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# Exit Options and Dividend Policy under Liquidity Constraints\*

## Abstract

We introduce a post-entry liquidity constraint to the standard model of a firm with stochastic cash flow and irreversible exit decision. We assume that a firm with no cash holdings and negative cash flow is forced to exit regardless of its future prospects. This creates a precautionary motive for holding cash, which must be traded off against the liquidity cost of holding cash. We characterize the optimal exit and dividend policy and analyze numerically its comparative statics properties. The firm pays dividends when it is in a sufficiently strong position in terms of cash flow and cash holdings, and the firm almost surely exits voluntarily to pre-empt forced exit. The direct effect of the liquidity constraint is to impose inefficient exit, but in industry equilibrium it also creates a price distortion that leads to inefficient survival.

**JEL Classification:** D81, D92, G35

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# 1 Introduction

In the standard entry-exit problem of a firm with stochastic cash flow, the optimal policy requires the firm to sustain negative cash flows indefinitely. Our main question is how firms should behave if they have a limited capability of paying for losses. The solution is a policy for exit and dividend payments that depends on the current levels of both cash flow and cash holdings.

In the standard problem, the potential for future profits and the irreversibility of exit make it optimal for a firm to accept negative cash flows up to some point.<sup>1</sup> In the absence of financial constraints cash holdings are irrelevant, and the optimal policy is simply a negative threshold level of cash flow below which the firm exits. However, the value of continuation is partly due to future paths where cash flow remains negative for arbitrarily long periods of time. It seems realistic in many contexts that a firm with a long history of losses would find it difficult to keep raising more funds. But as soon as there is a limit to a firm's ability to sustain losses the firm's problem changes in a fundamental way.

To make our point clear, we initially model the liquidity constraint as the complete inability to raise new funds. The firm has an initial stock of cash that can only be augmented with retained earnings. A firm without cash and with a negative cash flow is forced to exit immediately regardless of its future prospects, so firms have an incentive to hoard cash in order to avoid inefficient exit in the future. This precautionary saving is costly due to the liquidity premium—cash holdings earn interest at a rate below the discount rate. Therefore, if the firm is sufficiently safe from forced exit—with sufficiently high cash flow and/or cash holdings—it is optimal to pay out some of the cash to the owners. Thus, besides affecting the optimal exit policy, the liquidity constraint also generates the optimal dividend policy. We characterize the optimal policy and analyze its dependence on the properties of the cash flow process. Our numerical results show that a small liquidity premium has a large impact on optimal firm behavior.

We do not explicitly model the causes behind the liquidity constraint. One natural cause of liquidity constraints is asymmetric information: it can be difficult for a firm or a manager to credibly convey to investors that it has potential for profits.<sup>2</sup> Aside

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<sup>1</sup>See e.g. Chapter 7 in Dixit and Pindyck (1993).

<sup>2</sup>For evidence on the importance of liquidity constraints for firms, see for example Evans and

from the liquidity constraint, our model has no other imperfections such as agency problems. The optimal policy maximizes the value of the firm to its owners, taking as given the lack of further cash injections by the owners. In an extension to our model we assume that raising external funds incurs a transaction cost; in effect the basic model assumes that this cost is prohibitive.

Our model builds on elements from the literature on the optimal exercise of options, where the seminal papers are by McDonald and Siegel (1986) who model the optimal timing of investment under uncertain cash flow, and by Dixit (1989) who analyzes the firm's optimal entry and exit decisions in the same framework. A large number of extensions to various directions is summarized by Dixit and Pindyck (1993). Our paper extends this line of research further by adding a liquidity constraint that may prevent the firm from covering operating losses. As we are interested in environments where the firm's future prospects vary over time, we need a state variable for the firm's current cash flow (income level). On the other hand, to model a constraint for covering losses, we need another state variable for the firm's current cash holdings (wealth). The state-variable in our problem is inherently two-dimensional: any simplification that reduces the problem to one state-variable would assume away the problem we are interested in.<sup>3</sup>

Our model leads to a free boundary partial differential equation problem that does not have an analytical solution. Instead of attempting to solve the free boundary problem directly, we formulate it as a recursive dynamic programming problem in discrete time. We show that the problem can be easily solved by value function iteration. The solution has an intuitive interpretation and we illustrate its comparative statics properties graphically.

There are two related papers that address the effects of liquidity constraints on optimal exercise of real options. Boyle and Guthrie (2003) analyze the optimal timing of investment when uncertain wealth prior to the investment affects the firm's ability to finance the investment. Our paper, by contrast, focuses on post-investment cash flow uncertainty and its effects on dividends and exit. A special case of our model, where we assume away the liquidity premium, has close resemblance to the problem of

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Jovanovic (1989), Holtz-Eakin, Joulfaian and Rosen (1994), and Zingales (1998).

<sup>3</sup>By contrast, in setups where the wealth increments are i.i.d., the firm's future prospects never change so the problem becomes one-dimensional and yields closed-form solutions; see e.g. Radner and Shepp (1996) or Décamps and Villeneuve (2007).

a financially constrained firm in Mello and Parsons (2000), who analyze the optimal hedging policy for a firm that cannot raise new funds. The firm’s problem includes choosing the optimal exit policy, but there is no incentive to ever pay out dividends.

Our setup is also to some extent related to the models of precautionary saving. The seminal papers on precautionary saving by Zeldes (1989) and Deaton (1991) analyze the problem of optimal lifetime consumption. Under serially correlated income shocks the state space is two-dimensional (savings and expected income) as in our model; the key difference is that consumers do not face an exit decision. For consumers, precautionary saving results from the convexity of marginal utility, whereas in our model it results from the threat of forced exit.

We also analyze the impact of the liquidity constraint on market equilibrium when cash flow uncertainty faced by individual firms is due to idiosyncratic productivity shocks. Our concept of competitive industry equilibrium with entry and exit of firms is essentially that of Hopenhayn (1992). In our setup there is an obvious post-entry overselectivity effect in terms of productivity: some marginally productive firms that should survive a temporary negative cash flow exit due to insufficient funds (or more accurately, as we’ll see, to preempt forced exit). However, the liquidity constraint also creates a price distortion which causes some formerly productive firms with sufficient cash to stay on even when their productivity falls below the socially efficient exit threshold. This is a type of “survival of the fittest” as coined by Zingales (1998). We show that when the entry cost is sufficiently low the liquidity constraint in fact lowers the average productivity of firms in the industry.

In the next section we characterize the problem of the firm, and then in section 3 we solve the firm’s optimal policy under the liquidity constraint and analyze its comparative statics. The implications of the liquidity constraint for a competitive industry are analyzed in section 4.

## 2 The Problem of the Firm

The firm faces a stochastic revenue  $x$  that follows geometric Brownian motion:

$$dx = \mu x dt + \sigma x dw, \tag{1}$$

where  $dw$  is the increment of a standardized Wiener process (i.e., with mean zero and variance  $dt$ ). The firm earns a profit flow  $\pi = x - c$  where the fixed cost  $c$  is a

positive constant. Exit is irreversible and without an additional exit cost. (The entry decision will only show up in industry equilibrium.) The objective is to maximize the expected present value of the income to the owners, discounted at rate  $\rho > \mu$ .

