present up to the first time at which it is subsequently adjusted. Applying the strong Markov property of diffusions now implies that (2.7) can be re-expressed for $p \in (p_l,p_u)$ as

$$J(k,p) = G(p)\frac{F'(k)}{p} + \mathbb{E}_p \left[ e^{-r\tau_{p_u}} \left( q_+ \right. \right.$$ 

$$\left. - G(p_{\tau_{p_l}})F'(k) \right) ; \tau_{p_l} < \tau_{p_u} \Big] 
$$

$$\left. - \mathbb{E}_p \left[ e^{-r\tau_{p_u}} \left( G(p_{\tau_{p_u}})F'(k) - q_+ \right) ; \tau_{p_l} > \tau_{p_u} \Big] \right),$$

where $\tau_{p_i} = \inf\{t \geq 0: p_t = p_i\}$ denotes the first hitting time of the underlying price process to the boundary $p_i$, $i = l, u$. This representation characterizes how the value of a marginal unit of capital can be decomposed into the values of the available options for the firm (cf. Abel, Dixit, Eberly, and Pindyck (1996)). Whenever the firm acquires new productive stock it exercises its option to expand capacity and, therefore, loses the associated option value. Similarly, whenever the firm exercises its option to sell capacity it simultaneously acquires an option to expand its capacity later in the future should the economic situation improve sufficiently (i.e. should the unit price increase enough).

### 3 The Associated Irreversible Accumulation Problems

In light of the decomposition (2.8) it is clear that in the presence of costly reversibility the total value of a marginal unit of capacity is constituted by three different values. Namely, the expected cumulative present value of revenue product it generates, the value of the option to disinvest it later on the future, and the value of the subsequent expansion options. It is clear from (2.8) that in the general setting these factors are interdependent and should be considered simultaneously. However, it is naturally of
importance for the sake of comparison to consider the extreme cases where the firm can either invest or disinvest. In accordance with this argument, we first analyze the two associated optimal timing problems

\[ Q(k, p) = G(p)F'(k) + \sup_{\tau} E_p \left[ e^{-r\tau} (q_+ - G(p_\tau)F'(k)) \right] \]  

(3.1)

and

\[ H(k, p) = G(p)F'(k) - \sup_{\tau} E_p \left[ e^{-r\tau} (G(p_\tau)F'(k) - q_-) \right] . \]  

(3.2)

The optimal timing problem (3.1) constitutes the sum of the expected cumulative present value of the marginal revenue product of the current capital stock and the value of the embedded opportunity to sell a marginal unit of capacity at an optimally chosen date. Hence the embedded disinvestment option can be interpreted as a perpetual put option with strike price \( q_- \) and written on the expected cumulative present value of the marginal revenue product of capital. The optimal timing problem (3.2) in turn constitutes the difference between the expected cumulative present value of the marginal revenue product of the current capital stock and the value of the embedded opportunity to acquire a marginal unit of capacity at an optimally chosen date. Consequently, the value of the embedded option to expand productive capacity later in the future can be interpreted as the value of a perpetual call option with strike price \( q_+ \) and written on the expected cumulative present value of the marginal revenue product of capital. Our main results characterizing both the values and optimal exercise strategies of these associated option values are now summarized in our next theorem.

**Theorem 3.1.** (A) The value of the associated optimal timing problem characterizing the value of a disinvested marginal unit of capacity reads as

\[
Q(k, p) = \begin{cases} 
\left[ G(p) - \frac{G'(p_k)}{p'(p_k)} \varphi(p) \right] F'(k) & p > \bar{p}_k \\
q_- & p \leq \bar{p}_k,
\end{cases}
\]

(3.3)

where the optimal disinvestment threshold

\[
\bar{p}_k = \arg\max_{p \in \mathbb{R}_+} \left\{ \frac{q_- - G(p)F'(k)}{\varphi(p)} \right\} \in (0, rq_-/F'(k))
\]

(3.4)
constitutes the unique root of the ordinary first order condition

\[ G(\hat{p}_k) - \frac{G'(\hat{p}_k)}{\phi'(\hat{p}_k)} \phi(\hat{p}_k) = \frac{q_-}{F'(k)}. \]  

(B) The value of the associated optimal timing problem characterizing the value of a acquired marginal unit of capacity reads as

\[ H(k, p) = \begin{cases} 
q_+ & p \geq \hat{p}_k \\
[G(p) - \frac{G'(\hat{p}_k)}{\phi'(\hat{p}_k)} \phi(p)] F'(k) & p < \hat{p}_k 
\end{cases} \]  

where the optimal investment threshold

\[ \hat{p}_k = \arg\max_{p \in \mathbb{R}_+} \left\{ \frac{G(p) F'(k) - q_+}{\phi(p)} \right\} \in (rq_+/F'(k), \infty) \]  

constitutes the unique root of the ordinary first order condition

\[ G(\hat{p}_k) - \frac{G'(\hat{p}_k)}{\phi'(\hat{p}_k)} \phi(\hat{p}_k) = \frac{q_-}{F'(k)}. \]  

The value \( H(k, p) \) is non-decreasing as a function of the unit price \( p \) and non-increasing as a function of the capital stock \( k \). Moreover, the optimal investment threshold \( \hat{p}_k \) is monotonically increasing and satisfies the boundary conditions \( \lim_{k \to \infty} \hat{p}_k = \infty \) and \( \lim_{k \downarrow 0} \hat{p}_k = 0 \).

Proof. See Appendix A.  

Theorem 3.1 characterizes the values of a marginal unit of capital in the cases where the firm can either invest or disinvest a marginal unit of productive capacity and characterizes the optimal boundaries at which an irreversible investment or disinvestment decision should be exercised. According to Theorem 3.1 the optimal boundaries can be characterized by two ordinary first order conditions (3.5) and (3.8) guaranteeing that the proposed boundary maximizes the expected cumulative net present value of the revenue.
product generated by the disinvested or acquired marginal unit of capacity. Interestingly, Theorem 3.1 demonstrates that the optimal boundaries are increasing functions of the productive stock in both cases. However, since $\bar{p}_k < r q_\cdot / F'(k) < r q_+ / F'(k) < \hat{p}_k$ our results demonstrate that the boundary at which irreversible disinvestment is optimal is below the boundary at which an incremental unit of capacity should be acquired.

The results of Theorem 3.1 can be interpreted in terms of the ratio between the value of a marginal unit of capital and its acquisition cost, that is, in terms of Tobin’s marginal $q$. In the case where disinvesting is irreversible this quantity reads as $q_\cdot / Q(k, p)$ and, therefore, in that case the optimal rule essentially states that the firm should postpone disinvestment as long as marginal $q$ falls short of unity. In the latter case where investing is irreversible, the associated marginal $q$ reads as $H(k, p) / q_+$. Again, the optimal acquisition rule requires that investment should be postponed as long as the marginal $q$ falls short of unity.

It is worth noticing that the optimal accumulation rules characterized in Theorem 3.1 can be interpreted in terms of the classical balance identity requiring that at the optimum the project value has to coincide with its full costs. In the case of irreversible disinvestment we observe that at the optimum the revenue accrued from disinvesting a marginal unit of stock has to coincide with the sum of the expected cumulative present value of the revenue product it generates and the lost option value of disinvesting sometime later in the future. Analogously, in the irreversible investment case we find that at the optimum the expected cumulative present value of the revenue product generated by the acquired marginal unit of capacity has to coincide with the sum of its acquisition cost and the option value of the lost expansion opportunity.

An interesting characterization of the optimal policies in terms of the required excess returns is now summarized in the following.

**Theorem 3.2.** (A) Along the optimal irreversible disinvestment path

$$\frac{\bar{p}_k F'(k)}{r q_\cdot} = \frac{1}{1 + \rho_\cdot(\bar{p}_k)}, \quad (3.9)$$

where $\rho_\cdot(\bar{p}_k) > 0$. 

11
(B) Along the optimal irreversible investment path

\[
\frac{\hat{p}_k F'(k)}{rq_+} = \frac{1}{1 - \rho_+ (\hat{p}_k)}
\]  

(3.10)

where \( \rho_+ (\hat{p}_k) \in (0, 1) \).

**Proof.** See Appendix B.

When investment is reversible, the optimal accumulation policy is characterized by a rule requiring that the capital stock should be maintained at a level where its marginal revenue product coincides with its marginal user cost. According to Theorem 3.2 this conclusion is no longer true in the presence of irreversibility. More precisely, Theorem 3.2 shows that along the optimal disinvestment boundary, the ratio between the marginal user cost and the marginal revenue product of capital exceeds unity by a factor \( \rho_+ (\hat{p}_k) \) capturing the required excess return due to the irreversibility of the disinvestment decision. Along the findings of Alvarez (2006a) Theorem 3.2 also proves that along the optimal investment boundary the required rate of return accrued from the acquisition of a marginal unit of capacity has to exceed unity by a factor \( \rho_+ (\hat{p}_k) / (1 - \rho_+ (\hat{p}_k)) \) measuring the excess rate of return in that case.

