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# Patents, Product Cycles and Non-Diversifiable Risk

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## Abstract

This study examines patents in an economy where R&D firms innovate and imitate and households face non-diversifiable risk. Some characteristics of patents postpone the expected time a patent will be imitated (e.g. increase patent length), while the others protect the patentee's profits after a successful imitation (e.g. increase patent breadth). It is shown that the less patient households, the shorter and broader the welfare-maximizing patents.

**JEL Classification:** L11, L16, O31, O34.

**Keywords:** Patents, Imitation, Innovation, Product Cycles.

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# 1 Introduction

In models of technological change with uncertainty, the assumption of the diversifiable risk simplifies the analysis considerably. With that assumption, firms can borrow any amount for R&D at a given interest rate and households are protected from uncertainty through diversification in the market portfolio. Consequently, the optimal length and breadth of patents can be judged by the present value of investment projects independently of household preferences. In contrast, I assume that households cannot wholly diversify their investment risk. With that assumption, firms finance their R&D through issuing shares and households purchasing these shares face the uncertainty associated with investment. Consequently, there is a trade-off between patent length and patent breadth through the proportions of innovating and imitating firms in the entire product cycle and the welfare-maximizing shape of patents depends on household preferences.

In a growth model of creative destruction, firms can step forward in the quality ladders of technology by investing in R&D.<sup>1</sup> If imitation is possible as well, then economic growth is subject to product cycles as follows. Through the development of new products, an innovator achieves a temporary advantage earning monopoly profits. This advantage ends when an imitator succeeds in copying the innovation, enters the market and starts competing with the innovator. The structure of a product cycle model is characterized in Fig. 1. Let 0 be the starting time at which an innovation occurs,  $a$  the time at which the innovation is imitated,  $a + b$  the time at which an innovation of the next generation occurs. The innovator possesses the whole market during the imitative period  $[0, a)$  and the share  $\phi_1 \in (0, 1)$  of the market during the innovative period  $[a, a + b)$ , while the imitator possesses nothing during  $[0, a)$  and the share  $1 - \phi_1$  of the market during  $[a, a + b)$ . If imitation is serially uncorrelated, then the probability of a successful imitation is equal to the inverse of the time of the imitation,  $1/a$ . If innovation is serially uncorrelated, then the probability of a successful innovation is equal to the inverse of the time of the innovation,  $1/b$ .

A patent is an innovation that is protected by the authorities. In models with no endogenous growth, the definition of patent length is commonly

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<sup>1</sup>Cf. Grossman and Helpman (1991) (in ch. 4), and Aghion and Howitt (1998).

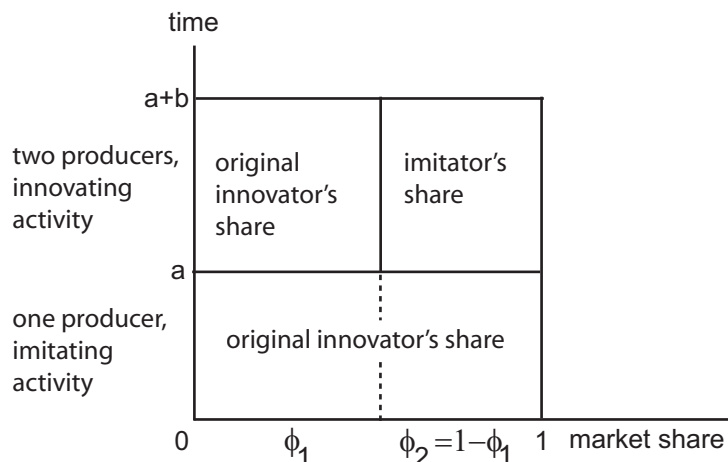


Figure 1: Stages of the product cycle.

equal to patent life. In a model of endogenous growth, this is however not adequate. Any patent will be first imitated and then replaced by a patent of the next generation, usually before its lifetime. Thus, I define patent length as the time the patentee earns full monopoly profit. In Fig. 1, this is given by  $a$ .<sup>2</sup> Following Denicolo (1996), Takalo (1998) and Kannianen and Stenbacka (2000), patent breadth (or width) can be proxied by the difference between the flows of profits of an innovator and a successful imitator. On the assumption that profits are proportional to the market share, Fig. 1 tells that an innovator earns 1 immediately after the innovation and an imitator earns  $1 - \phi_1$  immediately after the imitation. Patent breadth is then  $1 - (1 - \phi) = \phi_1$ . The broader patents (i.e. the bigger  $\phi_1$ ), the less an incumbent firm's profit falls with the entry of a new firm (i.e. the smaller  $1 - \phi_1$ ).

Patents generate economic growth at the level of the whole economy, but they are mainly a vehicle of taking over the market at the level of a single firm. Because duopolists benefit but monopolies do not benefit from a new innovation, an increase in the proportion of duopoly industries promotes innovative R&D and economic growth. If patents are long (i.e. a high  $a$  in Fig. 1), then the probability of a successful imitation,  $1/a$ , is small, and if patents are broad (i.e.  $\phi_1$  is large in Fig. 1), then an imitator's profit  $1 - \phi_1$

<sup>2</sup>This definition of patent length is also equivalent to Horii and Iwaisako's (2007) concept of Intellectual Property Rights (IPR): If IPR is strengthened, the probability of a successful imitation  $1/a$  declines.

is low after a successful imitation. Thus, there are two instruments, patent length and patent breadth, by which the government can control innovative and imitative R&D, economic growth and social welfare.

In my earlier work on growth and competition policy (Palokangas 2008), I have extended Wälde's (1999a, 1999b) growth model with non-diversifiable risk for a multi-sector economy and incorporated Segerstrom (1991) and Mukoyama's (2003) ideas on product cycles and cumulative technology into it. In this study, I modify that model for patent policy and introduce a benevolent government that controls patent length and patent breadth. As a result, I obtain Pareto-optimal patent policy for an economy with product cycles. The remainder of this paper is organized as follows. The structure of the model is presented in section 2, the existence of the equilibrium is proven in section 3 and the product cycle is constructed in section 4. In sections 5 and 6, welfare-maximizing patents are established as functions of the rate of time preference.

## 2 The model

There is a fixed number  $N$  of households, each of which supplies one labor unit. All labor is homogeneous and can be used both in production and in R&D. Because the labor market is competitive, aggregate labor supply  $N$  is equal to employment in production,  $x$ , and labor devoted to R&D,  $l$ :

$$N = x + l. \quad (1)$$

Because in the model there is no money that would pin down the nominal price level at any time, it is convenient to normalize the households' total spending in consumption at unity:

$$Py = 1, \quad y \doteq \sum_{\iota=1}^N C_{\iota}, \quad (2)$$

where  $y$  aggregate consumption,  $P$  the consumption price and  $C_{\iota}$  consumption by household  $\iota \in \{1, \dots, N\}$ . The utility for a single household  $\iota \in \{1, \dots, N\}$  from an infinite stream of consumption  $C_{\iota}$  beginning at time  $T$  is

$$U(C_{\iota}, T) = E \int_T^{\infty} C_{\iota}^{\sigma} e^{-\rho(t-T)} dt \quad \text{with } 0 < \sigma < 1 \text{ and } \rho > 0, \quad (3)$$

where  $t$  is time,  $E$  the expectation operator,  $\rho$  the rate of time preference and  $(1 - \sigma)$  is the constant rate of relative risk aversion.

Because R&D firms finance their expenditure by issuing shares and the households save only in these shares, aggregate income is equal to the value of consumption,  $Py$ , plus wages paid in R&D,  $wl$ , where  $w$  is the wage and  $l$  labor devoted to R&D. Given (2), it is then true that

$$\sum_{\iota=1}^N A_{\iota} = wl + Py = wl + 1, \quad (4)$$

where  $A_{\iota}$  is the income of household  $\iota \in \{1, \dots, N\}$  and  $\sum_{\iota=1}^N A_{\iota}$  aggregate income. All households are risk averters and share the same preferences.

Competitive firms produce the consumption good from a great number of intermediate goods that are evenly placed over the limit  $[0, 1]$ . Each intermediate good  $j \in [0, 1]$  is a composite good of the products of a number  $n_j$  of firms in industry  $j \in [0, 1]$ . Because a broader variety of products provides more services to households, an increase in the total number of firms  $n_j$  in any industry  $j$  raises every household's welfare.<sup>3</sup> Aggregate consumption is then produced from the products  $x_{j\kappa}$  of all firms  $\kappa \in \{1, \dots, n_j\}$  in all industries  $j \in [0, 1]$  through Cobb-Douglas technology as follows:

$$\begin{aligned} \log y &= \int_0^1 \log(n_j^{\epsilon} B_j x_j) dj = \int_0^1 \log(B_j x_j) dj + \epsilon \int_0^1 \log n_j dj, \quad \epsilon > 0, \\ \log x_j &= \sum_{\kappa=1}^{n_j} \phi_{\kappa} \log\left(\frac{x_{j\kappa}}{\phi_{\kappa}}\right), \quad \phi_{\kappa} > \phi_{\kappa+1} \text{ for } \kappa < n_j, \quad \sum_{\kappa=1}^{n_j} \phi_{\kappa} = 1, \end{aligned} \quad (5)$$

where  $n_j$  is the number of firms in industry  $j$ ,  $B_j$  the productivity parameter in industry  $j$ ,  $x_j$  the quantity of intermediate good  $j$ ,  $\phi_{\kappa}$  the constant weight of the  $\kappa$ th firm's product  $x_{j\kappa}$  in the production of intermediate good  $j$  in industry  $j$ , and  $\epsilon$  is a constant.

In any industry  $j$ , the first firm is always an innovator, while the rest  $\kappa = 2, \dots, n_j$  are imitators. In equilibrium, by (5), firm  $\kappa$ 's market share is equal to  $\phi_{\kappa}$ . The entry of new firms through successful imitations decreases

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<sup>3</sup>In general, the property that product variety increases welfare is commonly established through a CES production function. In this study, the replacement of the Cobb-Douglas function (5) by a CES function would excessively complicate the analysis.

the innovator's market share.<sup>4</sup> I assume that patent breadth  $\varphi$  is the government's policy variable that decreases the imitators' market shares:

$$\phi_\kappa(\varphi) \text{ with } \phi'_\kappa < 0 \text{ for } \kappa > 1. \quad (6)$$

When patents are broader, the imitators  $\kappa > 1$  have less properties in their products and they are able to supply less variants of the innovated good. Consequently, their market shares  $\phi_\kappa$  fall. From (5) and (6) imply that the innovator's market share  $\phi_1$  is an increasing function of patent breadth  $\varphi$ .