There are two fundamentally different cases. An unconstrained firm can accumulate negative profits indefinitely if needed. The problem of an unconstrained firm is described by the standard real option model of optimal exit. The sole decision is to choose the exit threshold for  $x$ , so there is no meaningful decision for when (if at all) to retain cash or pay dividends. This is the efficient benchmark for our analysis.

A constrained firm has to worry about its ability to cover negative profits, because it is forced to exit if it has no cash while it faces a negative cash flow. The optimal exit policy depends both on revenue  $x$  and cash holdings  $s$ . The firm's cash holdings are augmented by the profit flow and by the interest earned on the cash holdings at rate  $r \leq \rho$ . If  $r < \rho$  then the cash held by the firm is less productive than other assets available to the owners, so the firm faces a meaningful decision of how to pay dividends. The downside of paying dividends is that reduced cash holdings lower the capability to cover any future losses. We start by assuming that the liquidity constraint is very stark in the sense that it is not possible to inject more cash into the firm. We then extend the model to the case where new funds may be raised at some transaction cost; the basic version can be thought of as a special case in which such transaction costs are prohibitive.

## 2.1 Unconstrained Firm

The unconstrained firm will exit if the cash flow becomes too negative. The value function for the unconstrained firm is defined by the familiar differential equation:

$$\rho V = x - c + \mu x V_x + \frac{\sigma^2}{2} x^2 V_{xx} \quad (2)$$

(see e.g. Dixit and Pindyck 1993, Chapter 7) with the constraints that  $V_x$  be continuous ("smooth pasting") and have a finite limit. This ODE has a well-known closed-form solution. The optimal exit threshold is

$$x^* = \frac{\beta}{1 + \beta} \left(1 - \frac{\mu}{\rho}\right) c, \quad (3)$$

where  $\beta = -\frac{1}{2} + \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} > 0$ .

The unconstrained value function is

$$V^*(x) = \begin{cases} \frac{x^*}{\beta(\rho-\mu)} \left(\frac{x^*}{x}\right)^\beta + \frac{x}{\rho-\mu} - \frac{c}{\rho} & \text{for } x \geq x^*, \\ 0 & \text{for } x < x^*. \end{cases} \quad (4)$$

## 2.2 Constrained Firm

The constrained firm has an initial cash balance that is exogenous to the problem. Cash earns interest at rate  $r \leq \rho$ . When the firm is not paying dividends, the cash flow is the sum of the profit flow and the interest income flow

$$\frac{ds}{dt} = x - c + rs. \quad (5)$$

The firm is forced to exit if  $x \leq c$  and  $s = 0$ . If the firm chooses to exit when  $s > 0$ , then the remaining cash is paid out as the liquidation value.

The firm may at any point in time choose from three policy options. First, the firm may exit, which is irreversible, and results in the exit value  $s$ . Second, the firm may continue while paying a positive dividend to the owners. Third, the firm can continue without paying dividend. The solution to the firm's problem is a division of the  $(x, s)$ –space into regions in each of which one of the three policy options is optimal.

Figure 1 shows a schematic view of the optimal policy (we will explain shortly why it must look like this). Given the policy, the life span of the firm is a stochastic path in the  $(x, s)$ –space. The firm mainly ventures inside the *continuation region*, where its law of motion is given by equations (1) and (5). The firm never ventures inside the *dividend region*, because dividend payments move it immediately down along  $s$ -axis to the boundary of that region. When  $x$  is sufficiently high, the dividend region reaches all the way to the  $s = 0$  line, where the firm operates with zero cash holdings and continually pays out all of the profit flow as dividends. The life span ends when the firm hits the boundary of the *exit region* for the first time.

We will now explain why the optimal policy takes the form depicted in Figure 1.

[ Figure 1 here ]

### Continuation Region

The point in accumulating cash is to use it as a buffer that prevents inefficient exit. To see this, consider a situation where the firm's current cash holding  $s$  is small but

strictly positive, and where the profit flow is exactly zero, i.e.  $x = c$ . The firm is not currently making losses and there is a positive option value associated with future profits, so it can not be optimal to exit. Neither can it be optimal to pay out  $s$  as dividends, because this would cause the firm to immediately move down to the point  $(x = c, s = 0)$ , which means that the firm is forced to exit within the "next instant" thus losing the option value. Therefore, there must be a non-empty continuation region, where it is optimal to retain cash inside the firm despite the difference between the discount rate and the rate of return on cash holdings.

Now let's consider the properties of the value function in the continuation region. Define the value of the constrained firm  $V(x, s)$  as gross of the cash holdings, so the value at the time of exit is  $V(x, s) = s$ . Using Ito's lemma, we can write the differential  $dV$  as:<sup>4</sup>

$$dV = V_s ds + V_x dx + \frac{1}{2} V_{xx} (dx)^2. \quad (6)$$

Taking the expectation and letting  $dt$  be small yields:

$$E(dV) = V_s ds + V_x \mu x dt + \frac{1}{2} V_{xx} \sigma^2 x^2 dt,$$

where  $ds$  is from (5). The Bellman equation is  $V(x, s) = E(V + dV) / (1 + \rho dt)$ , which can be solved for  $\rho V dt = E(dV)$ , leading to the following PDE:

$$\rho V = (x - c + rs) V_s + \mu x V_x + \frac{\sigma^2}{2} x^2 V_{xx}. \quad (7)$$

Note that this PDE does not contain a cash flow term. The reason is that in the continuation region, the cash flow between the firm and its owners is zero: Positive cash flow adds to the cash balance and negative flow subtracts from it.

The PDE (7) does not have a closed-form solution. Further, it is valid only in the continuation region, the boundaries of which must be optimally chosen as part of the solution. We will next discuss the properties of these boundaries, which constitute the optimal exit and dividend policies. The numerical solution of the problem is discussed in Section 3.

## Exit Policy

The liquidity constraint can only reduce the continuation value of the firm, so the constrained firm should certainly exit whenever the unconstrained would, i.e., when

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<sup>4</sup>In the Appendix we show how this derivation is done more intuitively, if in a lengthier fashion, using the limiting case of a discrete-time binary process.

$x \leq x^*$ . In addition, the firm is forced to exit when it has no cash to cover the current loss, i.e., when  $(x \leq c, s = 0)$ . This gives a fixed boundary for the value of the firm:

$$V(x, 0) = 0 \text{ for } x \leq c. \quad (8)$$

On the other hand, the firm should clearly never exit while current profits are positive ( $x > c$ ). Now, consider a firm with a very small  $s$  and with  $x$  slightly above  $x^*$ . This firm could in principle continue. However, as  $ds/dt < 0$ , the firm is just about to run out of cash and be forced to exit at the next instant. For sufficiently small  $s$  the firm is so unlikely to bounce back to a positive cash flow before  $s$  hits zero that it is better off exiting immediately and just taking the remaining  $s$ . Thus, there must be a boundary between exit and continuation regions that lies strictly above  $s = 0$  for  $x < c$ . We denote this *exit threshold* by  $\tilde{s}(x)$ , defined in  $x \in [x_{\min}, c]$  where  $x_{\min}$  is, practically defined, the lowest revenue at which the firm ever continues. For  $x < c$ , the lower is  $x$ , the less valuable the continuation value of the firm, and thus the higher the  $s$  required for continuation to be optimal, so  $\tilde{s}'(x) < 0$  in  $x \in (x_{\min}, c)$ . We call exiting when  $x > x^*$  and  $s > 0$  *precautionary exit*.