The values of the associated optimal policies are now characterized explicitly in the following theorem.

**Theorem 3.3.** (A) The optimal irreversible disinvestment policy reads as \( k^- = \min(k, \hat{k}_p) \), where \( \hat{k}_p = \bar{p}_p^{-1} \) denotes the optimal disinvestment boundary in terms of the underlying price and \( l_t = \inf \{ p_s ; s \leq t \} \) denotes the minimum price attained up to date \( t \). Along the optimal irreversible disinvestment path the value of a marginal unit of capacity reads as \( Q(k, p) = V_-(k, p) \), where the value of the optimal disinvestment policy

\[
V^-(k, p) = \sup_{k^- \in \Lambda} E_p \int_0^\infty e^{-rs} (p_s F(k_s) ds + q_- dk_s^-)
\]  

(3.11)

can be expressed explicitly as

\[
V^-(k, p) = \begin{cases} 
q_- (k - \hat{k}_p) + V^- (\hat{k}_p, p) & k \geq \hat{k}_p \\
G(p) F(k) - \varphi(p) \int_0^k \frac{G'(p_y) F'(y)}{\varphi'(p_y)} dy & k < \hat{k}_p.
\end{cases}
\]  

(3.12)
The optimal irreversible investment policy reads as \( k^+_i = \max(k, \hat{k}_{ui}) \), where \( \hat{k}_p = \hat{p}^{-1} \) denotes the optimal investment boundary in terms of the underlying price and \( u_t = \sup\{p_s; s \leq t\} \) denotes the maximum price attained up to date \( t \). Along the optimal irreversible investment path the value of a marginal unit of capacity reads as \( H(k, p) = V^+_k(k, p) \), where the value of the optimal investment policy

\[
V^+(k, p) = \sup_{k^+ \in \Lambda} E_p \int_0^\infty e^{-rs}(p_s F(k_s)ds - q_+dk^+_s)
\]  

(3.13)
can be expressed explicitly as

\[
V^+(k, p) = \begin{cases} 
G(p)F(k) + \psi(p) \int_k^\infty \frac{G'(p_y)F'(y)}{\psi'(p_y)}dy & k > \hat{k}_p \\
q_+(k - \hat{k}_p) + V^+(\hat{k}_p, p) & k \leq \hat{k}_p 
\end{cases}
\]  

(3.14)

**Proof.** See Appendix C.

Theorem 3.3 characterizes both the optimal policies and their values in the two associated cases. As intuitively is clear, the optimal irreversible disinvestment policy is such that the capital stock is maintained above the optimal disinvestment boundary by disinvesting a marginal unit of capacity each time the underlying price reaches the optimal boundary \( \bar{p}_k \). In a similar fashion, the optimal irreversible investment policy can be characterized as a rule requiring that an incremental unit of stock is acquired each time the underlying price reaches the optimal capital dependent boundary \( \hat{p}_k \). In that case the optimal policy is to maintain the stock above the critical boundary at all times.

As our results in the characterization of the value of a marginal unit of stock in Theorem 3.1 already indicated, the values of the optimal irreversible policies can be expressed in terms of the values of the available options for the firm. If disinvestment is irreversible and the initial stock is above the optimal capacity then the firm instantaneously disinvests the excess stock \( k - \bar{k}_p \) yielding the return \( q(k - \bar{k}_p) \) and the value \( V^-(\bar{k}_p, p) \) measuring the sum of the expected cumulative present value of the revenue product of the optimal operational stock and the option value of the subsequent opportunities to disinvest. If investment is irreversible and the initial stock is below the
optimal capacity then the firm instantaneously makes a lump sum investment \( k - \hat{k}_p \) and, therefore, incurs the instantaneous sunk cost \(-q(\hat{k}_p - k)\). In this way the firm obtains the value \( V^+(\hat{k}_p, p) \) capturing the sum of the expected cumulative present value of the revenue product of the optimal capacity and the value of the opportunities to expand capacity later in the future.

4 The Impact of Costly Reversibility

Having considered the associated valuations, we now return to the original accumulation problem and investigate the determination of the optimal policy. In order to accomplish this task, we observe that since the associated value (2.8) depends on the particular choice of the boundaries \( p_l \) and \( p_u \) it is natural to ask whether these boundaries can be chosen in a way which would maximize this value. More precisely, it is important to investigate whether there is a pair of boundaries \( \hat{p}_l < \hat{p}_u \) yielding a value \( J^*(k, p) \) which dominates all the values resulting from other similar admissible boundary policies characterized by two separate thresholds. Interestingly, it turns out that the answer to this question is positive as is established in our following theorem.

**Theorem 4.1.** There is a unique optimal pair \((\hat{p}_l, \hat{p}_u) \in (\bar{p}_k, rq_+/F'(k)) \times (rq_+/F'(k), \hat{p}_k)\) satisfying the ordinary first order optimality conditions

\[
\begin{align*}
\int_{\hat{p}_l}^{\hat{p}_u} \psi(y)yF'(k)m'(y)dy &= q_+ \frac{\psi'(p_u^*)}{S'(p_u^*)} - q_- \frac{\psi'(p_l^*)}{S'(p_l^*)} \quad (4.1) \\
\int_{\hat{p}_l}^{\hat{p}_u} \phi(y)yF'(k)m'(y)dy &= q_+ \frac{\phi'(p_u^*)}{S'(p_u^*)} - q_- \frac{\phi'(p_l^*)}{S'(p_l^*)}. \quad (4.2)
\end{align*}
\]

In this case the value of a marginal unit of capacity reads for all \( p \in (\hat{p}_l, \hat{p}_u) \) as

\[
J^*(k, p) = G(p)F'(k) + \frac{\hat{\psi}(p)}{\hat{\phi}(\hat{p}_l^*)} \left[ q_+ - G(p_l^*)F'(k) \right] - \frac{\hat{\psi}(p)}{\hat{\phi}(p_u^*)} \left[ G(p_u^*)F'(k) - q_+ \right], \quad (4.3)
\]

where \( \hat{\phi}(p) = \phi(p) - \phi(p_u^*)\psi(p)/\psi(p_u^*) \) and \( \hat{\psi}(p) = \psi(p) - \psi(p_l^*)\phi(p)/\phi(p_l^*) \). Especially, the optimal thresholds are increasing functions of capital (i.e. \( \partial p_u^*/\partial k > 0 \) and \( \partial p_l^*/\partial k > 0 \)) and satisfy the limiting conditions \( \lim_{k \to 0} p_l^* = \lim_{k \to 0} p_u^* = 0 \) and \( \lim_{k \to \infty} p_l^* = \lim_{k \to \infty} p_u^* = \infty \). Moreover, the value \( J^*(k, p) \) satisfies the value matching
conditions \( \lim_{p \downarrow p^*} J^*(k,p) = q_- \), \( \lim_{p \downarrow p^*} J^*(k,p) = q_+ \) as well as the smooth fit conditions \( \lim_{p \downarrow p^*} J^*_p(k,p) = \lim_{p \uparrow p^*} J^*_p(k,p) = 0 \).

**Proof.** See Appendix D.

Theorem 4.1 characterizes those circumstances under which a unique pair of optimal boundaries exists and states a pair of ordinary first order conditions from which these boundaries can be determined. An important implication of Theorem 4.1 is that the optimal boundaries are homogeneous of degree zero in the marginal product of capital \( F'(k) \) and the prices \( q_- \) and \( q_+ \). Hence, equal percentage changes in the marginal product of capital and the prices \( q_- \) and \( q_+ \) leave the optimal boundaries unchanged.

Theorem 4.1 also shows how the value of a marginal unit of capacity can be decomposed in terms of the available opportunities for the optimally investing firm. According to (4.3) the value value of a marginal unit of capital constitutes the sum of the expected cumulative present value of the revenue product it generates, the option value of the future opportunity to disinvest, and the value of the subsequent expansion options. It is worth emphasizing that the multiplicatively separable form of the optimality conditions (4.1) and (4.2) implies that the optimal boundaries have to satisfy the identity

\[
\int_{p^*_u}^{p^*_d} \psi(y)ym'(y)dy \left[ q_- \frac{\varphi'(p^*_u)}{S'(p^*_u)} - q_- \frac{\varphi'(p^*_d)}{S'(p^*_d)} \right] = \int_{p^*_l}^{p^*_u} \varphi'(y)ym'(y)dy \left[ q_+ \frac{\psi'(p^*_u)}{S'(p^*_u)} - \frac{\psi'(p^*_d)}{S'(p^*_d)} \right]
\]

which is independent of the capital stock. This observation is of interest since it demonstrates that even though the optimal boundaries are functionally dependent on each other, the size of the installed stock does not affect this dependence.

Interestingly, the optimal accumulation rule can again be interpreted in terms of Tobin’s marginal \( q \) associated with the particular policy. In this case the optimal investment rule requires that capacity should be increased as soon as its marginal \( V_k(k,p)/q_+ \) is equal to one. Analogously, disinvestment is optimal as soon as the marginal \( q_-/V_k(k,p) \) associated to the disinvestment opportunity becomes equal to one.