The productivity parameter in industry  $j$  [Cf. (5)] is determined by

$$B_j \doteq \mu^{\tau_j}, \quad \mu > 1, \quad (7)$$

where  $\mu$  is a parameter and  $\tau_j$  an index of technology in industry  $j$ . The invention of a new technology in industry  $j$  raises the index  $\tau_j$  by one and the level of productivity by  $\mu > 1$ .

There are two types of R&D in any industry: *innovative R&D* that aims at creating a new state-of-the-art product in the industry; and *imitative R&D* that aims at creating a close substitute for the incumbent state-of-the-art product at the same level of technology. Following Palokangas (2008), I assume that there are constant returns to scale in imitative, but decreasing returns to scale in innovative R&D, for simplicity. Following Horii and Iwaisako (2007), this can be justified by the possibility of duplication: when two workers innovate in the same industry, they produce very likely less than a double amount of innovations. With constant returns to scale for all R&D, households would invest entirely in either innovating or imitating R&D depending on which one of these yields a higher rate of return.

When firm  $\kappa$  in industry  $j$  innovates, its technological change follows a Poisson process  $q_{j\kappa}$  in which the arrival rate of innovations,  $\Lambda_{j\kappa}$ , is

$$\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\nu}, \quad \lambda > 0, \quad 0 < \nu < 1, \quad (8)$$

where  $l_{j\kappa}$  is the firm's labor input and  $\lambda$  and  $\nu$  are constants. During a short time interval  $dt$ , there is an innovation  $dq_{j\kappa} = 1$  in firm  $\kappa$  with probability  $\Lambda_{j\kappa}dt$ , and no innovation  $dq_{j\kappa} = 0$  with probability  $1 - \Lambda_{j\kappa}dt$ .

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<sup>4</sup>I ignore the possibility that firms  $3, \dots, n_j$  crowd out the market share of the second firm, for simplicity. Since in equilibrium there will be only two producers per industry, this would only complicate the model without any change in the results.

I use the relative productivity between imitative and innovative R&D,  $a$ , as a proxy for patent length. The bigger  $a$ , the more difficult it is to imitate and the longer time it takes to produce a successful imitation for an invention. When firm  $\kappa$  in industry  $j$  imitates, its technological change follows a Poisson process  $Q_{j\kappa}$  in which the arrival rate of imitations is in fixed proportion  $\lambda/a$  to the firm's own labor input  $l_{j\kappa}$ :

$$\Gamma_{j\kappa} = (\lambda/a)l_{j\kappa}, \quad a > 0. \quad (9)$$

During a short time interval  $dt$ , there is an imitation  $dQ_{j\kappa} = 1$  with probability  $\Gamma_{j\kappa}dt$ , and no imitation  $dQ_{j\kappa} = 0$  with probability  $1 - \Gamma_{j\kappa}dt$ .

Each R&D firm distributes its profit among those who had financed it in proportion to their investment in the firm. Because both innovation and imitation follow a Poisson process, the values of shares in R&D projects are random variables and household  $\iota \in \{1, \dots, N\}$  maximizes its utility (3) subject to the random development of these values.

### 3 The steady-state equilibrium

In this section, I prove the existence of the following equilibrium:

**Definition.** *The economy is in a stationary-state equilibrium, if the following properties are satisfied:*

- (i) *The industries  $j$  are run either by monopolies ( $n_j = 1$ ) or duopolies ( $n_j = 2$ ). Non-producing outsiders imitate to enter any of the monopoly industries and the incumbent duopolists innovate to become a monopoly in the same industry.*
- (ii) *The proportions of monopoly and duopoly industries in the economy (denoted  $\alpha$  and  $\beta$ , respectively) are constants over time. Every time a new superior-quality product is discovered in some industry, changing this from a duopoly into a monopoly, imitation must occur in some other industry, changing this from a monopoly into a duopoly.*
- (iii) *The profits of a typical monopoly and a typical duopolist are constant over time.*

(iv) The wage  $w$ , total labor in production,  $x$ , and total labor in R&D,  $l$ , are constants over time.

(v) The labor input  $\eta$  of a typical innovating firm in R&D, the labor input  $\psi$  of a typical imitating firm in R&D, and the average growth rate of consumption,  $g$ , are constants over time.

### 3.1 The production of goods

The representative consumption-good firm maximizes its profit

$$\Pi^c \doteq Py - \int_{j \in [0,1]} \sum_{\kappa=1}^{n_j} p_{j\kappa} x_{j\kappa} dj$$

subject to technology (5), given the output price  $P$  and the input prices  $p_{j\kappa}$ . Noting (2), this implies

$$\Pi^c = 0, \quad p_{j\kappa} = P \frac{\partial y}{\partial x_{j\kappa}} = \phi_\kappa P \frac{y}{x_{j\kappa}} = \frac{\phi_\kappa}{x_{j\kappa}} \text{ for all } \kappa. \quad (10)$$

All intermediate-good firms produce one unit of their output from one labor unit. The product of the newest generation provides exactly the constant  $\mu > 1$  times as many services as that of earlier generation. A firm of earlier generation earns the profit  $\Pi_{j\kappa}^o = (p_{j\kappa}^o - w)x_{j\kappa}^o$ , where  $p_{j\kappa}^o$  is its output price and  $x_{j\kappa}^o$  its output. Every firm with the newest technology pushes and keeps the firms with older technology out of the market by choosing its price  $p_{j\kappa}$  so that these earn no profit,  $\Pi_{j\kappa}^o = 0$  and  $p_{j\kappa}^o = w$ . This yields  $p_{j\kappa}/\mu = p_{j\kappa}^o = w$ . Noting this, (5) and (10), one obtains the equilibrium conditions:

$$\begin{aligned} p_{j\kappa} &= \mu w, \quad x_{j\kappa} = \frac{\phi_\kappa}{p_{j\kappa}} = \frac{\phi_\kappa}{\mu w}, \quad \Pi_{j\kappa} = (p_{j\kappa} - w)x_{j\kappa} = \phi_\kappa \Pi, \quad x_j = \frac{1}{\mu w} \text{ and} \\ x &\doteq \int_{j \in [0,1]} x_j dj = \frac{1}{\mu w} \text{ for all } j \text{ and } \kappa; \quad \Pi_{j1} = \Pi \doteq 1 - \frac{1}{\mu} > 0 \text{ for } n_j = 1; \end{aligned} \quad (11)$$

where  $x$  is total employment in production [cf. (1)]. Thus, *the property (iii) of a stationary-state equilibrium is proven.*

Noting (11), one can observe the following. First, the innovator will earn the constant profit  $\Pi$  as long as it remains the monopoly producer in

the industry. Because a household holds the share of all firms in its same portfolio, it does not invest in innovative R&D in the monopoly industries. Second, if anyone invests in imitative R&D to enter a monopoly industry  $j$ , then its prospective profit is  $\Pi_{j2}$ , but if it does that (with the same cost) to enter an industry  $j$  with  $\kappa > 1$  producers, then its prospective profit is  $\Pi_{j,\kappa+1} < \Pi_{j2}$ . Thus, it invests in imitative R&D only to enter a monopoly industry, but not to enter an oligopoly industry. This means that there can be at most two producers in an industry. From (a) and (b) above it follows that in equilibrium there are only monopoly industries with imitative R&D or duopoly industries with innovative R&D. Thus, *the property (i) of a stationary-state equilibrium is proven.*

I denote the set of monopoly industries by  $\Theta \subset [0, 1]$ . The relative proportion of duopoly industries,  $\beta$ , and the relative proportion of monopoly industries,  $\alpha$ , are then given by

$$\beta \doteq \int_{j \notin \Theta} dj, \quad \alpha = \int_{j \in \Theta} dj = 1 - \beta. \quad (12)$$

Thus, *the property (ii) of a stationary-state equilibrium is proven.* Following Mukoyama (2003) and Tang and Wälde (2001), I define the proportion of duopoly industries in the economy,  $\beta$ , as the *scale of competition*.

### 3.2 Employment, output and growth

Given (1) and (5), the wage  $w$  becomes a function of total labor in R&D,  $l$ :

$$w = \frac{1}{\mu x} = \frac{1}{\mu} \frac{1}{N - l}, \quad w' > 0. \quad (13)$$

This *proves the property (iv) of a stationary-state equilibrium.* Higher demand for labor in R&D (i.e. a bigger  $l$ ) raises the wage  $w$ .

According to the properties (i) and (ii) of a stationary-state equilibrium, duopolists labeled 1 and 2 innovate and none imitates in duopoly industries  $j \notin \Theta$ , while outsiders imitate and none innovates in monopoly industries  $j \in \Theta$ . Because according to technology (9) imitation yields constant returns to scale, all outsiders in monopoly industry  $j \in \Theta$  behave as if there were a single outsider firm labeled 0. The structure of industries is given by Fig. 2.

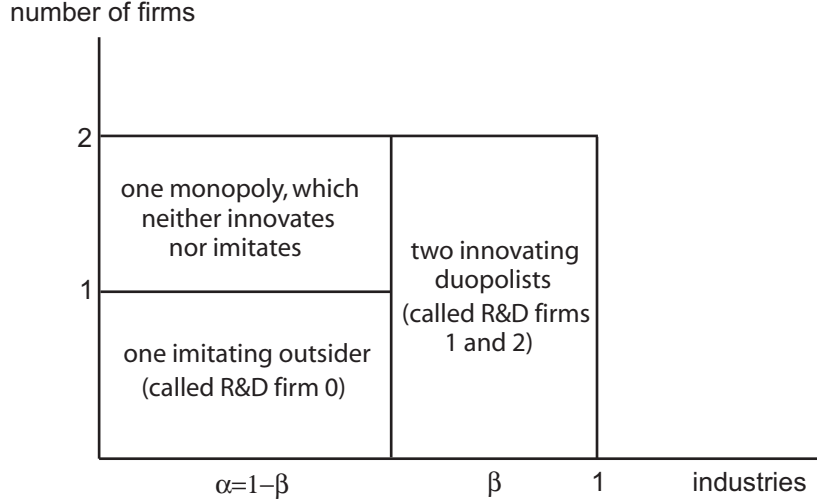


Figure 2: Competition and the number of firms in the economy.