Inside the continuation region the value of the firm must exceed the exit value  $s$ . At the exit boundary the firm is indifferent between taking the exit value and the continuation value, so

$$V(x, \tilde{s}(x)) = s. \quad (9)$$

Smooth pasting at the exit policy requires

$$V_s(x, \tilde{s}(x)) = 1, \quad (10)$$

$$V_x(x, \tilde{s}(x)) = 0. \quad (11)$$

The only way in which a firm following the optimal policy can extinguish all funds is to hit exactly the zero-flow-zero-stock point for cash,  $\{x, s\} = \{c, 0\}$ . Thus the constrained firm will experience a forced exit with probability 0.<sup>5</sup> All exit by liquidity constrained firms is precautionary.

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<sup>5</sup>The firm's position in  $(x, s)$ -space cannot evolve along the boundaries of the continuation region because, if  $s = 0$  and  $x > c$  then  $ds > 0$ , and if  $x < c$  then the firm exits if it hits the boundary  $\{x, \tilde{s}(x)\}$ .

## Dividend Policy

When  $r < \rho$ , holding cash is costly. The benefit of holding cash is that it may allow the firm to avoid a forced exit in the future when the option value of continuation would still be positive. This benefit is bounded above by  $V^*(c)$ , the unconstrained continuation value at the zero profit flow. Since the cost of holding cash increases without bound in  $s$ , there exists, for any  $x$ , some  $s$  high enough so that it is better to stop accumulating cash. This threshold value, denoted  $\hat{s}(x)$ , defines the boundary between the continuation region and the dividend region. We call it the *dividend threshold*. The value of the firm above the dividend threshold must be:

$$V(x, s) = V(x, \hat{s}(x)) + (s - \hat{s}(x)), \text{ when } s > \hat{s}(x).$$

For sufficiently high  $x$  the possibility of forced exit is so remote that it is not worth holding on to any cash. We denote the threshold above which it is optimal to not hold any cash by  $\hat{x}_{\max}$ . In the limit  $x \rightarrow \infty$ , the prospect of forced exit becomes irrelevant, and thus the value of the firm must converge to the value of the unconstrained:

$$\lim_{x \rightarrow \infty} V(x, s) = V^*(x) + s. \quad (12)$$

At the dividend threshold, cash is equally valuable inside as it is outside the firm, where one dollar is of course worth one dollar. Thus, the value matching condition

$$V_s(x, \hat{s}(x)) = 1 \quad (13)$$

must hold at the dividend threshold. The associated smooth-pasting condition requires

$$V_{ss}(x, \hat{s}(x)) = 0, \quad (14)$$

$$V_{xs}(x, \hat{s}(x)) = 0. \quad (15)$$

The firm is constrained at the margin only in the continuation region; there having a dollar more would increase the value of the firm by more than a dollar:  $V_s(x, s) > 1$ .

When the firm hits the dividend threshold from inside it pays out just enough cash to not cross the boundary. However, if the firm starts at  $s > \hat{s}(x)$ , then it immediately gives the excess  $s - \hat{s}(x)$  as a lump sum dividend. A lump sum dividend is also paid out as the liquidation value upon precautionary exit. Note that if a firm that enters the industry at revenue level  $x = x_0$  then it can choose its initial cash holdings and  $s_0 = \hat{s}(x_0)$  is the optimal choice.

**Special Case:  $r = \rho$** 

Consider now the special case in which there is no liquidity premium:  $r = \rho$ . Hoarding cash is now costless, so it can never be strictly optimal to pay dividends. The optimal policy is thus defined by dividing the  $(x, s)$  – space between the exit region and the continuation region. The qualitative properties of the exit region and the exit threshold  $\tilde{s}(x)$  are the same as with  $r < \rho$ .

Note that holding cash inside the firm can be strictly optimal only when there is a positive probability of being forced to exit in the future. Of course, no matter how high  $x$ , falling below  $x^*$  remains possible. But the firm would become irreversibly unconstrained if it were to accumulate so much cash that it could use the interest income from its cash holdings to cover what would be the worst-case losses under the optimal unconstrained policy. The worst-case loss under the unconstrained policy is  $x^* - c$  forever, so this *escape level of cash* is

$$s^* = -\frac{x^* - c}{r}. \quad (16)$$

This means that a fixed boundary condition

$$V(x, s^*) = V^*(x) + s^* \quad (17)$$

now replaces the free boundary  $\hat{s}(x)$  seen in the  $r < \rho$  case. For  $s \geq s^*$ , the firm is indifferent between paying dividends or not and  $V(x, s) = V^*(x) + s$ . Since the firm is then in effect unconstrained, the exit policy is the same as for an unconstrained firm: exit if and only if  $x \leq x^*$ .

This case without liquidity costs is similar to the setup of a financially constrained firm in Mello and Parsons (2000), save for minor differences.<sup>6</sup> Instead of a boundary condition like (17), they have a limiting condition by which the value of the firm approaches that of the unconstrained firm as the cash balance approaches infinity. We believe that they err by not taking into account that the firm would become unconstrained at a finite level of cash balances.

In most dynamic agency models either expected productivity is constant or saving by the agent is assumed away outright. However, DeMarzo and Sannikov (2008) assume that the agent is able to save and her expected productivity varies stochastically. They show that the principal finds it optimal to impose a liquidity constraint

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<sup>6</sup>They study optimal hedging, namely how firms should use futures contracts on an asset that is correlated with their profits to reduce the risk of inefficient exit.

on the agent that may cause her to exit in states where it is not first-best optimal, which makes the environment ostensibly similar to ours. However, precautionary exit does not arise in their setup because the expected cash flow faced by the agent is assumed to be always positive.<sup>7</sup>

### 2.3 Generalization: New Cash Injections

Next we extend the model by allowing the firm to increase its cash holdings at some transaction cost. Specifically, the firm can at any point in time raise any amount  $s$  of new cash at cost  $\xi + (\gamma + 1)s$ , where  $\xi > 0$  is the fixed and  $\gamma \geq 0$  the marginal transaction cost. In terms of Figure 1, the raising of new capital allows the firm to jump directly upwards in the state space  $(x, s)$ . This could only be optimal when the firm would otherwise face immediate forced exit ( $s = 0$  and  $x < 0$ ) because otherwise the transaction cost can still be postponed and, with luck, even avoided.

If the firm decides to incur the transaction cost, then its optimal target level of capital is

$$s^+(x) = \arg \max_s \{V(x, s) - (1 + \gamma)s\}. \quad (18)$$

The target level  $s^+$  equalizes the marginal cost of new cash and its marginal value at the firm,  $V_s(x, s^+(x)) = 1 + \gamma$ . Since transaction costs are independent of  $x$ , the region where capital is raised is an interval  $\{s = 0, x \in [x_{\min}^+, 0]\}$ , where  $x_{\min}^+ \in (x^*, 0)$ . The lowest  $x$  where the firm replenishes its capital,  $x_{\min}^+$ , is the point where the value of exit (zero on the  $s = 0$  line) is equal to the value of continuing from  $\{x, s^+(x)\}$ , net of the transaction cost of moving there.

$$V(x_{\min}^+, 0) = V(x_{\min}^+, s^+(x_{\min}^+)) - \xi - (1 + \gamma)s^+ = 0. \quad (19)$$

Figure 2 depicts the optimal policy for a firm that faces positive but not prohibitive transaction costs. The qualitative difference to Figure 1 is the capital-raising line and the associated target curve  $s^+(x)$  directly above. Still, for sufficiently low cash flow  $x$  the firm will find it optimal to exit rather than add capital.