It is worth emphasizing that the conclusions of Theorem 4.1 are important from the point of view of other related stochastic control problems as well. First, given the
close connection of the considered accumulation problem to optimal switching problems encountered in studies considering optimal entry and exit (cf. Dixit (1989)) and in studies considering the optimal operation of a mine (cf. Brennan and Schwartz (1985)) our existence and uniqueness proof extends to those modeling frameworks as well. Similarly, studies considering rational hiring and firing policies of competitive firms are mathematically based on a completely analogous approach as this study and, therefore, our findings are applicable within that framework as well (cf. Bentolila and Bertola (1990)).

An interesting implication of Theorem 4.1 restating the optimality conditions in an alternative form is now summarized in the following.

**Corollary 4.2.** The optimality conditions (4.1) and (4.2) can be rewritten as

\[
C_1(p^*_l, q-) = C_1(p^*_u, q_+) \quad \text{and} \quad C_2(p^*_l, q-) = C_2(p^*_u, q_+),
\]

where

\[
C_1(p, q) = \frac{1}{BS'(p)} \left( \varphi'(p)(G(p)F'(k) - q) - \varphi(p)G'(p)F'(k) \right) 
\]

(4.4)

\[
C_2(p, q) = \frac{1}{BS'(p)} \left( \psi'(p)(q - G(p)F'(k)) + \psi(p)G'(p)F'(k) \right). 
\]

(4.5)

**Proof.** See Appendix E.

Corollary 4.2 states an alternative representation of the optimality conditions (4.1) and (4.2) by combining the proven value matching and smooth fit conditions. It is worth noticing that this reformulation of the optimality conditions emphasizes the role of the values of the underlying disinvestment and expansion options as the principal determinants of the optimal accumulation policy since \(C_1(p, q)\) is associated to the growth rate of the function \((G(p)F'(k) - q)/\varphi(p)\) and \(C_2(p, q)\) is associated to the growth rate of the function \((G(p)F'(k) - q)/\psi(p)\).

In accordance with our observations in the associated problems considered in the previous section, Theorem 4.1 proves that along the optimal investment boundary the marginal revenue product of capital has to exceed the interest on the marginal acquisition cost \(q_+\) and along the optimal disinvestment boundary the marginal revenue product of capital has to fall short the interest on the marginal disinvestment price \(q_-\). Our main conclusions on the magnitude of this “hysteresis” effect (cf. Dixit (1989) and Dixit (1992)) are now stated in our following theorem.
Theorem 4.3. Along the optimal capital accumulation path we have

\[
\frac{1}{1 - \rho_+ (\hat{p}_k)} = \frac{\hat{p}_k F'(k)}{rq_+} > \frac{p^*_u F'(k)}{rq_+} = \frac{1}{1 - E^+(p^*_l, p^*_u)} \tag{4.6}
\]

\[
\frac{1}{1 + \rho_- (\hat{p}_k)} = \frac{\hat{p}_k F'(k)}{rq_-} < \frac{p^*_l F'(k)}{rq_-} = \frac{1}{1 + E^-(p^*_l, p^*_u)} \tag{4.7}
\]

where \(0 < E^+(p^*_l, p^*_u) < \min(1 - p^*_l/p^*_u, \rho_+ (\hat{p}_k))\) and \(0 < E^-(p^*_l, p^*_u) < \min(p^*_u/p^*_l - 1, \rho_- (\hat{p}_k))\). Moreover, \(\lim_{p_u \uparrow 1} E^+(p_l, p_u) = \rho_+(p_u)\) and \(\lim_{p_u \to \infty} E^-(p_l, p_u) = \rho_-(p_l)\).

**Proof.** See Appendix F. \(\square\)

Theorem 4.3 extends the findings of Theorem 3.2 to the case where disinvestment is possible but only at a cost. As intuitively is clear, the required excess returns associated to the available disinvestment and investment options are in this case more complex and depend on both optimal boundaries. In this case, we observe that the required excess return from acquiring a further marginal unit of capacity reads as \(\frac{E^+(p^*_l, p^*_u)}{1 - E^+(p^*_l, p^*_u)}\). As is established in Theorem 4.3 this return increases towards the excess return associated with the case where only investment is possible and the disinvestment boundary tends to zero. Similarly, the required excess return from disinvesting a marginal unit of capacity reads in the present case as \(E^-(p^*_l, p^*_u)\) which again increases towards the excess return associated with the case where only irreversible disinvestment is possible and the investment boundary tends to infinity. Since both these values can be attained by either setting the investment cost \(q_+ = \infty\) or the disinvestment return \(q_- = 0\) we find that the required excess returns associated to the problems considered in the previous section can be attained by letting the parameters \(q_+\) and \(q_-\) tend towards the above mentioned critical levels. An economically interesting and intuitively appealing implication of these observations is that even though the presence of costly reversibility diminishes the magnitude of hysteresis in comparison with the complete irreversibility case, it does not remove it from the optimal accumulation rule.

Having characterized the optimal accumulation policy by focusing on the marginal decision, we can now establish the following theorem characterizing the value of the optimal capital accumulation policy and its option components.
Theorem 4.4. Under the optimal capital accumulation policy the value of capital reads as

\[
V(k, p) = \begin{cases} 
q_-(k - k^*) + V(k^*, p) & k \geq k^* \\
G(p) F(k) + \int_{k^-}^k \pi_-(y, p) dy + \int_{k}^{k^*} \pi_+(y, p) dy & k^* < k < k^* \\
q_+(k - k^+) + V(k^+, p) & k \leq k^+
\end{cases} \tag{4.8}
\]

where \( k^- = p^*^{-1}(p) \) denotes the boundary at which disinvestment is optimal, \( k^+ = p^*^{-1}(p) \) denotes the boundary at which investment is optimal,

\[
\pi_-(k, p) = \frac{\tilde{\varphi}(p)}{\varphi(p^*)} [q_--G(p^*) F'(k)] \tag{4.9}
\]

denotes the option value of the embedded disinvestment opportunity and

\[
\pi_+(k, p) = \frac{\tilde{\psi}(p)}{\psi(p^*)} [G(p^*) F'(y) - q_+] \tag{4.10}
\]

denotes the option value of the opportunity to expand productive capacity.

Proof. See Appendix G.

Theorem 4.4 characterizes the value of the optimal accumulation policy. As intuitively is clear, on the continuation region where altering the current capacity is suboptimal the value of the optimal policy is essentially constituted by three factors. Namely, the expected cumulative present value of the revenue product generated by the current installed productive stock, the option value of the future disinvestment opportunities, and the value of the options to expand productive capacity later in the future. In contrast to the cases where the firm can only either invest or disinvest, we now observe that the values of the embedded investment and disinvestment opportunities are interdependent and, therefore, that the chosen investment policy affects the optimal disinvestment policy and vice versa.

It would be of interest to characterize the impact of increased volatility on the optimal boundaries and, therefore, on the required exercise premia. Unfortunately, establishing simple easily verifiable conditions under which the sign of the relationship between higher volatility and the optimal boundaries could be unambiguously
characterized in the general setting is extremely difficult, if possible at all. To motivate this argument, we observe that Theorem 4.1 indicates that $J^*_p(k, p^*_u) = 2(rq_+ - p^*_u F'(k))/\sigma^2(p^*_u) > 0$ and $J^*_p(k, p^*_l) = 2(rq_+ - p^*_l F'(k))/\sigma^2(p^*_l) < 0$ demonstrating that the marginal value of capital is locally convex at the optimal disinvestment boundary and locally concave at the optimal investment boundary. On the other hand, the value matching conditions $J^*_p(k, p^*_u) = J^*_p(k, p^*_l) = 0$ and Rolle’s theorem implies that there has to be at least one price $\hat{p} \in (p^*_l, p^*_u)$ such that $J^*_p(k, \hat{p}) = 0$. Thus, our results indicate that typically the sensitivity $J^*_p(k, p)$ of the marginal value of capital with respect to changes in the underlying price is not monotonic and, therefore, that the impact of increased volatility on $J^*(k, p)$ is ambiguous. However, given the local characterization of the value function in the neighborhood of the optimal boundaries, we conjecture that higher price volatility typically increases the optimal investment boundary $p^*_u$ and decreases the optimal disinvestment boundary $p^*_l$.