In duopoly industries  $j \notin \Theta$  the two producers employ  $l_{j1} + l_{j2}$  and in monopoly industries  $j \in \Theta$  the outsider employs  $l_{j0}$  labor units in R&D. Total employment in R&D,  $l$ , is the sum of all firms' employment in R&D:

$$l \doteq \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj + \int_{j \in \Theta} l_j dj. \quad (14)$$

Given (7), the average productivity in the economy,  $B$ , is defined as a function of the technologies  $\tau_j$  of all industries  $j \in [0, 1]$  as follows:

$$\log B \doteq \int_0^1 \log B_j dj = (\log \mu) \int_0^1 \tau_j dj. \quad (15)$$

Noting (1), (5), (11), (12) and (15), aggregate consumption  $y$  takes the form

$$y = e^{\epsilon[\alpha \log 1 + \beta \log 2]} xB = e^{\epsilon \beta \log 2} xB = e^{\delta \beta} (N - l)B, \quad \delta \doteq \epsilon \log 2 > 0. \quad (16)$$

The arrival rate of innovations in duopoly industry  $j \notin \Theta$  is the sum of the arrival rates of both duopolists in that industry,  $\Lambda_{j1} + \Lambda_{j2}$  [Cf., (8)]. Because only duopoly industries  $j \notin \Theta$  innovate, then the average growth rate of the average productivity  $B(\{t_k\})$  in the stationary state is given by

$$g \doteq (\log \mu) \int_0^1 \Pr(\tau_j \text{ increases by one}) dj = (\log \mu) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj, \quad (17)$$

where  $\Pr(\cdot)$  denotes the probability.

### 3.3 Innovation and imitation

In monopoly industry  $j \in \Theta$  outsider 0 and in industry  $j \notin \Theta$  duopolists 1 and 2 issue shares to finance their labor expenditure in R&D. Because the households  $\iota \in \{1, \dots, N\}$  invest in these shares, one obtains

$$\sum_{\iota=1}^N S_{\iota j 0} = w l_{j 0} \text{ for } j \in \Theta, \quad \sum_{\iota=1}^N S_{\iota j \kappa} = w l_{j \kappa} \text{ for } \kappa \in \{1, 2\} \text{ and } j \notin \Theta, \quad (18)$$

where  $w l_{j 0}$  is the imitative expenditure of outsider 0 in monopoly industry  $j \in \Theta$ ,  $w l_{j \kappa}$  the innovative expenditure of duopolist  $\kappa \in \{1, 2\}$  in industry  $j \notin \Theta$ ,  $S_{\iota j 0}$  household  $\iota$ 's investment in outsider firm 0 in monopoly industry  $j \in \Theta$ ,  $S_{\iota j \kappa}$  household  $\iota$ 's investment in duopolist  $\kappa$  in industry  $j \notin \Theta$ ,  $\sum_{\iota=1}^N S_{\iota j 0}$  aggregate investment in outsider firm 0 in monopoly industry  $j \in \Theta$ , and  $\sum_{\iota=1}^N S_{\iota j \kappa}$  aggregate investment in duopolist  $\kappa$  in industry  $j \notin \Theta$ . Household  $\iota$ 's relative investment shares in outsiders 0 and duopolists  $\kappa \in \{1, 2\}$  are

$$i_{\iota j 0} \doteq \frac{S_{\iota j 0}}{w l_{j 0}} \text{ for } j \in \Theta; \quad i_{\iota j \kappa} \doteq \frac{S_{\iota j \kappa}}{w l_{j \kappa}} \text{ for } j \notin \Theta. \quad (19)$$

When household  $\iota$  has financed a successful R&D firm, it acquires the right to the firm's profit in proportion to its relative investment share. Noting (11), the profit sharing in the economy can then be characterized as follows:

- $s_{\iota j \kappa}$  household  $\iota$ 's profit from duopolist  $\kappa \in \{1, 2\}$  in industry  $j \notin \Theta$ ;
- $i_{\iota j \kappa}$  household  $\iota$ 's investment share in duopolist  $\kappa \in \{1, 2\}$  in industry  $j \notin \Theta$  [cf. (19)];
- $\Pi_{j \kappa}$  the profit that duopolist  $\kappa \in \{1, 2\}$  in industry  $j \notin \Theta$  shall earn after innovation has changed it into a monopoly;
- $\Pi_{j \kappa} i_{\iota j \kappa}$  the profit that household  $\iota$  shall get from duopolist  $\kappa \in \{1, 2\}$  in industry  $j \notin \Theta$  after innovation has changed this into a monopoly;
- $s_{\iota j 0}$  household  $\iota$ 's profit from outsider 0 in industry  $j \in \Theta$ ;
- $i_{\iota j 0}$  household  $\iota$ 's investment share in outsider 0 in industry  $j \in \Theta$  [cf. (19)];
- $\Pi_{j 2}$  the profit that outsider 0 in industry  $j \in \Theta$  shall earn after imitation has changed it as the second duopolist;

$\Pi_{j2}i_{lj0}$  the profit that household  $l$  shall get from outsider 0 in industry  $j \in \Theta$  after imitation has changed it into the second duopolist.

The changes in the profits of firms in industry  $j$  are functions of the increments  $(dq_{j1}, dq_{j2}, dQ_{j0})$  of Poisson processes  $(q_{j1}, q_{j2}, Q_{j0})$  as follows:<sup>5</sup>

$$\begin{aligned} ds_{lj\kappa} &= (\Pi_{j\kappa}i_{lj\kappa} - s_{lj\kappa})dq_{j\kappa} - s_{lj\kappa}dq_{j(\zeta \neq \kappa)} \quad \text{when } j \notin \Theta; \\ ds_{lj0} &= (\Pi_{j2}i_{lj0} - s_{lj0})dQ_{j0} \quad \text{when } j \in \Theta. \end{aligned} \quad (20)$$

These functions can be explained as follows. If a household invests in innovating duopolist  $\kappa$  in industry  $j \notin \Theta$ , then, in the advent of a success for that duopolist,  $dq_{j\kappa} = 1$ , the amount of its share holdings rises up to  $\Pi_{j1}i_{lj\kappa}$ ,  $ds_{lj\kappa} = \Pi_{j1}i_{lj\kappa} - s_{lj\kappa}$ , but in the advent of success for the other duopolist  $\zeta \neq \kappa$ , its share holdings in duopolist  $\kappa$  fall down to zero,  $ds_{lj\kappa} = -s_{lj\kappa}$ . If a household invests in imitating outsider 0 in monopoly industry  $j \in \Theta$ , then, in the advent of a success for firm 0,  $dQ_{j0} = 1$ , the amount of its share holdings rises up to  $\Pi_{j2}i_{lj0}$ ,  $ds_{lj0} = \Pi_{j2}i_{lj0} - s_{lj0}$ .

### 3.4 Households

Because investment in shares in R&D firms is the only form of saving in the model, the budget constraint of household  $l$  is given by

$$A_l = PC_l + \int_{j \in \Theta} S_{lj0}dj + \int_{j \notin \Theta} (S_{lj1} + S_{lj2})dj, \quad (21)$$

where  $A_l$  is the household's total income,  $C_l$  its consumption,  $P$  the consumption price,  $S_{lj0}$  the household's investment in outsider firm 0 in monopoly industry  $j \in \Theta$ , and  $S_{lj\kappa}$  the household's investment in duopolist  $\kappa$  in industry  $j \notin \Theta$ . Household  $l$ 's total income  $A_l$  consists of its wage income  $w$  (the household supplies one labor unit), its profits  $s_{lj1}$  from the monopoly in each industry  $j \in \Theta$  and its profits  $s_{lj1}$  and  $s_{lj2}$  from the duopolists 1 and 2 in each industry  $j \notin \Theta$ . This yields

$$A_l = w + \int_{j \in \Theta} s_{lj1}dj + \int_{j \notin \Theta} (s_{lj1} + s_{lj2})dj. \quad (22)$$

<sup>5</sup>This extends the idea of Wälde (1999a, 1999b).

Household  $\iota$  maximizes its utility (3) by its investment,  $\{S_{\iota j0}\}$  for  $j \in \Theta$  and  $\{S_{\iota j1}, S_{\iota j2}\}$  for  $j \notin \Theta$ , subject to its budget constraint (21), the stochastic changes (20) in its profits, the composition of its income, (22), and the determination of its relative investment shares, (19), given the arrival rates  $\{\Lambda_{j\kappa}, \Gamma_{j0}\}$ , the wage  $w$  and the consumption price  $P$ . In Appendix A, this maximization problem is solved by a similar dynamic program as in Palokangas (2008), with the following result.<sup>6</sup> In the households' stationary equilibrium in which the allocation of resources is invariable across technologies, the following conditions are satisfied (cf. Appendix A)

$$\begin{aligned} l_{j\kappa} &= \eta(a, \varphi) && \text{for } j \notin \Theta, && \frac{\partial \eta}{\partial a} > 0, && \frac{\partial \eta}{\partial \varphi} > 0, \\ l_{j0} &= \psi \doteq \frac{1}{1-\beta} [l - 2\beta\eta(a, \varphi)] && \text{for } j \in \Theta, \end{aligned} \quad (23)$$

$$\rho + \frac{1 - \mu^\sigma}{\log \mu} g = \frac{\mu^\sigma \lambda \Delta(l)}{\eta(a, \varphi)^\nu} \text{ with } \Delta(l) \doteq \frac{(N-l)^2 \mu}{N/\Pi - l},$$

$$\frac{\Delta'}{\Delta} = \frac{d \log \Delta}{dl} = \frac{1}{N/\Pi - l} - \frac{2}{N-l} < 0 \text{ and } 1 - \frac{l\Delta'}{\Delta} > \frac{N}{N-l}, \quad (24)$$

$$g = (2\lambda \log \mu) \beta \eta(a, \varphi)^{1-\nu} \quad (25)$$

According to (23), with longer or broader patents (i.e. a bigger  $a$  or a bigger  $\varphi$ ), households invest more in each innovative firm to escape the competition (i.e.  $\eta$  rises). For given innovative investment  $\eta$  per firm, there is a trade-off between patent length  $a$  and patent breadth  $\varphi$ . According to (24), a household's subjective discount factor  $\rho + \frac{1-\mu^\sigma}{\log \mu} g$  is equal to the marginal rate of return to savings,  $\mu^\sigma \lambda \Delta(l) / \eta(a, \varphi)^\nu$ , which is a decreasing function of patent length  $a$ , patent breadth  $\varphi$  and total employment in R&D,  $l$ . According to (25), an increase in patent length  $a$ , patent breadth  $\varphi$  or the scale of competition  $\beta$  increases the growth rate  $g$ . Given (23) and (25), *the property (v) of a stationary-state equilibrium is proven.*