[ Figure 2 ]

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<sup>7</sup>Negative realizations of cash flow are possible in their setup in which *cumulative* cash flow follows Brownian motion and expected cash flow reacts to realized cash flows, but they rule out expected losses by assuming that the efficient exit threshold level of profits is positive.

The fixed cost  $\xi$  induces the firm to raise new cash in lumps. The liquidity cost of holding cash makes it desirable to limit the cash holdings, so without a fixed cost firms raise new capital continuously just to offset any negative profit flow. However, a marginal cost  $\gamma > 0$  still reduces the value of continuation and distorts the exit threshold above  $x^*$ .

In the absence of a marginal transaction cost it is optimal to "jump" all the way to the dividend boundary, where  $V_s = 1$  by definition. The unconstrained case is the limiting case where both the fixed cost  $\xi$  and the marginal transaction cost  $\gamma$  of raising capital are zero. The constrained case where the firm will never raise additional capital results when the costs parameters are sufficiently high. This happens when  $(\gamma, \xi)$  are such that

$$\max_s \{V(0, s) - (1 + \gamma)s - \xi\} \leq 0. \quad (20)$$

Hence this setup encompasses both the constrained and unconstrained cases of the basic model.

A literal interpretation of the model is of a risk-neutral owner-entrepreneur who allocates her wealth between two assets; one liquid that can be used to pay off possible losses, and another illiquid asset that yields a higher rate of return but can only be turned into liquid form at a transaction cost. The entrepreneur has deep pockets in terms of the illiquid asset, but the transaction cost makes it desirable to hold some liquid assets as well and in some circumstances rather fold the firm than pay another transaction cost.

### 3 Solving the Firm's Optimal Policy

The PDE defined by (7) and the various free boundary conditions cannot be solved analytically. To solve the firm's problem we turn to a discrete-time approximation of the problem and solve it numerically. In the binomial process approximation of geometric Brownian motion the evolution of  $x$  is governed by

$$x(t + \Delta) = \begin{cases} x(t) e^{\sigma\sqrt{\Delta}} & \text{with probability } q = \frac{1}{2} \left( 1 + \frac{\mu - \frac{\sigma^2}{2}}{\sigma} \sqrt{\Delta} \right) \\ x(t) e^{-\sigma\sqrt{\Delta}} & \text{with probability } 1 - q \end{cases} \quad (21)$$

where  $\Delta$  is the length of the time period.<sup>8</sup> The evolution of the cash balance is now

$$s(t + \Delta) = (s(t) - \delta(t))(1 + r\Delta) + (x(t) - c)\Delta \quad (22)$$

where  $\delta(t) \in [0, \hat{s}(t)]$  is the dividend paid at time  $t$ . The dividend cannot be so high as to make the cash holdings negative at any point in time, so the maximum feasible dividend, restricted by  $\min\{s(t + \Delta), s(t)\} \geq 0$ , is  $\hat{s}(t) \equiv s(t) + \min\{0, (x(t) - c)\Delta / (1 + r\Delta)\}$ .

The value function of the firm, stated in recursive form, is

$$V(x(t), s(t) | t) = \max \left\{ \begin{array}{l} s(t), \\ \max_{\delta \in [0, \hat{s}]} \left\{ \delta + \frac{1}{1 + \rho\Delta} [EV(x(t + \Delta), s(t + \Delta) | t + \Delta)] \right\}, \\ \max_{s^+ \in [s(t), \infty)} \{V(x(t), s^+ | t) - \xi - (1 + \gamma)(s^+ - s(t))\} \end{array} \right\} \quad (23)$$

where  $s(t + \Delta)$  is from (22).

The recursion in (23) satisfies Blackwell's sufficient conditions so  $V(x, s | t)$  is a contraction mapping. Thus it can be solved by iterating backwards in time: Starting from an arbitrary  $V_T(x, s | T)$  the value function converges to the unique solution  $V(x, s)$ .<sup>9</sup>

### Marginal value of cash holdings

In order to understand the nature of the firm's problem, it is useful to digress for a moment and consider the marginal value of cash holdings. Think of the cash holdings as a stockpile of dollars, and the firm as using the last-in-first-out principle in managing this stockpile (i.e., the firm only ever touches the top of the pile). Given that there is a liquidity cost to holding cash it may seem surprising that the firm runs out of cash with probability zero. After all, at every point in time, the bottom dollar incurs the same liquidity cost as any other dollar. Why keep a dollar that is "almost surely" never used? Why not use an otherwise similar policy but with slightly lower cash holdings, thus saving the liquidity cost on the last dollar while allowing the cash

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<sup>8</sup>This way of discretizing geometric Brownian motion was inspired by Cox, Ross and Rubinstein (1979).

<sup>9</sup>A natural starting point for the backward induction is  $V(x, s | T) = s$ . This means that the problem is turned into a finite-horizon problem with forced exit in the last period. By increasing  $T$  the value function at  $t = 0$  converges to that of the infinite horizon problem.

to run out with positive probability? To understand the answer, note that the “last” dollar would only ever be called upon in the vicinity of  $\{c, 0\}$ , i.e., when the flow profit is zero and the cash holdings are down to the last dollar. At this point the marginal contribution of cash to the value of the firm is extremely high. In fact, considering ever smaller  $\varepsilon$ ,  $V_s$  approaches infinity at  $\{c, -\varepsilon/2\}$  because an “epsilon” more of cash would allow the firm to continue, while without cash it is forced to quit and take zero value. Note that by surviving to  $\{c, \varepsilon/2\}$  the firm enters a region where  $ds > 0$  so being able to continue immediately gives a significant chance of drifting away from the danger zone. Figure 3 shows selected derivatives of the value function, including  $V_s$  in the top-left panel<sup>10</sup>. Similarly, the cross-partial  $V_{sx}$  exhibits an extreme reversal from large positive to large negative values near  $\{c, 0\}$ . To the left of the zero-profit zero-cash point, a slight improvement of cash flow leads to an extremely large increase in the marginal value of cash, as it improves substantially the probability that even a very small cash reserve will suffice to save the firm from running out of cash in the immediate future. To the right of that point, the firm is drifting toward safety as  $ds > 0$  so it is suddenly much less likely that the firm would end up needing the “last” dollar and the impact of better cash flow on the marginal value of cash is extremely negative.

[ Figure 3 here ]

### 3.1 Comparative Statics of Optimal Policy

Next we investigate how the firm’s optimal policy depends on the parameters  $r, \mu, \sigma$ . We do this comparison by varying one parameter at a time from a set of baseline parameters,  $r = 0.05, \rho = 0.1, \mu = 0, \sigma = 0.25$ . (Transaction cost parameters  $\gamma$  and  $\xi$  are, for now, assumed to be prohibitively high). The results are depicted in Figure 4. The solid lines mark the borders of the continuation region in the liquidity constrained case, and the dashed line marks the optimal exit threshold in the unconstrained case.<sup>11</sup>

The left hand panel of Figure 4 shows the impact of varying the return on firm’s cash holdings,  $r$ . As  $r$  gets smaller it becomes costlier to hold cash so continuation is

<sup>10</sup>The figure is calculated under the baseline parameters (see Section 3.1), but it remains qualitatively similar as long as  $r < \rho$ .