5 Illustration

5.1 Exponential Price Dynamics

We begin the illustration of our general findings by relying on the standard geometric characterization of the underlying unit price of output. More precisely, we assume that this price evolves according to a geometric Brownian motion described by the stochastic differential equation

$$dp_t = \mu p_t dt + \sigma p_t dW_t, \quad p_0 = p,$$

where both the drift coefficient $\mu$ and the volatility coefficient $\sigma$ are assumed to be exogenously determined constants. It is well-known that in this case the fundamental solutions read as $\psi(p) = p^\alpha$ and $\varphi(p) = p^\beta$, where

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0.$$
denotes the positive and
\[
\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0
\]
the negative root of the characteristic equation \(\sigma^2 x(x-1) + 2\mu x - 2r = 0\). Since now 
\(S'(p) = p^{\alpha+\beta-1}\) and \(m'(p) = 2p^{-\alpha-\beta-1}/\sigma^2\) we observe that the optimality conditions presented in Corollary 4.2 can in the present example be expressed as (provided that the condition \(r > \mu\) is satisfied)
\[
(\beta - 1)p_u^{1-\alpha} \frac{F'(k)}{r - \mu} - q_+ p_u^{1-\alpha} = (\beta - 1)p_l^{1-\alpha} \frac{F'(k)}{r - \mu} - q_- p_l^{1-\alpha}
\]
and
\[
(1 - \alpha)p_u^{1-\beta} \frac{F'(k)}{r - \mu} + \alpha q_+ p_u^{1-\beta} = (1 - \alpha)p_l^{1-\beta} \frac{F'(k)}{r - \mu} + \alpha q_- p_l^{1-\beta}.
\]
By collecting terms, we find that these optimality conditions can be expressed in the alternative form
\[
(\beta - 1)p_u^{1-\alpha} \frac{F'(k)}{r - \mu} [R - R^\alpha] = \beta [q_+ - q_- R^\alpha]
\]
and
\[
(1 - \alpha)p_u^{1-\beta} \frac{F'(k)}{r - \mu} [R - R^\beta] = \alpha [q_- R^\beta - q_+]
\]
where \(R = p_u^*/p_l^*\) denotes the price ratio of the optimal boundaries. Combining equations (5.4) and (5.5) finally demonstrates that the ratio \(R\) constitutes the root of equation (cf. Abel and Eberly (1996))
\[
(\alpha - \beta)(q_+ R^{1-\beta} + q_- R^\alpha) + \alpha(\beta - 1)(q_- R + q_+ R^\alpha - \beta) + \beta(1 - \alpha)(q_- R^{\alpha-\beta-1} + q_+) = 0
\]
which is independent of capital and the particular choice of the production function. Moreover, in the present example the critical boundaries of the associated irreversible investment and irreversible disinvestment problems read as
\[
\hat{p}_k = \left(1 - \frac{1}{\beta}\right) \frac{rq_+}{F'(k)}
\]
and
\[ \hat{p}_k = \left( 1 - \frac{1}{\alpha} \right) \frac{rq_-}{F'(k)}. \]

Interestingly, in the present example we observe that
\[ \frac{\partial \hat{p}_k}{\partial \sigma} = \frac{2(\beta - 1)}{\beta(\alpha - \beta)} \frac{rq_+}{F'(k)} > 0 \]

and
\[ \frac{\partial \bar{p}_k}{\partial \sigma} = \frac{2(1 - \alpha)}{\alpha(\alpha - \beta)} \frac{rq_-}{F'(k)} < 0. \]

proving that increased price volatility expands the continuation set where exercising an irreversible investment or disinvestment decision is suboptimal.

As is clear from the optimality conditions (5.2) and (5.3) solving the optimal boundaries explicitly is typically impossible and numerical techniques are needed in order to illustrate the optimal boundaries. We illustrate these boundaries in Figure 1 on the \((k, p)\)-plane under the parameter specifications \(\mu = 0.025, r = 0.035, q_+ = 30, q_- = 10, \sigma = 0.1\), and the assumption that the production function is of the standard Cobb-Douglas type \(F(k) = k^\theta\) with \(\theta = 0.75\). As Figure 1 clearly indicates, the difference

![Figure 1: The Optimal Boundaries](image_url)

between the two optimal boundaries increases as the capital stock becomes larger. Thus, even though the price at which divestment becomes optimal increases as productive capacity becomes larger, it does not increases as fast as the price at which the acquisition of new capital is optimal. As is clear from our Theorem 4.3 the main reason for this
observation is naturally the difference between the acquisition cost \( q_+ \) and the selling price \( q_- \) of productive capacity. The larger the difference \( q_+ - q_- \) is, the greater the difference between the optimal boundaries becomes.

The ratio of the optimal boundaries characterized in (5.6) is illustrated as a function of the volatility of the underlying price in Figure 2 under the parameter specifications \( \mu = 0.025, r = 0.035, q_+ = 11, \) and \( q_- = 10. \) As Figure 2 clearly shows, the ratio \( p_*/p_l^* \) is an increasing function of volatility. Thus, our numerical illustrations indicate that increased volatility expands the continuation region where investment is suboptimal by increasing the relative difference between the two optimal boundaries.

Figure 2: The Optimal Price Ratio \( p_*/p_l^* \)

5.2 Mean Reverting Price Dynamics

In order to illustrate our findings explicitly within a mean reverting setting, we now assume that the underlying price dynamics evolve according to the random dynamics characterized by the stochastic differential equation (a radial Ornstein-Uhlenbeck process, cf. Borodin and Salminen (2002), p. 72)

\[
dp_t = (a - bp_t)dt + \sigma \sqrt{p_t}dW_t, \quad p_0 = p,
\]

(5.7)

where \( a, b, \sigma \in \mathbb{R}_+ \) are exogenously determined constants. It is worth noticing that in this case the lower boundary 0 is unattainable as long as the condition \( 2a \geq \sigma^2 \) is
satisfied. Otherwise 0 is regular for the stochastic price dynamics and in that case the stochastic differential equation (5.7) is a proper description of the underlying stochastic dynamics only up to the first hitting time of \( p_t \) to 0. Moreover, the price dynamics tends towards the long run stationary Gamma-distributed value \( p_\infty \) characterized by the probability density
\[
P(z) = \frac{1}{\Gamma(2a/\sigma^2)} \left( \frac{2b}{\sigma^2} \right)^{2a/\sigma^2} z^{2a/\sigma^2-1} e^{-2bz/\sigma^2}.
\]
Consequently, \( \text{var}[p_\infty] = \sigma^2 a/(2b^2) \).

The linearity of the drift coefficient implies that in the present example the expected price reads as
\[
E_p[p_t] = \frac{a}{b} + e^{-bt} \left( p - \frac{a}{b} \right).
\]
Therefore, the expected cumulative present value of the flow \( p \) can be expressed as
\[
G(p) = \frac{p}{r + b} + \frac{a}{r(r + b)}.
\]
The fundamental solutions of the ordinary second order differential equation \( \sigma^2 p u''(p) + 2(a - bp) u'(p) - 2ru(p) = 0 \) now read as (cf. Borodin and Salminen (2002), pp. 138–140)
\[
\psi(p) = M \left( \frac{r}{b}, 2a/\sigma^2, 2bp/\sigma^2 \right)
\]
and
\[
\varphi(p) = U \left( \frac{r}{b}, 2a/\sigma^2, 2bp/\sigma^2 \right)
\]
where \( M \) denotes the confluent hypergeometric function of the first type and \( U \) denotes the confluent hypergeometric function of the second type.

Unfortunately, given the nature of the fundamental solutions solving the optimal boundaries explicitly from the optimality conditions is impossible. Thus, we illustrate the optimal boundaries numerically in Figure 3 on the \((k, p)\)-plane under the parameter specifications \( a = 0.01, b = 0.05, r = 0.04, \sigma = 0.1, q_+ = 11, q_- = 10, \) and the assumption that the production function is of the standard Cobb-Douglas type \( F(k) = k^\theta \) with \( \theta = 0.75 \). As in the case where the underlying price evolved according to a geometric Brownian motion, we again observe that the optimal boundaries are increasing and concave as functions of the capital stock. The impact of increased volatility on the optimal
boundaries characterizing the optimal accumulation policy is illustrated in Figure 4 under the same parameter specification as above. In accordance with our findings in the previous section, we again observe that increased volatility expands the continuation region where adjusting the capital stock is suboptimal by increasing the ratio between the optimal investment boundary and the optimal disinvestment boundary. However, in contrast with our findings in the case where the underlying price evolved according to a geometric Brownian motion, the ratio between the optimal boundaries $p^*_u/p^*_l$ appears to be non-linear and, therefore, characterizing it as a function of volatility is not possible in this particular case.
6 Conclusions

In this study we developed a general model of optimal capital accumulation under price uncertainty and costly reversibility of investment. We characterized the optimal accumulation policy of a firm facing decreasing return to scale for a broad class of underlying price dynamics and extended previous studies addressing this issue. In accordance with previous studies considering capital accumulation in the presence of uncertainty and costly reversibility, we found that the optimal accumulation policy can be characterized in terms of two boundaries at which the productive stock is optimally adjusted. One of the boundary characterizes the price and capital stock combinations at which investment is optimal and a further marginal unit of capacity should be acquired. The other boundary, in turn, characterizes the price and capital stock combinations at which disinvestment is optimal and an incremental unit of stock should be sold. As intuitively is clear, the optimal accumulation policy is such that the stock is maintained between these two critical boundaries at all times. Moreover, since the capital stock was assumed to be non-depreciating and the optimal policy was shown to be incremental, the optimal accumulation path consists typically of potentially long periods of time where the firm operates with the prevailing stock after which the firm enters into regimes of very intense investing or disinvesting. We also found that even though the possibility of selling capital increases the operational flexibility of the firm and in this way naturally decreases the excess returns associated to the optimal decision in comparison with the case where investment or disinvestment are irreversible, it does not remove completely these excess returns. More precisely, our results demonstrated that at the optimal investment boundary the revenue product of a marginal unit of capacity exceeds the interest on the marginal acquisition cost. Similarly, we found that at the optimal disinvestment boundary the interest on the selling price of a marginal unit of capacity exceeds the revenue product it would generate if it would not be sold. Therefore, our findings proved that hysteresis prevails in the general setting as well. We also characterized the value of the optimal policy and found that the value can be expressed in terms of the values of three factors. Namely, the expected cumulative present value of
the revenue product generated by current capacity, the option value of the future disinvestment opportunities, and the value of the options to expand productive capacity later in the future.