## 4 The product cycle

Given the property (ii) of the stationary-state equilibrium, the rate at which industries leave the group of duopoly industries  $k \notin \Theta$  in a small interval  $dt$ ,  $\beta(\Lambda_{j1} + \Lambda_{j2})dt$ , is then equal to the rate at which the industries leave the

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<sup>6</sup>A detailed proof will be delivered to a reader on request.

group of monopoly industries  $j \in \Theta$ ,  $\alpha\Gamma_{j0}dt$  in that interval  $dt$ :

$$\beta(\Lambda_{k1} + \Lambda_{k2}) = \alpha\Gamma_{j0} \text{ for } k \notin \Theta \text{ and } j \in \Theta. \quad (26)$$

From equations (8), (9), (12), (23) and (26) it follows that

$$1 = \frac{\Lambda_{k1} + \Lambda_{k2}}{\alpha\Gamma_{j0}/\beta} = \frac{a(l_{k1}^{1-\nu} + l_{k2}^{1-\nu})}{(1-\beta)l_{j0}/\beta} = \frac{2a\eta^{1-\nu}}{l/\beta - 2\eta}.$$

Solving for the scale of competition,  $\beta$ , yields

$$\beta = (\eta + a\eta^{1-\nu})^{-1}l/2. \quad (27)$$

Given (24), the inverse of the function  $\Delta$  is

$$l = \Delta^{-1}\left(\left(\rho + \frac{1-\mu^\sigma}{\log \mu}g\right)\frac{\eta^\nu}{\lambda\mu^\sigma}\right). \quad (28)$$

Inserting (27) and (28) into (25) and noting (7), (8), and (24), one obtains

$$\begin{aligned} g &= \frac{\lambda \log \mu}{\eta^\nu + a} \Delta^{-1}\left(\left(\rho + \frac{1-\mu^\sigma}{\log \mu}g\right)\frac{\eta^\nu}{\lambda\mu^\sigma}\right) \doteq J(g, a, \eta, \rho), \\ \frac{\partial J}{\partial g} &= \frac{1-\mu^\sigma}{\log \mu} \frac{\eta^\nu g}{\lambda\mu^\sigma l \Delta'} > 0, \quad \frac{\partial J}{\partial \rho} = \frac{\eta^\nu g}{\lambda\mu^\sigma l \Delta'} < 0, \quad \frac{\partial J}{\partial a} = -\frac{g}{\eta^\nu + a} < 0, \\ \frac{\partial J}{\partial \eta} &= \frac{\nu g \eta^{\nu-1}}{\lambda\mu^\sigma l \Delta'} \left(\rho + \frac{1-\mu^\sigma}{\log \mu}g\right) - \frac{\nu g \eta^{\nu-1}}{\eta^\nu + a} = \left(\rho + \frac{1-\mu^\sigma}{\log \mu}g\right) \frac{\nu}{\eta} \frac{\partial J}{\partial \rho} + \nu \eta^{\nu-1} \frac{\partial J}{\partial a} \\ &< 0. \end{aligned} \quad (29)$$

The equation (29) defines the growth rate  $g$  as a function of  $\chi$ . Unfortunately, the variable  $g$  appears in both sides of the equation, which makes this dependence mathematically ambiguous. This ambiguity can be eliminated by the stability properties of the model. Assume that vector  $(\chi, \eta, \lambda)$  changes so that  $J(g, \chi, \eta, \lambda)$  increases.<sup>7</sup> This raises the growth rate  $g$  by the same amount, which generates a further increase  $\partial J/\partial g$  in  $J$ . If  $\partial J/\partial g < 1$ , there will be a sequence of dampening increases in  $g$  until a new equilibrium is attained. If  $\partial J/\partial g > 1$ , then there will be ever accelerating increases in  $g$  and

<sup>7</sup>Formally, this can be proven as follows. Assume that an increase in the growth rate  $g$  is in fixed proportion  $\varpi > 0$  to the perturbation  $G(g, \pi, \beta) - g$  from the equilibrium:  $\dot{g} = \varpi[J(g, a, \varphi, \sigma) - g]$ . This system has a stable equilibrium only if  $\partial \dot{g}/\partial g = \varpi[\partial J/\partial g - 1] < 0$ , which is equivalent to  $\partial J/\partial g < 1$ .

the system will never end up with an equilibrium. Because the comparative static properties of a constant-growth equilibrium cannot be analyzed by an unstable model,<sup>8</sup> I assume  $\partial J/\partial g < 1$ . Given  $0 < \partial J/\partial g < 1$  and (24), the comparative statics of the equation (29) implies the function

$$\begin{aligned}
g = G(a, \eta, \rho), \quad \frac{\partial G}{\partial a} &\doteq \frac{\partial J}{\partial a} \bigg/ \left(1 - \frac{\partial J}{\partial g}\right) < 0, \quad \frac{\partial G}{\partial \rho} \doteq \frac{\partial J}{\partial \rho} \bigg/ \left(1 - \frac{\partial J}{\partial g}\right) < 0, \\
\frac{\partial G}{\partial a} \bigg/ \frac{\partial G}{\partial \rho} &= \frac{\partial J}{\partial a} \bigg/ \frac{\partial J}{\partial \rho} = -\frac{\lambda \mu^\sigma l \Delta'}{(\eta^\nu + a)\eta^\nu}, \\
\frac{\partial G}{\partial \eta} &\doteq \frac{\partial J}{\partial \eta} \bigg/ \left(1 - \frac{\partial J}{\partial g}\right) = \left(\rho + \frac{1 - \mu^\sigma}{\log \mu} g\right) \frac{\nu}{\eta} \frac{\partial G}{\partial \rho} + \nu \eta^{\nu-1} \frac{\partial G}{\partial a} < 0. \quad (30)
\end{aligned}$$

The results (30) can be explained as follows. When households are less patient (i.e. the rate of time preference,  $\rho$ , is higher), their subjective discount factor  $\rho + \frac{1 - \mu^\sigma}{\log \mu} g$  is higher, they save less and invest less in R&D [i.e.  $l$  falls, cf. (24)]. With a lower level of R&D, the growth rate  $g$  falls. Assume that patent length  $a$  or patent breadth  $\varphi$  decreases (i.e.  $a$  or  $\eta(a, \varphi)$  decreases). This reduces households' incentives to invest in innovative R&D. In that case, a smaller proportion  $\alpha$  of industries will end up as monopolies and a larger proportion  $\beta$  will remain as duopolies [cf. (27)]. With a larger proportion  $\beta$  of innovating duopolies, the growth rate  $g$  will be higher [cf. (25)].

## 5 The government

Given (27), (28) and (30), there is one to-one-correspondence from patent length  $a$  and patent breadth  $\varphi$  to the growth rate  $g$  and the scale of competition,  $\beta$ . This means that I can solve the optimum in two stages. First, I replace  $(a, \varphi)$  by  $(g, \beta)$  as the control variables in the government's optimal program and obtain the government's optimum in terms of  $(g, \beta)$ . Second, in subsection 6.3, I transform the optimum so that it can be expressed in terms of patent length  $a$  and patent breadth  $\varphi$ .

From a household's equilibrium conditions (24) and (25) it follows that

$$\lambda \mu^\sigma \Delta(l) \left[ \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right]^{-1} = \eta^\nu = \left[ \frac{g}{(2\lambda \log \mu)\beta} \right]^{\nu/(1-\nu)}.$$

<sup>8</sup>Only in a stable system, a small change of the vector  $(\varphi, a, \sigma)$  generates a small change in the equilibrium value of the endogenous variable  $g$ .

Noting (8) and (24), this defines total employment in R&D,  $l$ , as the following function of the growth rate  $g$  and the scale of competition,  $\beta$ :

$$l(g, \beta, \rho), \quad \frac{\partial l}{\partial \beta} = \frac{\nu/\beta}{\nu-1} \frac{\Delta(l)}{\Delta'(l)} > 0, \quad \frac{\partial l}{\partial \rho} = \frac{\Delta(l)}{\Delta'(l)} \left/ \left( \rho + \frac{1-\mu^\sigma}{\log \mu} g \right) \right. < 0,$$

$$\frac{\partial l}{\partial g} = \left[ \frac{\nu/g}{1-\nu} + \frac{1-\mu^\sigma}{\log \mu} \left( \rho + \frac{1-\mu^\sigma}{\log \mu} g \right)^{-1} \right] \frac{\Delta(l)}{\Delta'(l)}. \quad (31)$$

Given (2), the symmetry across the households  $\iota = 1, \dots, n$  yields  $C_\iota = y/N$ . Noting  $C_\iota = y/N$ , (16) and (31), a single household's consumption relative to the level of productivity,  $c \doteq C_\iota/B(\{t_k\})$ , can be written as follows:

$$c(g, \beta, \rho) \doteq \frac{C_\iota}{B(\{t_k\})} = \frac{y/N}{B(\{t_k\})} = e^{\delta\beta} \left[ 1 - \frac{l(g, \beta, \rho)}{N} \right],$$

$$\frac{1}{c} \frac{\partial c}{\partial g} = \frac{1}{l-N} \frac{\partial l}{\partial g} < 0, \quad \frac{1}{c} \frac{\partial c}{\partial \beta} = \delta + \frac{1}{l-N} \frac{\partial l}{\partial \beta}. \quad (32)$$

Noting this, a single household's utility function (3) takes the form

$$U(C_\iota, T) = E \int_T^\infty c(g, \beta, \rho)^\sigma B(\{t_k\})^\sigma e^{-\rho(\nu-T)} d\nu. \quad (33)$$