<sup>11</sup>The program for solving the optimal policy is available at <http://www.hse-econ.fi/murto>.

everywhere less attractive and the continuation region shrinks. The optimal dividend policy is extremely sensitive to  $r$  for values near  $\rho$ . The case  $r = \rho = 0.1$  results in the escape level of cash  $s^* = 3.24$  from (16). This is much higher than the highest cash holdings that the firm would ever keep even at  $r = 0.0999$ .<sup>12</sup> I.e., the optimal policy approaches the limiting case  $r = \rho$  very slowly. This is understandable because the limiting case is qualitatively different. The value  $\hat{x}_{\max}$  above which the firm optimally holds no cash is finite for all  $r < \rho$ , but if  $r = \rho$  then the firm will not stop holding cash no matter how high  $x$ . The dividend boundary  $\hat{s}$  hits the  $x$ -axis at a finite value  $\hat{x}_{\max}$  for all  $r < \rho$  but the limiting value of  $\hat{x}_{\max}$  when  $r \rightarrow \rho$  is infinity and the dividend boundary limits to a horizontal half-line that begins at  $\{x^*, s^*\}$ . The high sensitivity of optimal policy to  $r$  near  $\rho$  means that, even when the liquidity premium is approximately zero, the optimal behavior of firms is not approximated by a model where the liquidity cost is completely assumed away.

The top right panel of Figure 4 shows the relation of the optimal policy and the volatility of the cash flow process. As is typical, higher volatility makes it optimal to accept bigger losses because it increases the upside potential while the downside is still protected by the exit option. In terms of the optimal policy, the increased option value shows up as an enlarged continuation region. This is already visible in the unconstrained problem, where the exit threshold  $x^*$  is decreasing in  $\sigma$ . In the constrained problem, the dividend boundary shifts out to the right because, at any given  $x$ , higher volatility also increases the risk of facing forced exit within any given period of time.

The bottom right panel shows the effect of varying  $\mu$ , the percentage drift of the cash flow process. Higher  $\mu$  increases the option value at any given level of losses, as the firm is more likely to bounce back to positive profits within any given period of time. However, as higher  $\mu$  also makes the firm safer at any given point—by making it less likely that forced exit would threaten it within any given time—it is not obvious that a higher  $\mu$  should also shift out the dividend boundary. However, we have found no examples of the opposite.

[ Figure 4 here ]

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<sup>12</sup>The numerical solution converges extremely slowly when  $r$  is near  $\rho$ . This limits our ability to solve the optimal policy for values of  $r$  closer to but strictly below  $\rho$ .

## 3.2 Comparative Statics with Cash Injections

Next we investigate the impact of transaction costs on optimal policy while holding other parameters constant at the baseline levels. The optimal policy is depicted in Figure 5 under four combinations of the transaction cost parameters  $(\gamma, \xi)$ . The basic model is equivalent to any combination of  $\gamma$  and  $\xi$  where transaction costs are prohibitive in the sense that the firm would never add new capital; the optimal policy under prohibitive costs is depicted by a dotted curve for reference.<sup>13</sup>

The top left panel of Figure 5 shows a benchmark case where both fixed and marginal transaction costs are at intermediate levels. The top right panel considers an increase in fixed cost and the bottom left panel an increase in marginal cost, compared to the benchmark case. In each case the exit boundary is further left than under prohibitive costs, as the threat of forced exit is not as grave with the possibility to raise new capital. The lower the transaction costs, the further the exit boundary shifts towards the unconstrained exit threshold (which is depicted by the vertical dashed line). Similarly, the dividend boundary shifts down when it is cheaper to raise new capital. Intuitively, there is less need to hold cash (and pay the associated liquidity cost) as it can more cheaply be obtained later when necessary.

The bottom right panel shows the optimal policy when both  $\gamma$  and  $\xi$  are very low. There, as it is very cheap to add cash whenever it is necessary, it becomes possible to reduce the liquidity cost and never hold very much cash. At the same time, the exit threshold approaches the efficient threshold. Closer to the limiting case, where both transaction costs approach zero, the continuation region approaches the half-line  $\{x^* \leq x \leq 0, s = 0\}$ . The limiting case looks like the standard textbook case with no cash, except that zero is the strict optimum for cash holdings.

The behavior of the target cash curve  $\{x, s^+(x)\}$  is complicated by the opposing impacts of  $\gamma$  and  $\xi$ . The last acquired dollar of cash must match the marginal transaction cost, so  $V_s = 1 + \gamma$  must hold at the target curve. If  $\gamma = 0$  then the target cash curve coincides with a section of the dividend boundary, where  $V_s = 1$ . Higher  $\gamma$  means that the cash infusion should be smaller; in terms of the graph this means that the target curve is further below the dividend boundary. By contrast, higher  $\xi$  makes

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<sup>13</sup>Hennessy and Whited (2007) estimate that (financial companies excluded) the marginal cost of raising new equity is  $\gamma = 0.053$  for large companies and  $\gamma = 0.12$  for small, and that fixed costs are \$38,900 and \$95,100 respectively.

it attractive to get a bigger infusion of cash, in order to diminish and postpone the prospects of having to incur the fixed cost again. At the same time higher  $\xi$  reduces the continuation value so the interval where cash is raised contracts.

[ Figure 5 here ]

## 4 Industry Equilibrium

We saw in Section 2 how a liquidity constraint causes firms to exit at higher levels of current revenue compared to unconstrained firms. It might therefore seem obvious that, at the level of an entire industry, the liquidity constraint would cause there to be fewer but on average more productive firms. However, as we next show, this partial equilibrium reasoning is incorrect once we take into account the impact that the liquidity constraint has on the levels of revenue in competitive equilibrium.

In order to analyze the impact of the liquidity constraint on a competitive industry, we use the definition of industry equilibrium similar to Hopenhayn (1992).<sup>14</sup> We assume that revenue  $x$  depends on firm-specific output or “productivity”  $z$  and an endogenous industry-specific output price  $p$ , so that  $x = pz$ . Productivity  $z$  follows geometric Brownian motion  $dz = \mu z dt + \sigma z dw$ , with the shocks  $dw$  independent across firms. New firms of known productivity  $z_0$  can be established by paying an entry cost  $\phi$ .<sup>15</sup> In the constrained case new firms enter with initial cash holdings  $s_0$ , which we treat as a parameter of the problem. To guarantee the existence of steady state, we assume an exogenous “death rate”  $\lambda > \mu$  at which firms are forced to exit with their cash holdings as the exit value (see Appendix for details).<sup>16</sup> In steady state, both the dying and the endogenously exiting firms are replaced by new firms of type  $\{z_0, s_0\}$ .

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<sup>14</sup>Financial constraints are introduced to a similar setting by Gomes (2001) to study the relation of cash flow and investment, and by Cooley and Quadrini (2001) to study the age-conditional relation of growth and firm size. Miao (2005) analyzes the impact of a distortionary tax on optimal capital structure with a similar model.

<sup>15</sup>The value  $z_0 > 0$  can be chosen without loss of generality, as it amounts to setting the units of measurement of  $z$ .