Given the assumed infinite planning horizon of the optimally investing firm, it would naturally be of interest to analyze how the potential variability of the discount rate affects the optimal accumulation policy (cf. Alvarez (2006b)). Analogously, introducing technological progress into the model by letting the productivity of capital fluctuate in time would constitute a second economically interesting direction towards which the model could be developed (cf. Bertola (1998)). A third interesting extension of our problem would be to introduce fixed sunk costs into the investment and disinvestment decisions of the firm since that case typically leads to lump-sum investment and disinvestment policies (cf. Caballero and Leahy (1996)). Unfortunately, all these extensions are extremely challenging problems outside the scope of the present study.
References


A Proof of Lemma 3.1

Proof. (A) Consider the functional

$$\Pi(p, k) = \frac{q_0 - G(p)F'(k)}{\varphi(p)}.$$ 

Standard differentiation and application of the representation (2.6) now yields that

$$\Pi_p(p, k) = S'(p)L(p, k)/\varphi^2(p)$$

where

$$L(p, k) = \int_p^\infty \varphi(y)(rq_- - yF'(k))m'(y)dy.$$ 

It is clear that $L(p, k) < 0$ as long as $pF'(k) > rq_-$. Let $p_0 < rq_-/F'(k)$ be arbitrary and assume that $p < p_0$. Then the monotonicity of the function $rq_- - pF'(k)$ implies that

$$L(p, k) = \int_p^{p_0} \varphi(y)(rq_- - yF'(k))m'(y)dy + L(p_0, k) > \frac{(rq_- - p_0F'(k))}{r} \left[ \frac{\varphi'(p_0)}{S'(p_0)} - \frac{\varphi'(p)}{S'(p)} \right] + L(p_0, k).$$ 

Since $\varphi'(p)/S'(p) \downarrow -\infty$ as $p \downarrow 0$ we observe that equation $L(p, k) = 0$ has at least one root $\bar{p}_k \in (0, rq_-/F'(k))$. Since $L_p(p, k) = \varphi(p)(pF'(k) - rq_-)(p) < 0$ for all $p \in (0, rq_-/F'(k))$, we find that the root is unique. Moreover, since $L(p, k) \gtrless 0$ for $p \gtrless \bar{p}_k$ we also observe that

$$\bar{p}_k = \arg\max_{p \in \mathbb{R}^+} \left\{ \frac{q_0 - G(p)F'(k)}{\varphi(p)} \right\}.$$ 

Having established the existence and uniqueness of the optimal disinvestment threshold, denote now the proposed value function as $\hat{Q}(k, p)$. We first observe that the proposed value can be expressed as

$$\hat{Q}(k, p) = G(p)F'(k) + \mathbb{E}_p \left[ e^{-\tau t} \left( q_0 - G(p_\tau)F'(k) \right) \right],$$

where $\tau = \inf\{t \geq 0 : p_t \leq \bar{p}_k\}$ denotes the first exit time from the continuation set $(\bar{p}_k, \infty)$. Therefore, $\hat{Q}(k, p) \leq Q(k, p)$. To prove the opposite inequality we first observe that the proposed value function can be expressed as

$$\hat{Q}(k, p) = G(p)F'(k) + \varphi(p) \sup_{y \leq p} \left\{ \frac{q_0 - G(y)F'(k)}{\varphi(y)} \right\}.$$
proving that \( \tilde{Q}(k, p) \geq q_- \) for all \((k, p) \in \mathbb{R}_+^2\). We also notice that for any fixed stock \(k \in \mathbb{R}_+\) \(\tilde{Q}(k, p)\) is as a function of \(p\) continuously differentiable, twice continuously differentiable on \(\mathbb{R}_+ \setminus \{\bar{p}_k\}\), and satisfies the inequality \(|\tilde{Q}_{pp}(k, \bar{p}_k \pm)| < \infty\). Moreover, since \((A\tilde{Q})(k, p) - r\tilde{Q}(k, p) + pF'(k) = 0\) on \((\bar{p}_k, \infty)\) and
\[
(A\tilde{Q})(k, p) - r\tilde{Q}(k, p) + pF'(k) = pF'(k) - rq_- < 0
\]
on \((0, \bar{p}_k)\) we find that the proposed value satisfies the sufficient variational inequalities guaranteeing that \(\tilde{Q}(k, p) \geq Q(k, p)\) and, therefore, that \(\tilde{Q}(k, p) = Q(k, p)\).

Standard differentiation of the value function \(Q(k, p)\) on the continuation region where disinvestment is suboptimal yields
\[
Q_k(k, p) = \varphi(p) \left( \frac{G(p)}{\varphi(p)} - \frac{G(\bar{p}_k)}{\varphi(\bar{p}_k)} \right) F''(k)
\]
and
\[
Q_p(k, p) = \varphi'(p) \left( \frac{G'(p)}{\varphi'(p)} - \frac{G'(\bar{p}_k)}{\varphi'(\bar{p}_k)} \right) F'(k)
\]
Since
\[
\frac{d}{dp} \left[ \frac{G(p)}{\varphi(p)} \right] = \frac{S'(p)}{\varphi^2(p)} \int_{p}^{\infty} \varphi(y)ym'(y)dy > 0
\]
and
\[
\frac{d}{dp} \left[ \frac{G'(p)}{\varphi'(p)} \right] = \frac{2rS'(p)}{\sigma^2(p)\varphi^2(p)} \int_{p}^{\infty} \varphi(y)(p - y)m'(y)dy < 0
\]
we find that \(Q_k(k, p) < 0\) and \(Q_p(k, p) > 0\) on \(p > \bar{p}_k\). On the other hand, since \(Q(k, p) = q_-\) on \(p \leq \bar{p}_k\) the alleged results follow.

It remains to establish the alleged monotonicity of the threshold and its limiting values. Implicit differentiation yields
\[
\frac{\partial \bar{p}_k}{\partial k} = \frac{\int_{\bar{p}_k}^{\infty} \varphi(y)ym'(y)dyF''(k)}{(\bar{p}_kF'(k) - rq_-)\varphi(\bar{p}_k)m'(\bar{p}_k)} > 0
\]
proving the monotonicity of \(\bar{p}_k\). Since \(\lim_{k \to \infty} F'(k) = 0\) and \(\lim_{p \to \infty} \varphi'(p)/S'(p) = 0\) we observe that \(\lim_{k \to \infty} \bar{p}_k = \infty\). Similarly, since \(\lim_{k \downarrow 0} F'(k) = \infty\) and \(\lim_{p \downarrow 0} \varphi'(p)/S'(p) = -\infty\) we find that \(\lim_{k \downarrow 0} \bar{p}_k = 0\). Establishing part (B) of our theorem is entirely analogous. \(\square\)
B Proof of Theorem 3.2

Proof. (A) It is clear from the proof of Theorem 3.1 that the first order optimality condition can be restated by adding and subtracting \( \bar{p}_k \) from the term \( y \) and applying the identity

\[
-\frac{\varphi'(p)}{S'(p)} = r \int_p^\infty \varphi(y)m'(y)dy
\]
as

\[
\int_{\bar{p}_k}^{\infty} \int_{\bar{p}_k}^{y} \varphi(y)m'(y)dtdy - \frac{\bar{p}_k \varphi'(\bar{p}_k)}{r S'(\bar{p}_k)} = -\frac{q_- \varphi'(\bar{p}_k)}{F'(k) S'(\bar{p}_k)}
\]

A standard application of Fubini’s theorem implies that this condition can be re-expressed as

\[
\bar{p}_k \varphi'(\bar{p}_k) + \int_{\bar{p}_k}^{\infty} \frac{\varphi'(y)}{S'(y)}dy = -\frac{q_- \varphi'(\bar{p}_k)}{F'(k) S'(\bar{p}_k)}
\]

Making the change of variable \( v = y/\bar{p}_k \) in the integral expression and multiplying the resulting identity with the term \( S'(\bar{p}_k)F'(k)/\varphi'(\bar{p}_k) \) finally implies that

\[
\bar{p}_k F'(k) \left[ 1 + \frac{S'(\bar{p}_k)}{\varphi'(\bar{p}_k)} \int_{1}^{\infty} \frac{\varphi'(v\bar{p}_k)}{S'(v\bar{p}_k)}dv \right] = rq_-
\]

which completes the proof of part (A) of our theorem. In order to establish part (B) of our theorem we first observe that along the optimal accumulation boundary

\[
\hat{p}_k F'(k) \left[ 1 - \frac{S'(\hat{p}_k)}{\varphi'(\hat{p}_k)} \int_{0}^{1} \frac{\varphi'(v\hat{p}_k)}{S'(v\hat{p}_k)}dv \right] = rq_+
\]

The monotonicity and positivity of \( \psi'(p)/S'(p) \) then implies that

\[
0 < \frac{S'(\hat{p}_k)}{\varphi'(\hat{p}_k)} \int_{0}^{1} \frac{\varphi'(v\hat{p}_k)}{S'(v\hat{p}_k)}dv < 1
\]

completing the proof of our theorem.