The government chooses the growth rate  $g$  and the scale of competition,  $\beta$ , to maximize a household's welfare (33) subject to stochastic technological change (8). I denote by  $\Upsilon(\{t_k\})$  the value of any industry using current technology  $t_k$ , and by  $\Upsilon(t_j+1, \{t_{k \neq j}\})$  the value of industry  $j$  using technology  $t_j+1$ , when other industries  $k \neq j$  use current technology  $t_k$ . In each duopoly industry  $j \notin \Theta$ , the arrival rate of innovations that change technology from  $t_j$  to  $t_{j+1}$  is equal to  $\Lambda_{j1} + \Lambda_{j2}$ , while there are no innovations in monopoly industries  $j \in \Theta$ . Noting this, the Bellman equation for the government's maximization problem is given by<sup>9</sup>

$$\rho \Upsilon(\{t_k\}) = \max_{g, \beta} \mathcal{F}(g, \beta, \rho), \quad \text{where}$$

$$\mathcal{F}(g, \beta, \rho) \doteq \frac{c(g, \beta)^\sigma}{B(\{t_k\})^{-\sigma}} + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) [\Upsilon(t_j+1, \{t_{k \neq j}\}) - \Upsilon(\{t_k\})] dj.$$

Because in equilibrium technological change is symmetric throughout all innovating industries,

$$\Upsilon(t_j+1, \{t_{k \neq j}\}) - \Upsilon(\{t_k\}) = \Upsilon(t_\iota+1, \{t_{k \neq \iota}\}) - \Upsilon(\{t_k\}) \text{ for } j \notin \Theta,$$

<sup>9</sup>Cf. Dixit and Pindyck (1994).

then, noting (17), this Bellman equation changes into

$$\begin{aligned}\rho\Upsilon(\{t_k\}) &= \max_{g,\beta} \mathcal{F}(g, \beta, \rho), \quad \text{where} \\ \mathcal{F}(g, \beta, \rho) &= \frac{c(g, \beta, \rho)^\sigma}{B(\{t_k\})^{-\sigma}} + [\Upsilon(t_l + 1, \{t_{k \neq l}\}) - \Upsilon(\{t_k\})] \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj \\ &= \frac{c(g, \beta, \rho)^\sigma}{B(\{t_k\})^{-\sigma}} + [\Upsilon(t_l + 1, \{t_{k \neq l}\}) - \Upsilon(\{t_k\})] \frac{g}{\log \mu}.\end{aligned}\quad (34)$$

## 6 Public policy

### 6.1 The welfare-maximizing scale of competition

Noting (32) and (34), one obtains

$$\beta = \arg \max_{\beta} \mathcal{F}(g, \beta, \rho) = \arg \max_{\beta} c(g, \beta, \rho). \quad (35)$$

This can be rephrased as:

**Proposition 1** *The welfare-maximizing scale of competition is the scale  $\beta$  of competition that maximizes current consumption  $c$ .*

The change of duopoly industries into monopoly industries has two opposite effects on consumption:

- (a) It increases the variety of products and the index of consumption.
- (b) Because duopolies employ more than monopolies, it increases total investment in R&D and deprives these resources from the production of the consumption good.

The opposite effects (a) and (b) are in balance when innovative R&D per firm maximizes consumption.

The first-order condition corresponding to (35) is  $\partial c / \partial \beta = 0$ . Given (31) and (32), this condition is equivalent to

$$\frac{\nu/\beta}{\nu-1} \frac{\Delta(l)}{\Delta'(l)} = \frac{\partial l}{\partial \beta} = (N-l)\delta. \quad (36)$$

Noting (11) and (24), this equation defines that at the government's optimum the scale of competition,  $\beta$ , must be negatively associated with total employment in R&D,  $l$ :

$$\begin{aligned}\beta(l) &\doteq \frac{\nu/\delta}{\nu-1} \frac{\Delta(l)}{\Delta'(l)} \frac{1}{N-l} = \frac{\nu/\delta}{1-\nu} \left(1 + \frac{1-\Pi}{1-l\Pi/N}\right)^{-1} \quad \beta' < 0, \\ -\frac{\beta'}{\beta} &= \frac{(1-\Pi)\Pi/N}{[2-(1+l/N)\Pi](1-\Pi l/N)} < \frac{\Pi/N}{2(1-\Pi)}.\end{aligned}\quad (37)$$

Because duopolies employ more workers in production than monopolies, a transfer of labor from R&D to production (i.e. a smaller  $l$ ) is associated with a higher proportion  $\beta$  of duopoly industries.

## 6.2 The welfare-maximizing growth rate

I try the solution that the value function is of the form

$$\Upsilon(\{t_k\}) = c^\sigma B(\{t_k\})^\sigma / \vartheta \quad (38)$$

where  $\vartheta$  is independent of the endogenous variables of the system. From (7), (15) and (38) it then follows that

$$\frac{\Upsilon(t_j+1, \{t_{k \neq j}\})}{\Upsilon(\{t_k\})} = \left(\frac{B(t_j+1, \{t_{k \neq j}\})}{B(\{t_k\})}\right)^\sigma = \left(\frac{B_j(t_j+1)}{B_j(t_j)}\right)^\sigma = \mu^\sigma. \quad (39)$$

Inserting (38) and (39) into the Bellman equation (34), we obtain

$$\rho = \vartheta + (\mu^\sigma - 1) \int_{j \neq \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj = \vartheta + (\mu^\sigma - 1) \frac{g}{\log \mu}$$

and

$$\vartheta = \rho + \frac{1 - \mu^\sigma}{\log \mu} g. \quad (40)$$

Noting (31), (32), (36), (37), (38), (39) and (40), one obtains

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial g} &= \sigma c^{\sigma-1} B^\sigma \frac{\partial c}{\partial g} + \frac{\mu^\sigma - 1}{\log \mu} \Upsilon(\{t_k\}) = \left[ \vartheta \frac{g}{c} \frac{\partial c}{\partial g} + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right] \frac{\sigma}{g} \Upsilon(\{t_k\}) \\ &= \left[ \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \frac{g}{c} \frac{\partial c}{\partial g} + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right] \frac{\sigma}{g} \Upsilon(\{t_k\}) \\ &= \left[ \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \frac{g}{l - N} \frac{\partial l}{\partial g} + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right] \frac{\sigma}{g} \Upsilon(\{t_k\})\end{aligned}$$

$$\begin{aligned}
&= \left\{ \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \left[ \frac{\nu}{1 - \nu} + \frac{1 - \mu^\sigma}{\log \mu} g \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right)^{-1} \right] \frac{1}{l - N} \frac{\Delta(l)}{\Delta'(l)} \right. \\
&\quad \left. + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right\} \frac{\sigma}{g} \Upsilon(\{t_k\}) \\
&= \left\{ \left[ \frac{\nu}{1 - \nu} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) + \frac{1 - \mu^\sigma}{\log \mu} g \right] \frac{1}{l - N} \frac{\Delta(l)}{\Delta'(l)} + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right\} \frac{\sigma}{g} \Upsilon(\{t_k\}) \\
&= \left\{ \left[ \frac{\nu}{1 - \nu} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) + \frac{1 - \mu^\sigma}{\log \mu} g \right] \frac{1 - \nu}{\nu/\delta} \beta + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right\} \frac{\sigma}{g} \Upsilon(\{t_k\}) \\
&= \left\{ \left[ \rho + \frac{1 - \mu^\sigma}{\log \mu} g + \left( \frac{1}{\nu} - 1 \right) \frac{1 - \mu^\sigma}{\log \mu} g \right] \delta \beta + \frac{\mu^\sigma - 1}{\sigma \log \mu} g \right\} \frac{\sigma}{g} \Upsilon(\{t_k\}) \\
&= \left[ \frac{\log \mu}{\mu^\sigma - 1} \frac{\rho}{g} + \frac{1}{\delta \sigma \beta (\xi(a, \eta))} - \frac{1}{\nu} \right] \frac{\mu^\sigma - 1}{\log \mu} g \delta \beta \frac{\sigma}{g} \Upsilon(\{t_k\}). \tag{41}
\end{aligned}$$

According to (41), there is  $\partial \mathcal{F} / \partial g > 0$  in the case  $\sigma \delta \beta < \nu$ . Noting the structure of the function (5), this result can be rephrased also as follows:

**Proposition 2** *There can be a finite welfare-maximizing growth rate only if the households appreciate product variety high enough (i.e. if  $\delta > \nu / (\sigma \beta)$ ).*

If the households do not appreciate product variety high enough (i.e. if  $\delta \leq \nu / \sigma$ ), then it is socially optimal to allocate more labor in R&D and to speed up growth indefinitely. For the remainder of this paper, I focus on the case where there is a welfare-maximizing growth rate:

$$\delta > \nu / (\sigma \beta). \tag{42}$$

The first-order condition corresponding to the maximization  $\mathcal{F}$  in (34) by the growth rate  $g$  is  $\partial \mathcal{F} / \partial g = 0$ . This, (41) and (42) imply that the optimal growth rate  $g$  is an increasing function of the rate of time preference  $\rho$ :

$$g = \left[ \frac{1}{\nu} - \frac{1}{\delta \sigma \beta (\xi(a, \eta))} \right]^{-1} \frac{\rho \log \mu}{\mu^\sigma - 1}, \quad \frac{dg}{d\rho} = \frac{g}{\rho}, \quad \frac{\rho \log \mu}{(\mu^\sigma - 1)g} = \frac{1}{\nu} - \frac{1}{\delta \sigma \beta} < \frac{1}{\nu}. \tag{43}$$

### 6.3 The welfare-maximizing patent shape

In subsections 6.1 and 6.2, the government's optimum was defined in terms of the scale of competition,  $\beta$ , and the growth rate  $g$ . In this subsection, I define the patent shape  $(a, \varphi)$  that corresponds to that optimum.

Inserting (37) into the equation (27) yields  $l/\beta(l) = 2(\eta + a\eta^{1-\nu})$ . Noting (37), this equation defines that at the government's optimum total labor devoted to R&D,  $l$ , must be an increasing function of both patent length  $a$  and patent breadth  $\varphi$ :

$$l = \xi(a, \eta), \quad \frac{\partial \xi}{\partial a} = \left( \frac{1}{l} - \frac{\beta'}{\beta} \right)^{-1} \frac{1}{a + \eta^\nu} > 0,$$

$$\frac{\partial \xi}{\partial \eta} = \left( \frac{1}{l} - \frac{\beta'}{\beta} \right)^{-1} \frac{\eta^\nu + 1 - \nu}{(a + \eta^\nu)\eta} = \left( \eta^{\nu-1} + \frac{1 - \nu}{\eta} \right) \frac{\partial \xi}{\partial a} > 0. \quad (44)$$

With an increase in patent length  $a$  or patent breadth  $\varphi$ , a larger proportion  $\alpha$  of industries will end up as monopolies and a smaller proportion  $\beta$  as duopolies. Because monopolies employ less in production, this reduces the demand for labor in production. With lower employment in production, wages fall and R&D firms employ more labor (i.e.  $l$  increases).