<sup>16</sup>The risk of exogenous exit changes the firm’s optimal policy slightly compared to Section 2: the firms discount the future at rate  $\lambda + \rho$  instead of  $\rho$  and the Bellman equation of the constrained firm includes a term  $\lambda s$  on the right hand side of (7).

The industry faces a downward sloping demand curve for its output. Price  $p$  is determined by the entry condition of new firms: It must adjust to eliminate expected rents to entrants. Firms are atomistic, hence there is no aggregate uncertainty,  $p$  is constant, and individual firms in effect just face the revenue process (1). It follows that, for given parameters of the firm's problem, the optimal policy is fixed in the  $(x, s)$  state space. In the unconstrained case this means that the equilibrium price of output  $p^*$  is solved implicitly from  $V^*(p^*z_0) = \phi$ , where  $V^*$  is from (4). For constrained firms the value at entry must cover both the entry cost and the initial cash injection. Equilibrium  $p$  is determined from  $V(pz_0, s_0) = \phi + s_0$ , where  $V$  is obtained numerically as described in the previous section. The constraint reduces welfare so it distorts  $p$  upwards because, due to perfect competition, welfare is purely a matter of consumer surplus.<sup>17</sup> There are three possible channels for the distortion: higher aggregate entry cost (due to higher turnover), lower productivity, and higher liquidity costs.

To understand why the impact of the liquidity constraint on mean productivity is ambiguous, consider, for simplicity, a world where entering firms have no cash holdings ( $s_0 = 0$ ). The position of firms in  $(z, s)$ -space is illustrated in Figure 6. Entry level  $z_0$  is at the point to the right of the zero-profit level ( $z = c/p$ ) where the continuation value matches the entry cost. As price is distorted upwards, the lowest type to ever continue ( $z_{\min}$ ) is below the unconstrained exit threshold ( $z^*$ ), even though the associated revenue level is higher (Recall  $x_{\min} > x^*$  in Figure 1). The price distortion makes it optimal for firms with sufficient cash reserves to continue at productivity levels that would trigger exit in the unconstrained world. The light shaded region (*inefficient survival*) covers firms that would exit in the unconstrained solution but stay in under the liquidity constraint. The dark region (*inefficient exit*) covers firms that are more productive than the unconstrained exit threshold  $z^*$  but exit due to the liquidity constraint. Whether mean productivity is increased or decreased by a liquidity constraint depends on which of these two effects dominates.<sup>18</sup>

[ Figure 6 here ]

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<sup>17</sup>We do not model the shape of the demand curve, as we are not interested in the aggregate level of output, but rather in its distribution across firms. The industry as a whole has constant returns to scale, so demand only matters for the mass of firms.

<sup>18</sup>If  $s_0$  is sufficiently high and  $\phi$  not too high then  $z_0 \in (z^*, c/p)$  and the picture is more complicated, as some of inefficiently exiting firms are replaced by less productive firms.

**Numerical Results** Selected steady state outcomes are reported in Figure 7. Each outcome is reported for those combinations of entry cost  $\phi$  and starting cash  $s_0$  that result in firms entering inside the continuation region. Other parameters are held at the baseline levels used in Section 3.<sup>19</sup> The assumption that transaction costs are prohibitively high is made in order to obtain a clear contrast between the constrained and unconstrained cases: Varying  $\gamma$  and  $\xi$  between zero and prohibitive levels covers the entire ground between the two cases in a continuous manner, as discussed in Section 3.2. Blank regions correspond to  $s_0$  so high that entering firms would be in the dividend region; the outcomes for points in the blank region are thus exactly the same as in the highest colored point directly below. Values of  $\phi$  that are outside the figures result in such high  $p$  that, in terms of Figure 1, the position of entrants is to the right of  $x_{\max}$ .

The top panels show the output price and mean productivity of firms; the middle panels show the same values relative to the unconstrained benchmark. The liquidity constraint is harsher when  $s_0$  is small, so the relative distortion is always decreasing in  $s_0$  as the constraint becomes milder. However, there is a subtle interaction with the entry cost  $\phi$ . If  $\phi$  is small then  $p$  is low and the profit level of entering firms is low or even negative, so newborn firms enter near the exit boundary and immediately face an acute threat of exit. By contrast, when  $\phi$  is high then entrants have a large safety margin in terms of initial revenue making any liquidity constraint less important. The relative impact of the constraint is highest when both  $s_0$  and  $\phi$  are low: the constraint is harsh and the safety margin low. At high values of  $\phi$  the level contours are almost vertical, reflecting the safety margin effect that reduces the impact of the liquidity constraint.

[ Figure 7 here ]

Mean productivity is shown in the top right panels of Figure 7. The liquidity constraint has a negative impact on mean productivity at low levels of  $\phi$ . Thus we find a case of “survival of the fittest” when the entry cost is sufficiently low, with a

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<sup>19</sup>Baseline parameters are  $\mu = 0$ ,  $\sigma = 0.25$ ,  $\rho = 0.1$ ,  $r = 0.05$ ,  $\lambda = 0.1$ ,  $c = 1$ ,  $z_0 = 1$ . The combinations  $\{\gamma, \xi\}$  that result in prohibitive costs can be obtained by solving  $\xi(\gamma)$  implicitly from the equality in (20). For example,  $\gamma = 0.15$ ,  $\xi = 0.25$  results (just barely) in prohibitive costs.

magnitude of up to a 15% decrease in mean productivity. At higher levels of  $\phi$  the impact is positive but eventually the impact of the constraint is attenuated as the safety margin effect becomes overwhelming. Output is increasing in  $s_0$  at low levels of  $\phi$  and decreasing at high levels of  $\phi$ . This means that the output distortion generally gets smaller as the liquidity constraint gets milder.

Average cash holdings are depicted in the bottom-left panel of Figure 7. An increase in initial cash holdings naturally tends to increase the mean cash holdings of all firms in steady state, but, surprisingly, not always. When both  $\phi$  and  $s_0$  are low then an increase in  $s_0$  decreases average cash holdings. This is possible because entering firms have a narrow safety margin. When entrants' profit level is negative then young firms tend to have cash holdings further below  $s_0$ . The decrease in  $p$  caused by higher  $s_0$  further reduces the cash holdings of young firms, which have a high steady state population share precisely because many firms exit soon after entry.

For simplicity, we have treated initial cash holdings  $s_0$  as a parameter, but our setup allows it to be endogenized as the entering firms' optimal response to the transaction cost parameters. The lower-right panel of Figure 7 maps the implicit marginal transaction cost  $\gamma_0$  that would result in the given  $s_0$  being the optimal choice of the entering firms, assuming that entering firms can choose any  $s_0 \geq 0$  at a cost  $(1+\gamma_0)s_0$ , while the cost of raising more cash post-entry is still prohibitive. The dark shaded region covers the points that do not arise endogenously under any  $\{\gamma_0, \phi\}$ .