C Proof of Theorem 3.3

Proof. (A) Denote the proposed value function as \( \bar{V}^-(k, p) \). Since \( \bar{V}^- \in C^{1,2}(\mathbb{R}_+^2) \),

\[
(A\bar{V}^-)(k, p) - r\bar{V}^-(k, p) + pF(k) = 0 \quad \text{for} \quad k < \bar{k}_p = \bar{p}_p^{-1}, \quad \text{and} \quad d[ (A\bar{V}^-)(k, p) - r\bar{V}^-(k, p) + pF(k) ]/dk \leq 0 \quad \text{for all} \quad (k, p) \in \mathbb{R}_+^2
\]

we observe that \( \bar{V}^-_k(k, p) \geq q_- \) and \( (A\bar{V}^-)(k, p) - r\bar{V}^-(k, p) + pF(k) \leq 0 \) for all \( (k, p) \in \mathbb{R}_+^2 \). Thus, the proposed value
function satisfies the sufficient variational inequalities guaranteeing that it dominates the value of the optimal disinvestment policy and, therefore, that \( V^-(k, p) \geq V^-(k, p) \) for all \( (k, p) \in \mathbb{R}_+^2 \). However, since the value \( V^-(k, p) \) is attained by applying the admissible disinvestment policy \( k_t = \min(k, \hat{k}_t) \), where \( l_t = \inf\{p_s; s \leq t\} \) denotes the minimum price attained up to date \( t \) (i.e. the proposed policy solves the associated Skorokhod problem characterizing the optimal policy; cf. Kobila (1993) and Øksendal (2000)), we find that \( \bar{V}^- (k, p) \leq V^- (k, p) \) for all \( (k, p) \in \mathbb{R}_+^2 \) and, therefore, that \( \bar{V}^- (k, p) = V^- (k, p) \). Establishing part (B) is entirely analogous. \( \square \)

### D Proof of Theorem 4.1

**Proof.** Consider for \( p \in (p_l, p_u) \) the boundary value problem

\[
(\mathcal{A}J)(k, p) - rJ(k, p) + pF'(k) = 0, \\
J(k, p_l) = q_-, \quad J(k, p_u) = q_+ 
\tag{D.1}
\]

where \( p_l < p_u \) are two unknown boundaries. The general solution of the ordinary differential equation reads as

\[
J(k, p) = G(p)F'(k) + c_1 \psi(p) + c_2 \varphi(p). 
\]

Applying the boundary conditions \( J(k, p_l) = q_-, J(k, p_u) = q_+ \) now imply that

\[
c_1 = \frac{\varphi(p_l)(q_+ - G(p_u)F'(k)) - \varphi(p_u)(q_- - G(p_l)F'(k))}{\psi(p_u)\varphi(p_l) - \psi(p_l)\varphi(p_u)} \tag{D.2}
\]

\[
c_2 = \frac{\psi(p_u)(q_- - G(p_l)F'(k)) - \psi(p_l)(q_+ - G(p_u)F'(k))}{\psi(p_u)\varphi(p_l) - \psi(p_l)\varphi(p_u)} \tag{D.3}
\]

and, therefore, that the solution of the boundary value problem is

\[
J(k, p) = G(p)F'(k) + \frac{\varphi(p)}{\varphi(p_l)} \left[ q_- - G(p_l)F'(k) \right] + \frac{\psi(p)}{\psi(p_u)} \left[ q_+ - G(p_u)F'(k) \right], \tag{D.4}
\]

where \( \tilde{\varphi}(p) = \varphi(p) - \varphi(p_u)\psi(p)/\psi(p_u) \) denotes the decreasing and \( \tilde{\psi}(p) = \psi(p) - \psi(p_l)\varphi(p)/\varphi(p_l) \) denotes the increasing fundamental solution (unique up to a multiplicative constant) of the ordinary differential equation \( (\mathcal{A}u)(p) = ru(p) \) subject to \( \tilde{\psi}(p_l) = 0 \) and \( \tilde{\varphi}(p_u) = 0 \).
Having derived the solution of the boundary value problem (D.1) conditional on the two arbitrary boundaries \( p_l \) and \( p_u \), we now investigate whether there is a pair of boundaries \( p_l^* \) and \( p_u^* \) such that the marginal value (D.4) is maximized. Differentiating (D.4) with respect to \( p_l \) and \( p_u \), setting the resulting partial derivatives equal to zero, and simplifying now yields

\[
\frac{\tilde{\varphi}'(p_l)}{S'(p_l)}(l - G(p_l)F'(k)) + \frac{G'(p_l)}{S'(p_l)}F'(k)\tilde{\varphi}(p_l) = B \frac{G(p_u)F'(k) - q_+}{\psi(p_u)} \tag{D.5}
\]

\[
\frac{G'(p_u)}{S'(p_u)}F'(k)\tilde{\psi}(p_u) - \frac{\psi'(p_u)}{S'(p_u)}(G(p_u)F'(k) - q_+) = B \frac{q_- - G(p_l)F'(k)}{\varphi(p_l)} \tag{D.6}
\]

Applying now the representation (2.6) to these identities and simplifying finally yields that if an optimal pair \((p_l^*, p_u^*)\) exists, it has to satisfy the conditions

\[
\int_{p_l^*}^{p_u^*} \psi(y) y F'(k)m'(y) dy = q_+ \frac{\psi'(p_u^*)}{S'(p_u^*)} - q_- \frac{\psi'(p_l^*)}{S'(p_l^*)} \tag{D.7}
\]

\[
\int_{p_l^*}^{p_u^*} \varphi(y) y F'(k)m'(y) dy = q_+ \frac{\varphi'(p_u^*)}{S'(p_u^*)} - q_- \frac{\varphi'(p_l^*)}{S'(p_l^*)} \tag{D.8}
\]

In light of these optimality conditions, we now consider the continuously differentiable mappings

\[
L_1(p, p_l) = \int_{p_l}^{p} \psi(y) y F'(k)m'(y) dy - q_+ \frac{\psi'(p)}{S'(p)} + q_- \frac{\psi'(p_l)}{S'(p_l)} \tag{D.9}
\]

\[
L_2(p, p_u) = \int_{p}^{p_u} \varphi(y) y F'(k)m'(y) dy - q_+ \frac{\varphi'(p_u)}{S'(p_u)} + q_- \frac{\varphi'(p)}{S'(p)} \tag{D.10}
\]

Since 0 was assumed to be unattainable for the underlying price dynamics, we find by applying the canonical form (cf. Borodin and Salminen (2002), p. 18)

\[
\frac{\psi'(p)}{S'(p)} = r \int_{0}^{p} \psi(y)m'(y) dy
\]

that the functional \( L_1(p, p_l) \) can be rewritten as

\[
L_1(p, p_l) = \int_{p_l}^{p} \psi(y)(yF'(k) - rq_+)m'(y) dy + (q_- - q_+) \frac{\psi'(p_l)}{S'(p_l)}.
\]

Standard analysis yields that \( L_1(p_l, p_l) = (q_- - q_+)\psi'(p_l)/S'(p_l) < 0, \)

\[
\frac{\partial L_1}{\partial p}(p, p_l) = \psi(p)(pF'(k) - rq_+)m'(p) \geq 0, \quad p \geq \frac{rq_+}{F'(k)}
\]

\[
\frac{\partial L_1}{\partial p_l}(p, p_l) = -\psi(p_l)(p_lF'(k) - rq_-)m'(p_l) \geq 0, \quad p_l \leq \frac{rq_-}{F'(k)}.
\]
In light of these observations, it is clear that if \( p_l < p \leq rq_+/F'(k) \) then \( L_1(p, p_l) < 0 \). Assume therefore that \( p > \hat{p} > rq_+/F'(k) \). It is then clear that

\[
L_1(p, p_l) \geq L_1(\hat{p}, p_l) + \left( \frac{\hat{p}F'(k) - rq_+}{r} \right) \left( \frac{\psi'(p)}{S'(p)} - \frac{\psi'(\hat{p})}{S'(\hat{p})} \right).
\]