To obtain an unambiguous solution, I make the plausible assumption<sup>10</sup>

$$\nu > \left( 1 + 2 \frac{1 - \Pi}{\Pi} \frac{l}{N} \right)^{-1}, \quad (45)$$

where  $1 - \nu$  is the returns-to-scale parameter in innovative R&D [cf. (8)],  $l/N$  the proportion of labor devoted to R&D and  $\Pi/(1 - \Pi)$  the ratio of profits to wages in production. If the returns to scale in innovative R&D are too decreasing, so that (45) does not hold, then the results are unambiguous.

Given (24), (27), (30), (44) and (43), there is a system of two equations

$$g(\rho) = G(a, \eta, \rho), \quad \xi(a, \eta) = \Delta^{-1} \left( \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g(\rho) \right) \frac{\eta^\nu}{\lambda \mu^\sigma} \right), \quad (46)$$

where  $a$  and  $\varphi$  are unknown variables and  $\rho$  is an exogenous variable. Differentiating this system totally and noting (24), (44) and (43), one obtains [cf. Appendix B]

$$da/d\rho < 0, \quad d\varphi/d\rho > 0. \quad (47)$$

These results can be rephrased as follows:

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<sup>10</sup>In advanced economies, the proportion of research workers,  $l/N$ , is less than  $\frac{1}{5}$  and the ratio of profits to wages in production,  $\Pi/(1 - \Pi)$ , less than  $\frac{1}{2}$ . This implies  $\nu > \frac{1}{21}$  and  $1 - \nu < \frac{20}{21}$ . It is plausible to assume that the returns to scale parameter in innovative R&D,  $1 - \nu$ , is less than  $\frac{20}{21}$ .

**Proposition 3** *The less patient the households in the economy (i.e. the bigger  $\rho$ ), the shorter (i.e. the smaller  $a$ ) and the broader (i.e. the bigger  $\varphi$ ) the welfare-maximizing patent.*

The interpretation of proposition 3 is the following. When households are impatient (i.e.  $\rho$  is high), their subjective discount factor  $\rho + \frac{1-\mu^\sigma}{\log \mu} g$  is high. Provided that innovative investment per firm  $\eta(a, \varphi)$  is held constant, this decreases their savings, investment and employment  $l$  in R&D [cf. (24)]. In order to maintain full employment, the government must decrease patent length  $a$  to increase the proportion  $\beta$  of duopoly industries, because duopolies employ more workers in production than monopolies. Simultaneously, it must increase patent breadth  $\varphi$  to hold innovative investment per firm  $\eta(a, \varphi)$  constant [cf. (23)]. This shows that patents should be tailored short and wide for an economy with impatient households.

## 7 Conclusions

This study examines a multi-industry economy in which growth is generated by creative destruction. In each industry, a firm that creates the newest technology by a successful innovation crowds out the other firms with older technologies from the market and becomes the first producer of the industry. A firm creating a copy of the newest technology starts producing a close substitute for the innovator's product and establishes an innovation race with the first producer. Because systematic investment risk cannot be eliminated by diversification, the households hold the shares of all firms in their portfolios.

Innovations are protected by patents. Some characteristics of a patent postpone the expected time the patent will be imitated (e.g. increase patent length), while the others protect the patentee's profits after a successful imitation (i.e. increase patent breadth). With these two instruments, the government can regulate innovative and imitative R&D, economic growth and social welfare. The main findings of this study are as follows.

An increase in patent breadth or patent length increases innovative R&D per firm. This changes duopoly industries into monopoly industries, which has two opposite effects on consumption: It decreases the variety of products and lowers the index of consumption. On the other hand, because duopolies

employ more than monopolies, it decreases total investment in R&D and releases resources from R&D to consumption. These opposite effects are in balance when innovative investment per firm maximizes consumption. When this optimal consumption is maintained, there is a trade-off between patent breadth or patent length. This means that patents must be tailored either broad and short or narrow and long.

Because monopolies have no incentives to innovate, the growth rate increases with the proportion of duopoly industries. With non-diversifiable risk, there is a trade-off between patent length and patent breadth for given innovative investment per firm. When households are impatient, they save less and invest less in R&D. This, in turn, decreases employment in R&D. In order to maintain full employment, the government must decrease patent length to increase the proportion of duopoly industries, because duopolies employ more workers in production than monopolies. Simultaneously, it must increase patent breadth for given innovative investment per firm. Thus, patents should be tailored short and wide for with impatient households.

## Appendix

### *A. Results (23)-(25)*

I denote:

$\{s_{tkv}\}$  vector of  $s_{tkv}$  for  $k \in [0, 1]$  and  $v \in \{0, 1, 2\}$ ,

$\{s_{t(k \neq j)v}\}$  vector of  $s_{tkv}$  for  $k \in [0, 1]$ ,  $k \neq j$  and  $v \in \{0, 1, 2\}$ ,

$\{\tau_k\}$  vector of  $\tau_k$  for  $k \in [0, 1]$ ,

$\{\tau_{k \neq j}\}$  vector of  $\tau_k$  for  $k \in [0, 1]$  and  $k \neq j$ .

According to (11), the profit of a monopoly is  $\Pi$  and the profit of the first (second) duopolist  $\Pi_{j1} = \phi_1\Pi$  ( $\Pi_{j2} = \phi_2\Pi$ ). I define the following value functions:

$\Omega(\{s_{tkv}\}, \{\tau_k\})$  the value of receiving profits  $s_{tkv}$  from all firms  $v$  in all industries  $k$  using current technology  $\tau_k$ .

$\Omega(\Pi_{i_{j\kappa}}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\})$  the value of receiving the profit  $\Pi_{i_{j\kappa}}$  from firm  $\kappa$  in industry  $j \notin \Theta$  using technology  $\tau_j + 1$ , but receiving no profits from the other firm which was a producer in that industry when technology  $\tau_j$  was used, and receiving profits  $s_{\iota(k \neq j)v}$  from all firms  $v$  in other industries  $k \neq j$  with current technology  $\tau_k$ .

$\Omega(\phi_1 \Pi_{i_{j1}}, \phi_2 \Pi_{i_{j2}}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\})$  the value of receiving profits  $\phi_2 \Pi_{i_{j\kappa}}$  from firms  $\kappa \in \{1, 2\}$  in industry  $j \in \Theta$ , but receiving profits  $s_{\iota(k \neq j)v}$  from all firms  $v$  in the other industries  $k \neq j$  with current technology  $\tau_k$ .

Household  $\iota$  maximizes its utility (3) by its investment,  $\{S_{\iota j0}\}$  for  $j \in \Theta$  and  $\{S_{\iota j1}, S_{\iota j2}\}$  for  $j \notin \Theta$ , subject to (19), (20), (21) and (22), , taking  $w, P$  and  $\{\Lambda_{j\kappa}, \Gamma_{j0}\}$  for all  $j$  and  $\kappa$  as exogenous. The Bellman equation associated with the household's maximization is<sup>11</sup>

$$\rho \Omega(\{s_{\iota kv}\}, \{\tau_k\}) = \max_{S_{\iota j} \geq 0 \text{ for all } j} \Xi_{\iota} \quad (48)$$

with

$$\begin{aligned} \Xi_{\iota} \doteq & C_{\iota}^{\sigma} + \int_{j \in \Theta} \Gamma_{j0} \left[ \Omega(\phi_1 \Pi_{i_{j1}}, \phi_2 \Pi_{i_{j2}}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}) \right] dj \\ & + \int_{j \notin \Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \left[ \Omega(\Pi_{i_{j\kappa}}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}) \right] dj, \end{aligned} \quad (49)$$

where  $\rho$  is the rate of time preference (constant),  $C_{\iota}^{\sigma}$  the household's instantaneous utility,  $\Lambda_{j\kappa}$  the arrival rate of innovations that increases the value of profits from duopolist  $\kappa$  in industry  $j \notin \Theta$  by the amount

$$\Omega(\Pi_{i_{j\kappa}}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}),$$

and  $\Gamma_{j0}$  the arrival rate of imitations that increases the value of profits from outsider 0 in industry  $j \in \Theta$  by the amount

$$\Omega(\phi_1 \Pi_{i_{j1}}, \phi_2 \Pi_{i_{j2}}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}).$$

Because  $\partial C_{\iota} / \partial S_{\iota j\kappa} = -1/P$  by (21), the first-order conditions are given by

$$\begin{aligned} \Lambda_{j\kappa} \frac{d}{dS_{\iota j\kappa}} \left[ \Omega(\Pi_{i_{j\kappa}}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}) \right] &= \frac{\sigma}{P} C_{\iota}^{\sigma-1} \\ &\text{for } j \notin \Theta \text{ and } \kappa \in \{1, 2\}, \end{aligned} \quad (50)$$

<sup>11</sup>Cf. Dixit and Pindyck (1994).

$$\Gamma_{j0} \frac{d}{dS_{lj0}} \left[ \Omega(\phi_1 \Pi_{lj1}, \phi_2 \Pi_{lj2}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) - \Omega(\{s_{\iota kv}\}, \{\tau_k\}) \right] = \frac{\sigma}{P} C_{\iota}^{\sigma-1}$$

for  $j \in \Theta$ . (51)

I try the solution that for each household  $\iota$  the propensity to consume,  $h_{\iota}$ , and the subjective interest rate  $r_{\iota}$  are independent of income  $A_{\iota}$ , i.e.  $PC_{\iota} = h_{\iota} A_{\iota}$  and  $\Omega = C_{\iota}^{\sigma}/r_{\iota}$ . I denote variables depending on technology  $\tau_j$  by superscript  $\tau_j$  and a vector that consists of  $t_k$  for all  $k$  by  $\{t_k\}$ . Since according to (22) income  $A_{\iota}^{\{\tau_k\}}$  depends directly on variables  $\{s_{\iota k}^{\tau_k}\}$ , I denote  $A_{\iota}^{\{\tau_k\}}(\{s_{\iota k}^{\tau_k}\})$ . Assuming that  $h_{\iota}$  is invariant across technologies yields

$$P^{\{\tau_k\}} C_{\iota}^{\{\tau_k\}} = h_{\iota} A_{\iota}^{\{\tau_k\}}(\{s_{\iota k}^{\tau_k}\}). \quad (52)$$