The cross section of firms in our setup bears a resemblance to that in Gomes (2001), who analyzes industry equilibrium with a model where firms face a mean reverting productivity process and a cost of raising external funds. In his model firms are not able to hold cash, but use an excessive stock of physical capital in effect as a form of precautionary savings, in order to reduce the need for external finance in the future. Gomes shows that the nonlinearity of the optimal investment rule generates a spurious correlation between investment and cash flow, irrespective of whether there are liquidity constraints. In our model cash holdings have a purely precautionary motive while physical capital is fixed (its rental cost is included in  $c$ ). Now suppose that the observed value of capital includes assets that are held for precautionary reasons. It is clear from our results that the contribution of the precautionary motive to the relation of cash flow and accumulation of capital is then necessarily non-monotone. To see this, recall Figure 1. Firms with lowest  $x$  are spending their reserves on covering losses (and thus have  $E[dS|x] < 0$ ), firms with

intermediate  $x$  are on average accumulating cash ( $E[dS|x] > 0$ ), while at  $x > x_{\max}$  no cash is held and  $dS = 0$ .<sup>20</sup> Gomes' point is that the power of a cash flow variable in classic investment regressions arises spuriously when the data is generated in a structural model. Our model implies that, if the capital stock includes assets held for precautionary reasons, then the relation between "investment" and cash flow is nonlinear (indeed non-monotone) even if the relation of physical investment and cash flow were linear (as it is in our model).

## 5 Conclusion

We have analyzed the problem of a liquidity constrained firm that faces a stochastic cash flow. The firm may be forced to exit due to inability to absorb a negative cash flow, even when the possibility to rebound into profits conveys option value that would make continuation (socially) optimal. To prevent such inefficient exit, the firm engages in precautionary saving out of retained earnings, and to pre-empt it the firm will exit before actually running out of cash. The optimal policy includes both an exit policy and a dividend policy, which depend on current cash flow and cash holdings.

The obvious selection effect of pre-entry liquidity constraints is to increase the average productivity of firms in market equilibrium, because the standard for profitable entry is set too high. Similarly, in partial equilibrium, the post-entry liquidity constraint would seem to distort the average productivity upwards, by weeding out firms with upside potential that are facing a negative cash flow. We showed that post-entry liquidity constraints lead also to an opposite phenomenon, where unproductive firms that have a lot of cash (from earlier success) do not exit soon enough and end up reducing the average productivity below the efficient benchmark level. Our steady state calculations showed that the negative effect dominates when entry costs are sufficiently low.

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<sup>20</sup>The same non-monotonicity applies to  $E[dS|V]$  because the contour lines of  $V$  are downward-sloping in  $(x, s)$ -space.

# Appendix: Stationary distributions

## Unconstrained Case

In the unconstrained case, the steady-state firm distribution and its properties reported in Section 4 can be derived analytically as follows. Denote  $y \equiv \log z$ . The exit threshold is  $y^* = \log z^*$  and new firms are born at  $y_0 > y^*$ . Taking a discrete time approximation,  $y$  follows the binomial process:

$$y(t + \Delta) = \begin{cases} y(t) + \Delta y & \text{with probability } q \\ y(t) - \Delta y & \text{with probability } 1 - q \end{cases}$$

where  $\Delta$  is the length of a period,  $q = \frac{1}{2} \left( 1 + \frac{\mu - \sigma^2/2}{\sigma} \sqrt{\Delta} \right)$ , and  $\Delta y = \sigma \sqrt{\Delta}$ . The steady state condition gives a difference equation for the mass of firms located at an arbitrary state point  $y$ ,

$$(1 - \lambda \Delta) [q f(y - \Delta y) + (1 - q) f(y + \Delta y)] \Delta y + g(y) \Delta y = f(y) \Delta y,$$

where  $f(y) \Delta y$  is the mass of all firms and  $g(y) \Delta y$  is the mass of newborn firms at state point  $y$ . Taking the limit  $\Delta \rightarrow 0$  leads to a differential equation for the stationary firm density:<sup>21</sup>

$$\frac{1}{2} \sigma^2 f''(y) - (\mu - (1/2) \sigma^2) f'(y) - \lambda f(y) + g(y) = 0, \quad (24)$$

with  $f(y^*) = 0$  and  $\lim_{y \rightarrow \infty} f(y) = 0$  as boundary conditions. In our setup  $g(y)$  is positive at  $y_0$  and zero elsewhere. The point  $y_0$  splices the differential equation into two regions, with the  $f(y_0) = f_0$  as a boundary condition in the middle. ( $f$  is finite but not differentiable at  $y_0$ ). The value of  $f_0$  can be solved from the condition that total probability density integrates to one. Combining the boundary conditions with (24) yields the closed-form solution:

$$f(y) = \begin{cases} 0 & y \leq y^* \\ f_0 e^{-\frac{(\gamma+\eta)(y-y_0)}{2\sigma^2}} \frac{\left( \frac{\eta y}{e^{\sigma^2/2}} - e^{\frac{\eta y^*}{\sigma^2}} \right)}{\left( \frac{\eta y_0}{e^{\sigma^2/2}} - e^{\frac{\eta y^*}{\sigma^2}} \right)} & y^* < y \leq y_0 \\ f_0 e^{-\frac{(\gamma+\eta)(y-y_0)}{2\sigma^2}} & y_0 < y \end{cases} \quad (25)$$

<sup>21</sup>See Dixit and Pindyck (1993), chapter 8, section 4.c for more details.

where  $\gamma \equiv \sigma^2 - 2\mu$ ,  $\eta = \sqrt{8\lambda\sigma^2 + \gamma^2}$ , and

$$f_0 = \frac{2\lambda}{\eta} \frac{\left( e^{\frac{\eta y}{\sigma^2}} - e^{\frac{\eta y^*}{\sigma^2}} \right)}{\left( e^{\frac{\eta y_0}{\sigma^2}} - e^{-\frac{(\eta-\gamma)y^* + (\eta+\gamma)y_0}{2\sigma^2}} \right)}. \quad (26)$$

There is no economically sensible steady state unless  $z = e^y$  has a finite mean. Here  $\int_{y_0}^{\infty} e^y f(y) dy < \infty$  is a necessary and a sufficient condition for the finite mean. Taking out the terms that are independent of  $y$  in (25), the finite mean requirement becomes

$$\int_{y_0}^{\infty} e^{y - \frac{(\gamma+\eta)y}{2\sigma^2}} dy < \infty. \quad (27)$$

This holds if  $2\sigma^2 - \gamma - \eta < 0$ , which simplifies to  $\lambda > \mu$ .

### Constrained Case

The stationarity proof in the unconstrained case is sufficient for the stationarity of the distribution of  $z$  in the constrained process. As  $s$  is endogenously bounded by the optimal dividend policy and, firm by firm, depends deterministically on the history of  $z$ , the fact that  $z$  has a stationary distribution suffices for the stationarity of the joint distribution  $(z, s)$ . However, now the optimal policy has no closed-form solution so the steady state distribution must be computed numerically. In the discrete time approximation the life span of each individual firm is a Markov chain in the discretized state space. Therefore, the steady state distribution is obtained in a straightforward manner by first computing the optimal policy of an individual firm, and then, starting from some initial firm distribution, iterating the firm distribution according to the state transition equations associated with the policy (where a constant mass of new firms are established at the birth point within each iteration) until the firm distribution converges to the steady state. Mean output can then be readily computed.