Since \( \psi'(p)/S'(p) \to \infty \) as \( p \to \infty \) we find that \( \lim_{p \to \infty} L_1(p, p_l) = \infty \) and, therefore, that equation \( L_1(p, p_l) = 0 \) has for all \( p_l \in \mathbb{R}_+ \) a unique root \( p^*_u(p_l) \in (rq_+/F'(k), \infty) \). Moreover, in light of the analysis above, we observe that the root \( p^*_u(p_l) \) satisfies the limiting conditions \( \lim_{p_l \to \infty} p^*_u(p_l) = \infty \), \( \lim_{p_l \to 0} p^*_u(p_l) = p^*_u(0) \) (where \( p^*_u(0) \) coincides with the optimal investment threshold in the complete irreversibility case; cf. Alvarez (2005)), as well as the monotonicity condition

\[
p^*_u(p_l) = \frac{\psi(p_l)(p_lF'(k) - rq_-)m'(p_l)}{\psi(p^*_u(p_l))(p^*_u(p_l)F'(k) - rq_+)} \geq 0, \quad p_l \geq \frac{rq_-}{F'(k)}.
\]

Consequently, we find that the root \( p^*_u(p_l) \) attains a global minimum at \( rq_-/F'(k) \) and that \( p^*_u(p_l) \geq rq_+/F'(k) \) for all \( p_l \). By noticing that

\[
L_2(p, p_u) = \int_p^{p_u} \varphi(y)(yF'(k) - rq_-)m'(y)dy - (q_+ - q_-)\frac{\varphi'(p_u)}{S'(p_u)}
\]

one can establish in a completely analogous fashion that equation \( L_2(p, p_u) = 0 \) has for all \( p_u \in \mathbb{R}_+ \) a unique root \( p^*_l(p_u) \in (0, rq_-/F'(k)) \) that satisfies the monotonicity condition

\[
p^*_l(p_u) = \frac{\varphi(p_u)(p_uF'(k) - rq_-)m'(p_u)}{\varphi(p^*_l(p_u))(p^*_l(p_u)F'(k) - rq_-)} \leq 0, \quad p_u \leq \frac{rq_+}{F'(k)}
\]

and the limiting conditions \( \lim_{p_u \to 0} p^*_l(p_u) = 0 \) and \( \lim_{p_u \to \infty} p^*_l(p_u) = p^*_l(\infty) \), where \( p^*_l(\infty) \) constitutes the optimal exit threshold satisfying the condition (cf. Alvarez (1998))

\[
\int_{p^*_l(\infty)}^{\infty} \varphi(y)(yF'(k) - rq_-)m'(y)dy = 0.
\]

Given these observations, we notice that the roots \( p^*_l(p_u) \) and \( p^*_u(p_l) \) have a unique interception point on the set \( (p_l, p_u) \in (0, rq_-/F'(k)) \times (rq_+/F'(k), \infty) \). Thus, we find that the optimality conditions (D.7) and (D.8) have a unique root \( (p^*_l, p^*_u) \in \mathbb{R}_+^2 \).

In order to analyze the sensitivity of the optimal thresholds with respect to changes in the capital stock implicit differentiation of the optimality conditions (4.1) and (4.2)
yields

\[
\begin{bmatrix}
\psi(p_u^\ast)\pi_1(p_u^\ast, k)m'(p_u^\ast) & \psi(p_t^\ast)\pi_2(p_t^\ast, k)m'(p_t^\ast) \\
\varphi(p_u^\ast)\pi_1(p_u^\ast, k)m'(p_u^\ast) & \varphi(p_t^\ast)\pi_2(p_t^\ast, k)m'(p_t^\ast)
\end{bmatrix}
\begin{bmatrix}
dp_u^\ast/dk \\
dp_t^\ast/dk
\end{bmatrix}
= -\begin{bmatrix}
\int_{p_t^\ast}^{p_u^\ast} \psi(y)F''(k)m'(y)dy \\
\int_{p_t^\ast}^{p_u^\ast} \varphi(y)F''(k)m'(y)dy
\end{bmatrix}
\]

where \(\pi_1(p, k) = pF'(k) - rq_+\) and \(\pi_2(p, k) = rq_- - pF'(k)\). Since the matrix on the left hand side of the linear equation is non-singular we find that

\[
\frac{\partial p_u^\ast}{\partial k} = -\frac{F''(k)}{\pi_1(p_u^\ast, k)m'(p_u^\ast)\psi(p_u^\ast)}\int_{p_t^\ast}^{p_u^\ast} \tilde{\psi}(y)ym'(y)dy > 0
\]

and

\[
\frac{\partial p_t^\ast}{\partial k} = \frac{F''(k)}{\pi_2(p_t^\ast, k)m'(p_t^\ast)\tilde{\varphi}(p_t^\ast)}\int_{p_t^\ast}^{p_u^\ast} \tilde{\varphi}(y)ym'(y)dy > 0
\]

proving the alleged monotonicity of the critical thresholds. The alleged limiting conditions \(\lim_{k\downarrow 0}p_t^\ast = \lim_{k\downarrow 0}p_u^\ast = 0\) and \(\lim_{k\to\infty}p_t^\ast = \lim_{k\to\infty}p_u^\ast = \infty\) now follow from the result that \((p_t^\ast, p_u^\ast) \in (\bar{p}_k, rq_-/F'(k)) \times (rq_+/F'(k), \bar{p}_k)\) and Theorem 3.1. Finally, standard differentiation yields

\[
J_p(k, p) = G'(p)F'(k) + \frac{\tilde{\varphi}'(p)}{\tilde{\varphi}(p)} [q_- - G(p_t^\ast)F'(k)] + \frac{\tilde{\psi}'(p)}{\psi(p_u^\ast)} [q_+ - G(p_u^\ast)F'(k)].
\]

Letting \(p \to p_t^\ast\), \(p \to p_u^\ast\) and invoking the optimality conditions (D.7) and (D.8) then imply \(\lim_{p\to p_u^\ast}J_p^\ast(k, p) = \lim_{p\to p_t^\ast}J_p^\ast(k, p) = 0\) which completes the proof of our lemma.

\[\square\]

**E  Proof of Corollary 4.2**

*Proof.* As was established in Theorem 4.1 the value \(J^\ast(k, p)\) satisfies the smooth fit conditions \(\lim_{p\to p_u^\ast}J_p^\ast(k, p) = \lim_{p\to p_t^\ast}J_p^\ast(k, p) = 0\). Since \(J^\ast(k, p)\) satisfies the boundary value problem (D.1) as well, we find that the following conditions hold

\[
G(p_t^\ast)F'(k) + c_1\psi(p_t^\ast) + c_2\varphi(p_t^\ast) = q_- \quad (E.1)
\]

\[
G(p_u^\ast)F'(k) + c_1\psi(p_u^\ast) + c_2\varphi(p_u^\ast) = q_+ \quad (E.2)
\]

\[
G'(p_t^\ast)F'(k) + c_1\psi'(p_t^\ast) + c_2\varphi'(p_t^\ast) = 0 \quad (E.3)
\]

\[
G'(p_u^\ast)F'(k) + c_1\psi'(p_u^\ast) + c_2\varphi'(p_u^\ast) = 0. \quad (E.4)
\]
Solving $c_1$ and $c_2$ from equations (E.1) and (E.3) yields

\[
c_1 = \frac{1}{BS'(p_1^*)} \left[ \varphi'(p_1^*)(G(p_1^*)F'(k) - q_-) - \varphi(p_1^*)G'(p_1^*)F'(k) \right] \tag{E.5}
\]

\[
c_2 = \frac{1}{BS'(p_1^*)} \left[ \psi'(p_1^*)(q_- - G(p_1^*)F'(k)) + \psi(p_1^*)G'(p_1^*)F'(k) \right]. \tag{E.6}
\]

Analogously, solving $c_1$ and $c_2$ from equations (E.2) and (E.3) yields

\[
c_1 = \frac{1}{BS'(p_u^*)} \left[ \varphi'(p_u^*)(G(p_u^*)F'(k) - q_+) - \varphi(p_u^*)G'(p_u^*)F'(k) \right] \tag{E.7}
\]

\[
c_2 = \frac{1}{BS'(p_u^*)} \left[ \psi'(p_u^*)(q_+ - G(p_u^*)F'(k)) + \psi(p_u^*)G'(p_u^*)F'(k) \right]. \tag{E.8}
\]

Combining (E.5) with (E.7) and (E.6) with (E.8) proves the alleged result. \[\square\]

## F Proof of Theorem 4.3

**Proof.** The optimality conditions (4.1) and (4.2) can be re-expressed as

\[
\int_{p_1^*}^{p_u^*} \psi(y)ym'(y)dy = \frac{q_+}{F'(k)S'(p_u^*)} \psi'(p_u^*) - \frac{q_-}{F'(k)S'(p_1^*)} \psi'(p_1^*) \tag{F.1}
\]

\[
\int_{p_1^*}^{p_u^*} \varphi(y)ym'(y)dy = \frac{q_+}{F'(k)S'(p_u^*)} \varphi'(p_u^*) - \frac{q_-}{F'(k)S'(p_1^*)} \varphi'(p_1^*). \tag{F.2}
\]