The share in the next innovator  $\tau_j + 1$  is determined by investment under the present technology  $\tau_j$ ,  $s_{\iota j \kappa}^{\tau_j+1} = \Pi_{\iota j \kappa}^{\tau_j}$  for  $j \notin \Theta$ . The share in the next imitator is determined by investment under the same technology  $\tau_j$ ,  $s_{\iota j \kappa}^{\tau_j} = \phi_{\kappa} \Pi_{\iota j \kappa}^{\tau_j}$  for  $j \in \Theta$ . The value functions are then given by

$$\begin{aligned} \Omega(\{s_{\iota kv}\}, \{\tau_k\}) &= \Omega(\phi_1 \Pi_{lj1}, \phi_2 \Pi_{lj2}, \{s_{\iota(k \neq j)v}\}, \{\tau_k\}) = \frac{1}{r_{\iota}} (C_{\iota}^{\{\tau_k\}})^{\sigma}, \\ \Omega(\Pi_{\iota j \kappa}, 0, \{s_{\iota(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) &= \frac{1}{r_{\iota}} (C_{\iota}^{\tau_j+1, \{\tau_{k \neq j}\}})^{\sigma}. \end{aligned} \quad (53)$$

Given this, one obtains

$$\frac{\partial \Omega(\{s_{\iota kv}\}, \{\tau_k\})}{\partial S_{lj}^{\tau_j}} = 0. \quad (54)$$

From (19), (22), (52), (53),  $s_{\iota j \kappa}^{\tau_j+1} = \Pi_{\iota j \kappa}^{\tau_j}$  for  $j \notin \Theta$  and  $\kappa = 1, 2$ , and  $s_{\iota j 2}^{\tau_j} = \phi_2 \Pi_{\iota j 2}^{\tau_j}$  for  $j \in \Theta$  it follows that

$$\begin{aligned} \frac{\partial s_{\iota j \kappa}^{\tau_j+1}}{\partial i_{\iota j \kappa}^{\tau_j}} &= \Pi \text{ for } j \notin \Theta \text{ and } \kappa = 1, 2, & \frac{\partial s_{\iota j 2}^{\tau_j}}{\partial i_{\iota j 0}^{\tau_j}} &= \phi_2 \Pi \text{ for } j \in \Theta, \\ \frac{\partial A_{\iota}^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial s_{\iota j \kappa}^{\tau_j+1}} &= \frac{\partial A_{\iota}^{\{\tau_k\}}}{\partial s_{\iota j 0}^{\tau_j}} = 1, \\ \frac{\partial i_{\iota j 0}^{\tau_j}}{\partial S_{\iota j 0}^{\tau_j}} &= \frac{1}{w^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \text{ for } j \in \Theta, & \frac{\partial i_{\iota j \kappa}^{\tau_j}}{\partial S_{\iota j \kappa}^{\tau_j}} &= \frac{1}{w^{\{\tau_k\}} l_{j \kappa}^{\{\tau_k\}}} \text{ for } j \notin \Theta, \end{aligned}$$

$$\begin{aligned}
& \frac{d\Omega(\Pi i_{lj\kappa}, 0, \{s_{l(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\})}{dS_{lj\kappa}^{\tau_j}} \\
&= \frac{\sigma}{r_l} (C_l^{\tau_j+1, \{\tau_{k \neq j}\}})^{\sigma-1} \underbrace{\frac{\partial C_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}}}_{=h_l/P^{\tau_j+1, \{\tau_{k \neq j}\}}} \underbrace{\frac{\partial A_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{\partial s_{lj\kappa}^{\tau_j+1}}}_{=1} \underbrace{\frac{\partial s_{lj\kappa}^{\tau_j+1}}{\partial i_{lj\kappa}^{\tau_j}}}_{=\Pi} \frac{\partial i_{lj\kappa}^{\tau_j}}{\partial S_{lj\kappa}^{\tau_j}} \\
&= \frac{\Pi \sigma h_l (C_l^{\tau_j+1, \{\tau_{k \neq j}\}})^{\sigma-1}}{r_l P^{\tau_j+1, \{\tau_{k \neq j}\}}} \frac{\partial i_{lj\kappa}^{\tau_j}}{\partial S_{lj\kappa}^{\tau_j}} = \frac{\Pi h_l \sigma (C_l^{\tau_j+1, \{\tau_{k \neq j}\}})^{\sigma-1}}{r_l w^{\{\tau_k\}} P^{\tau_j+1, \{\tau_{k \neq j}\}} l_{j\kappa}^{\{\tau_k\}}} \quad \text{for } j \notin \Theta, \quad (55)
\end{aligned}$$

$$\begin{aligned}
& \frac{d\Omega(\phi_1 \Pi i_{lj1}, \phi_2 \Pi i_{lj2}, \{s_{l(k \neq j)v}\}, \{\tau_k\})}{dS_{lj0}^{\tau_j}} \\
&= \frac{\sigma}{r_l} (C_l^{\{\tau_k\}})^{\sigma-1} \underbrace{\frac{\partial C_l^{\{\tau_k\}}}{\partial A_l^{\{\tau_k\}}}}_{=h_l/P^{\{\tau_k\}}} \underbrace{\frac{\partial A_l^{\{\tau_k\}}}{\partial s_{lj0}^{\tau_j}}}_{=1} \underbrace{\frac{s_{lj0}^{\tau_j}}{\partial i_{lj0}^{\tau_j}}}_{=\phi_2 \Pi} \frac{\partial i_{lj0}^{\tau_j}}{\partial S_{lj0}^{\tau_j}} \\
&= \frac{\phi_2 \Pi \sigma h_l (C_l^{\{\tau_k\}})^{\sigma-1}}{r_l P^{\{\tau_k\}}} \frac{\partial i_{lj0}^{\tau_j}}{\partial S_{lj0}^{\tau_j}} = \frac{\phi_2 \Pi h_l \sigma (C_l^{\{\tau_k\}})^{\sigma-1}}{r_l w^{\{\tau_k\}} P^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \quad \text{for } j \in \Theta. \quad (56)
\end{aligned}$$

I focus on a stationary equilibrium where the growth rate  $g$  and the allocation of labor,  $(l_{j\kappa}, x)$ , are invariant across technologies. Given (1), (13), (15) and (16), this implies

$$\begin{aligned}
l_{j\kappa}^{\{\tau_k\}} &= l_{j\kappa}, \quad x^{\{\tau_k\}} = x = N - l, \quad w^{\{\tau_k\}} = w = 1/(x\mu), \\
\frac{P^{\{\tau_k\}}}{P^{\tau_j+1, \{\tau_{k \neq j}\}}} &= \frac{C_l^{\tau_j+1, \{\tau_{k \neq j}\}}}{C_l^{\{\tau_k\}}} = \frac{y^{\tau_j+1, \{\tau_{k \neq j}\}}}{y^{\{\tau_k\}}} = \frac{B^{\tau_j+1, \{\tau_{k \neq j}\}}}{B^{\{\tau_k\}}} = \mu. \quad (57)
\end{aligned}$$

Inserting (17), (49), (53) and (57) into (48) yields

$$\begin{aligned}
0 &= \left[ \rho + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj + \int_{j \in \Theta} \Gamma_{j0} dj \right] \Omega(\{s_{lkv}\}, \{\tau_k\}) - (C_l^{\{\tau_k\}})^\sigma \\
&\quad - \int_{j \notin \Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \Omega(\Pi i_{lj\kappa}, 0, \{s_{l(k \neq j)v}\}, \tau_j + 1, \{\tau_{k \neq j}\}) dj \\
&\quad - \int_{j \in \Theta} \Gamma_{j0} \Omega(\pi_2 i_{lj1}, \pi_2 i_{lj2}, \{s_{l(k \neq j)v}\}, \{\tau_k\}) dj \\
&= \left[ \rho + \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj \right] \frac{(C_l^{\{\tau_k\}})^\sigma}{r_l} - (C_l^{\{\tau_k\}})^\sigma - \int_{j \notin \Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \frac{\mu^\sigma}{r_l} (C_l^{\{\tau_k\}})^\sigma dj \\
&= \frac{1}{r_l} (C_l^{\{\tau_k\}})^\sigma \left[ \rho - r_l + \frac{1 - \mu^\sigma}{\log \mu} g \right].
\end{aligned}$$

This equation is equivalent to

$$r_\iota = \rho + \frac{1 - \mu^\sigma}{\log \mu} g. \quad (58)$$

Because there is symmetry throughout all households  $\iota$ , their propensity to consume is equal,  $h_\iota = h$ . This, (4), (18), (21) and (52) yield

$$\begin{aligned} wl &= w \int_{j \in \Theta} l_{j0} dj + w \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj = w \int_{j \in \Theta} l_{j0} dj + \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj \\ &= \sum_{\iota=1}^N \left[ \int_{j \in \Theta} S_{\iota j0} dj + \int_{j \notin \Theta} (S_{\iota j1} + S_{\iota j2}) dj \right] = \sum_{\iota=1}^N (A_\iota - PC_\iota) = (1 - h) \sum_{\iota=1}^N A_\iota \\ &= (1 - h)(1 + wl) \end{aligned}$$

and

$$h_\iota = h = (1 + wl)^{-1}. \quad (59)$$

Inserting (8), (9), (54), (55), (56), (57), (58) and (59) into (50) and (51), one obtains

$$\begin{aligned} \frac{\sigma \Pi h \mu^\sigma (C_\iota^{\{\tau_k\}})^{\sigma-1} \lambda}{\left(\rho + \frac{1-\mu^\sigma}{\log \mu} g\right) w P^{\{\tau_k\}}} l_{j\kappa}^{-\nu} &= \frac{\sigma \Pi h \mu^\sigma (C_\iota^{\{\tau_k\}})^{\sigma-1} \lambda l_{j\kappa}^{-\nu}}{r_\iota w P^{\{\tau_k\}}} = \frac{\sigma \Pi h_\iota \mu^\sigma (C_\iota^{\{\tau_k\}})^{\sigma-1} \Lambda_{j\kappa}}{r_\iota w l_{j\kappa} P^{\{\tau_k\}}} \\ &= \frac{\sigma \Pi h_\iota (C_\iota^{\tau_j+1, \{\tau_k \neq j\}})^{\sigma-1} \Lambda_{j\kappa}}{r_\iota w l_{j\kappa} P^{\tau_j+1, \{\tau_k \neq j\}}} = \Lambda_{j\kappa} \frac{d}{dS_{\iota j\kappa}} \Omega(\Pi_{\iota j}, \{s_{\iota(k \neq j)}\}, \tau_j + 1, \{\tau_k \neq j\}) \\ &= \frac{\sigma}{P^{\{\tau_k\}}} (C_\iota^{\{\tau_k\}})^{\sigma-1} \text{ for } j \notin \Theta \text{ and } \kappa \in \{1, 2\}, \end{aligned} \quad (60)$$

$$\begin{aligned} \frac{\phi_2(\varphi)}{a} \frac{\sigma \Pi h (C_\iota^{\{\tau_k\}})^{\sigma-1} \lambda}{\left(\rho + \frac{1-\mu^\sigma}{\log \mu} g\right) w P^{\{\tau_k\}}} &= \frac{\sigma \phi_2 \Pi h_\iota (C_\iota^{\{\tau_k\}})^{\sigma-1} \frac{\lambda}{a}}{r_\iota w P^{\{\tau_k\}}} = \frac{\sigma \phi_2 \Pi h_\iota (C_\iota^{\{\tau_k\}})^{\sigma-1} \Gamma_{j0}}{r_\iota w l_{j0} P^{\{\tau_k\}}} \\ &= \Gamma_{j0} \frac{d}{dS_{\iota j0}} \Omega(\phi_1 \Pi_{\iota j1}, \phi_2 \Pi_{\iota j2}, \{s_{\iota m(k \neq j)}\}, \{\tau_k\}) = \frac{\sigma}{P^{\{\tau_k\}}} (C_\iota^{\{\tau_k\}})^{\sigma-1} \text{ for } j \in \Theta. \end{aligned} \quad (61)$$

Given  $\mu > 1$ ,  $\pi_2 \leq \Pi/2$ , (60) and (61), one obtains

$$\begin{aligned} l_{j\kappa} &= \eta(a, \varphi) \quad \text{for } j \notin \Theta, \\ l_{j0} &= \psi \quad \text{for } j \in \Theta, \end{aligned} \quad \eta(a, \varphi) \doteq \left[ \frac{\mu^\sigma a}{\phi_2(\varphi)} \right]^{1/\nu} \quad \frac{\partial \eta}{\partial a} > 0, \quad \frac{\partial \eta}{\partial \varphi} > 0. \quad (62)$$

Equations (8), (11), (12), (13), (14), (17), (59), (61) and (62) yield

$$l = \int_{j \notin \Theta} (l_{j1} + l_{j2}) dj + \int_{j \in \Theta} l_j dj = 2\eta \int_{j \notin \Theta} dj + \psi \int_{j \in \Theta} dj = (1 - \beta)\psi + 2\beta\eta,$$

$$\psi = (l - 2\beta\eta)/(1 - \beta), \quad (63)$$

$$\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\nu} = \lambda \eta^{1-\nu} \text{ for } j \notin \Theta \text{ and } \kappa \in \{1, 2\},$$

$$g = (\log \mu) \int_{j \notin \Theta} (\Lambda_{j1} + \Lambda_{j2}) dj = (2 \log \mu) \beta \Lambda_{j\kappa} = (2\lambda \log \mu) \beta \eta^{1-\nu}, \quad (64)$$

$$\left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \frac{\eta^\nu}{\lambda \mu^\sigma} = \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \frac{a}{\lambda \phi_2} = \frac{h\Pi}{w} = \frac{\Pi}{(1 + wl)w}$$

$$= \frac{(N - l)^2 \mu \Pi}{N - l + l/\mu} = \frac{(N - l)^2 \mu \Pi}{N - l\Pi} = \frac{(N - l)^2 \mu}{N/\Pi - l} \doteq \Delta(l) \text{ with}$$

$$\frac{\Delta'}{\Delta} = \frac{d \log \Delta}{dl} = \frac{1}{N/\Pi - l} - \frac{2}{N - l} < \frac{1}{N - l} - \frac{2}{N - l} = -\frac{1}{N - l} < 0, \quad (65)$$

$$1 - \frac{l\Delta'}{\Delta} > 1 + \frac{l}{N - l} = \frac{N}{N - l}. \quad (66)$$

Relations (62), (64), (65) and (66) yield (23)-(25).  $\square$

## B. Results (47)

Differentiating the equations (46) totally and noting (24), (30), (37), (36), (43) and (45), one obtains

$$\mathcal{J} = \left| \begin{array}{cc} G_a & G_\eta \\ \xi_a & \xi_\eta - \frac{\nu \eta^{\nu-1}}{\lambda \mu^\sigma \Delta'} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \end{array} \right|$$

$$= \underbrace{G_a}_{-} \left[ \underbrace{(1 - \nu)}_{+} \underbrace{\left( \eta^{\nu-1} + \frac{1}{\eta} \right)}_{+} \underbrace{\xi_a}_{+} + \underbrace{\frac{\beta'/\beta}{1/l - \beta'/\beta}}_{-} \underbrace{\frac{\nu \eta^{\nu-1}}{\lambda \mu^\sigma \Delta'}}_{-} \underbrace{\left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right)}_{+} \right] < 0,$$

$$\frac{da}{d\rho} = -\frac{1}{\mathcal{J}} \left| \begin{array}{cc} G_\rho - g' & G_\eta \\ -\frac{\eta^\nu}{\lambda \mu^\sigma \Delta'} \left( 1 + \frac{1 - \mu^\sigma}{\log \mu} g' \right) & \xi_\eta - \frac{\nu \eta^{\nu-1}}{\lambda \mu^\sigma \Delta'} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \end{array} \right|$$

$$< -\frac{1}{\mathcal{J}} \left| \begin{array}{cc} G_\rho & G_\eta \\ -\frac{\eta^\nu}{\lambda \mu^\sigma \Delta'} \left( 1 + \frac{1 - \mu^\sigma}{\log \mu} g' \right) & \xi_\eta - \frac{\nu \eta^{\nu-1}}{\lambda \mu^\sigma \Delta'} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \end{array} \right|$$

$$< -\frac{1}{\mathcal{J}} \left| \begin{array}{cc} G_\rho & G_\eta \\ -\frac{\eta^\nu}{\lambda \mu^\sigma \Delta'} & \xi_\eta - \frac{\nu \eta^{\nu-1}}{\lambda \mu^\sigma \Delta'} \left( \rho + \frac{1 - \mu^\sigma}{\log \mu} g \right) \end{array} \right|$$

$$= -\frac{G_\rho}{\mathcal{J}} \left[ \left( \frac{1}{l} - \frac{\beta'}{\beta} \right)^{-1} \frac{\eta^\nu + 1 - \nu}{(a + \eta^\nu)\eta} - \frac{\nu \eta^{\nu-1} l}{\eta^\nu + a} \right]$$

$$\begin{aligned}
&= - \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \underbrace{\frac{\nu\eta^\nu}{(a+\eta^\nu)\eta}}_{+} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \left[ \underbrace{(1/\nu - 1)}_{+} \underbrace{(1 + \eta^{-\nu})}_{+} + l \frac{\beta'}{\beta} \right] \\
&< - \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \underbrace{\frac{\nu\eta^\nu}{(a+\eta^\nu)\eta}}_{+} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \left[ \frac{1}{\nu} - 1 + l \frac{\beta'}{\beta} \right] \\
&< - \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \underbrace{\frac{\nu\eta^\nu}{(a+\eta^\nu)\eta}}_{+} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \left[ \frac{1}{\nu} - 1 - \frac{1}{2} \frac{\Pi}{1 - \Pi} \frac{l}{N} \right] < 0, \\
\frac{d\eta}{d\rho} &= - \frac{1}{\mathcal{J}} \left| \frac{G_a}{\xi_a} - \frac{G_\rho - g'}{\lambda\mu^\sigma\Delta'} \left(1 + \frac{1-\mu^\sigma}{\log \mu} g'\right) \right| > - \frac{1}{\mathcal{J}} \left| \frac{G_a}{\xi_a} - \frac{G_\rho}{\lambda\mu^\sigma\Delta'} \left(1 + \frac{1-\mu^\sigma}{\log \mu} g'\right) \right| \\
&= \frac{G_\rho}{\mathcal{J}} \left[ - \left(1 + \frac{1-\mu^\sigma g}{\log \mu \rho}\right) \frac{l}{\eta^\mu + a} + \xi_a \right] \\
&= \frac{G_\rho}{\mathcal{J}} \left[ - \left(1 + \frac{1-\mu^\sigma g}{\log \mu \rho}\right) \frac{l}{\eta^\mu + a} + \frac{1}{a + \eta^\nu} \left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1} \right] \\
&> \frac{1}{\underbrace{a + \eta^\nu}_{+}} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \left[ \left(1 + \frac{1-\mu^\sigma g}{\log \mu \rho}\right) \left(-\frac{1}{2} \frac{l}{N} \frac{\Pi}{1 - \Pi} - 1\right) + 1 \right] \\
&= \frac{1}{\underbrace{a + \eta^\nu}_{+}} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \left[ \left(\frac{\mu^\sigma - 1}{\log \mu \rho} g - 1\right) \underbrace{\left(\frac{1}{2} \frac{l}{N} \frac{\Pi}{1 - \Pi} + 1\right)}_{+} + 1 \right] \\
&> \frac{1}{\underbrace{a + \eta^\nu}_{+}} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \left[ (\nu - 1) \underbrace{\left(\frac{1}{2} \frac{l}{N} \frac{\Pi}{1 - \Pi} + 1\right)}_{+} + 1 \right] \\
&= \frac{1 - \nu}{\underbrace{a + \eta^\nu}_{+}} \underbrace{\left(\frac{1}{l} - \frac{\beta'}{\beta}\right)^{-1}}_{+} \underbrace{\frac{G_\rho}{\mathcal{J}}}_{+} \left[ \underbrace{-\frac{1}{2} \frac{l}{N} \frac{\Pi}{1 - \Pi} - 1 + \frac{1}{1 - \nu}}_{+} \right] > 0.
\end{aligned}$$

From these inequalities and the definition of  $\eta$  in (23) it follows that

$$\frac{d\varphi}{d\rho} = \left( \underbrace{\frac{d\eta}{d\rho}}_{+} - \underbrace{\frac{\partial\eta}{\partial a}}_{+} \underbrace{\frac{da}{d\rho}}_{-} \right) / \underbrace{\frac{\partial\eta}{\partial\varphi}}_{+} > 0.$$

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