Since $d(\psi'(y)/S'(y)) = r\psi(y)m'(y)dy$ and $d(\varphi'(y)/S'(y)) = r\varphi(y)m'(y)dy$ ordinary integration by parts yields

\[
\left( p_u^* - \frac{rq_+}{F'(k)} \right) \frac{\psi'(p_u^*)}{S'(p_u^*)} - \left( p_1^* - \frac{rq_-}{F'(k)} \right) \frac{\psi'(p_1^*)}{S'(p_1^*)} = \int_{p_1^*}^{p_u^*} \frac{\psi'(y)}{S'(y)}dy \tag{F.3}
\]

\[
\left( p_u^* - \frac{rq_+}{F'(k)} \right) \frac{\varphi'(p_u^*)}{S'(p_u^*)} - \left( p_1^* - \frac{rq_-}{F'(k)} \right) \frac{\varphi'(p_1^*)}{S'(p_1^*)} = \int_{p_1^*}^{p_u^*} \frac{\varphi'(y)}{S'(y)}dy \tag{F.4}
\]

which implies that

\[
p_u^* - \frac{rq_+}{F'(k)} = \frac{S'(p_u^*)}{\psi'(p_1^*)\varphi'(p_u^*) - \psi'(p_u^*)\varphi'(p_1^*)} \left[ \psi'(p_1^*) \int_{p_1^*}^{p_u^*} \frac{\varphi'(y)}{S'(y)}dy - \varphi'(p_1^*) \int_{p_1^*}^{p_u^*} \frac{\psi'(y)}{S'(y)}dy \right]
\]

\[
p_1^* - \frac{rq_-}{F'(k)} = \frac{S'(p_1^*)}{\psi'(p_1^*)\varphi'(p_1^*) - \psi'(p_1^*)\varphi'(p_1^*)} \left[ \psi'(p_u^*) \int_{p_1^*}^{p_u^*} \frac{\varphi'(y)}{S'(y)}dy - \varphi'(p_u^*) \int_{p_1^*}^{p_u^*} \frac{\psi'(y)}{S'(y)}dy \right].
\]

Introducing the functions $\hat{\psi} : [p_1^*, p_u^*] \mapsto \mathbb{R}_+$ and $\hat{\varphi} : [p_1^*, p_u^*] \mapsto \mathbb{R}_+$ defined as $\hat{\psi}(p) = \psi'(p_1^*)\varphi(p) - \varphi'(p_1^*)\psi(p)$ and $\hat{\varphi}(p) = \psi'(p_u^*)\varphi(p) - \varphi'(p_u^*)\psi(p)$ (these functions constitute
the increasing and decreasing fundamental solutions of the ordinary differential equation $(Au)(p) = ru(p)$ defined with respect to the underlying price diffusion reflected at the optimal boundaries $p^+_t$ and $p^-_t$ and combining these definitions with the characterization above then demonstrates that

$$p^+_u - \frac{rq_+}{F'(k)} = \frac{S'(p^+_u)}{\psi'(p^+_u)} \int_{p^-}^{p^+_u} \frac{\psi'(y)}{S'(y)} dy$$

$$p^-_l - \frac{rq_-}{F'(k)} = -\frac{S'(p^-_l)}{\varphi'(p^-_l)} \int_{p^-}^{p^-_l} \frac{\varphi'(y)}{S'(y)} dy.$$ 

Making the change of variable $y = v/p^+_u$ in the first expression and the change of variable $y = v/p^+_l$ in the latter expression then finally yields

$$p^+_u F'(k)(1 - E^+(p^+_l, p^+_u)) = rq_+$$

$$p^-_l F'(k)(1 + E^-(p^-_l, p^-_u)) = rq_-,$$ 

where

$$E^+(p^+_l, p^-_u) = \frac{S'(p^+_u)}{\psi'(p^+_u)} \int_{p^-}^{1} \frac{\psi'(vp^+_u)}{S'(vp^+_u)} dv \in (0, 1 - p^-_l/p^+_u)$$

$$E^-(p^-_l, p^-_u) = \frac{S'(p^-_l)}{\varphi'(p^-_l)} \int_{0}^{p^-_l} \frac{\varphi'(vp^-_l)}{S'(vp^-_l)} dv \in (0, p^-_u/p^-_l - 1)$$

by the monotonicity of the functions $\psi'(y)/S'(y)$ and $\varphi'(y)/S'(y)$. The rest of the inequalities follow from Theorem 3.2 after noticing that according to Theorem 4.1 the optimal boundaries satisfy the inequalities $\bar{p}_k F'(k) < p^+_l F'(k) < rq_-$ and $rq_+ < p^-_u F'(k) < \hat{p}_k F'(k)$. 

\[\Box\]

G Proof of Theorem 4.4

Proof. Denoted the proposed value function as $\tilde{V}(k, p)$. We first observe from Theorem 4.1 that the monotonicity and boundary conditions of the optimal boundaries imply that the inverse mappings $k^+_s = p^+_u(p)$ and $k^-_s = p^-_l(p)$ exist and are well-defined and that the continuation region $\{(k, p) \in \mathbb{R}^2_+ : p^- < p < p^+_u\}$ can be analogously expressed in terms of the capital stock as $\{(k, p) \in \mathbb{R}^2_+ : k^+_s < k < k^-_s\}$. It is clear that $\tilde{V} \in C^{1,2}(\mathbb{R}^2_+)$, that $\tilde{V}_k(k, p) = q_+$ on $k \leq k^+_s$, that $\tilde{V}_k(k, p) = q_-$ on $k \geq k^-_s$, and that
Theorem 4.1 yields that 

\[ V \text{ path dependent incremental investment policy} \]

ity, we observe that the proposed value can be attained by applying the admissible 

therefore, dominates the value of the optimal policy. To prove the opposite inequal-

Thus, the proposed value function satisfies the sufficient variational inequalities and, 

completes the proof of our theorem.

\[ \hat{V}_k(k, p) = J^*(k, p) \text{ on } (k_+^*, k_-^*). \] 

Combining the proof of Corollary 4.2 with the proof of 

\[ \frac{d}{dp} \left[ \frac{J(k, p) - q_-}{\psi(p)} \right] = \frac{S'(p)}{\psi^2(p)} \left[ q_+ \frac{\psi'(p)}{S'(p)} - q_- \frac{\psi'(p^*)}{S'(p^*)} - \int_{p^*}^p \psi(y) y F'(k) m'(y) dy \right] > 0 \]

and

\[ \frac{d}{dp} \left[ \frac{J(k, p) - q_-}{\varphi(p)} \right] = \frac{S'(p)}{\varphi^2(p)} \left[ \int_{p^*}^p \varphi(y) y F'(k) m'(y) dy - q_+ \frac{\varphi'(p^*)}{S'(p^*)} + q_- \frac{\varphi'(p)}{S'(p)} \right] > 0 \]

for all \( p \in (p_1^*, p_2^*) \). Combining these inequalities with the value matching conditions

\[ \lim_{p \downarrow p^*_1} J(k, p) = q_- \text{ and } \lim_{p \uparrow p^*_2} J(k, p) = q_+ \]

then proves that \( q_- < \hat{V}_k(k, p) < q_+ \) for all \( k \in (k_+^*, k_-^*) \) and, therefore, that \( q_- \leq \hat{V}_k(k, p) \leq q_+ \) for all \( (k, p) \in \mathbb{R}_+^2 \). It remains to 

establish that the proposed value function satisfies the condition \( J(A \hat{V})(k, p) - r \hat{V}(k, p) + p F(k) \leq 0 \) for all \( (k, p) \in \mathbb{R}_+^2 \). To see that this is indeed the case we observe that

\[ (A \hat{V})(k, p) - r \hat{V}(k, p) + p F(k) = 0 \text{ for } k_+^* < k < k_-^* \]

and

\[ \frac{d}{dk} \left[ (A \hat{V})(k, p) - r \hat{V}(k, p) + p F(k) \right] = \begin{cases} 0 & k_+^* < k < k_-^* \\ p F'(k) - r q_- < 0 & k \geq k_-^* \\ p F'(k) - r q_+ > 0 & k \leq k_+^* \end{cases} \]

Thus, the proposed value function satisfies the sufficient variational inequalities and, 

therefore, dominates the value of the optimal policy. To prove the opposite inequal-

ity, we observe that the proposed value can be attained by applying the admissible 

path dependent incremental investment policy \( k_t^* = k + k_t^{+*} - k_t^{-*} \), where the optimal 

disinvestment and investment policies are characterized as \( k_0^{+*} = (p_0^{*-1}(p) - k)^+ \), \( k_0^{-*} = (k - p_1^{*-1}(p))^+ \), \( k_t^{+*} = p_u^{*-1}(u_t) \), \( k_t^{-*} = p_u^{*-1}(l_t) \), where \( u_t = \sup\{p_s; s \leq t\} \) and \( l_t = \inf\{p_s; s \leq t\} \) (these singular policies satisfy the associated Skorokhod-problem 

characterizing the optimal cumulative investment and disinvestment rule). This com-

pletes the proof of our theorem. 