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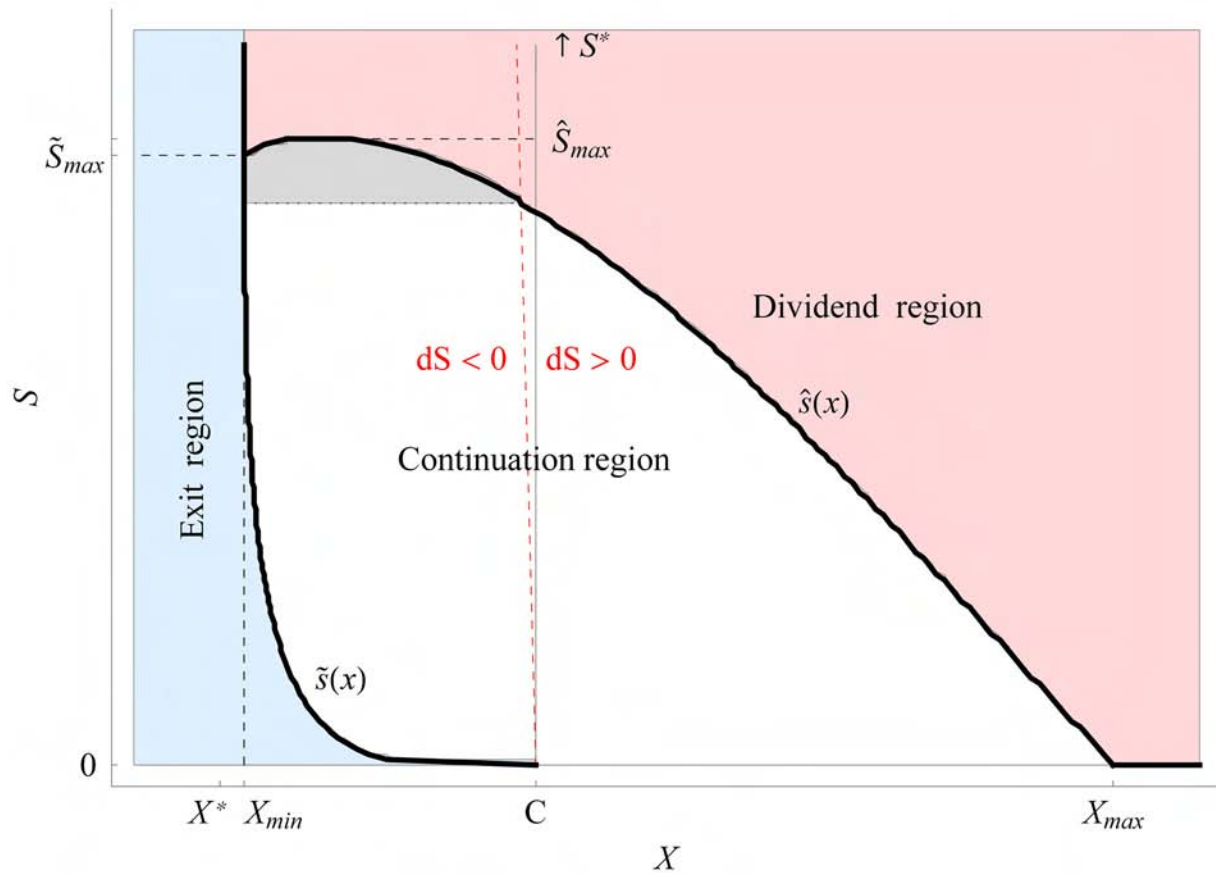


Figure 1. Optimal policy regions of a liquidity constrained firm.

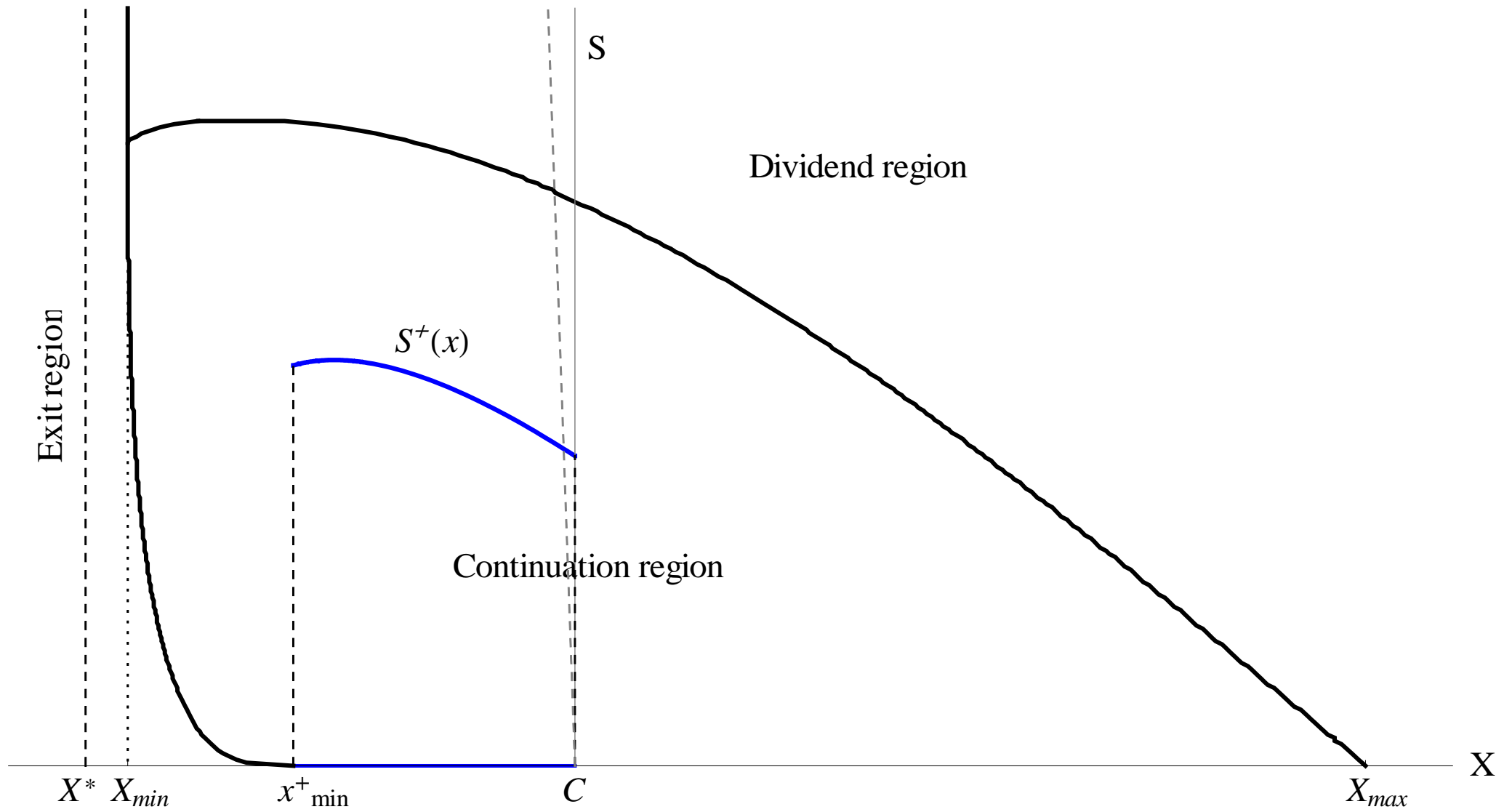


Figure 2. Optimal policy in the general model, where it is possible to raise new capital.

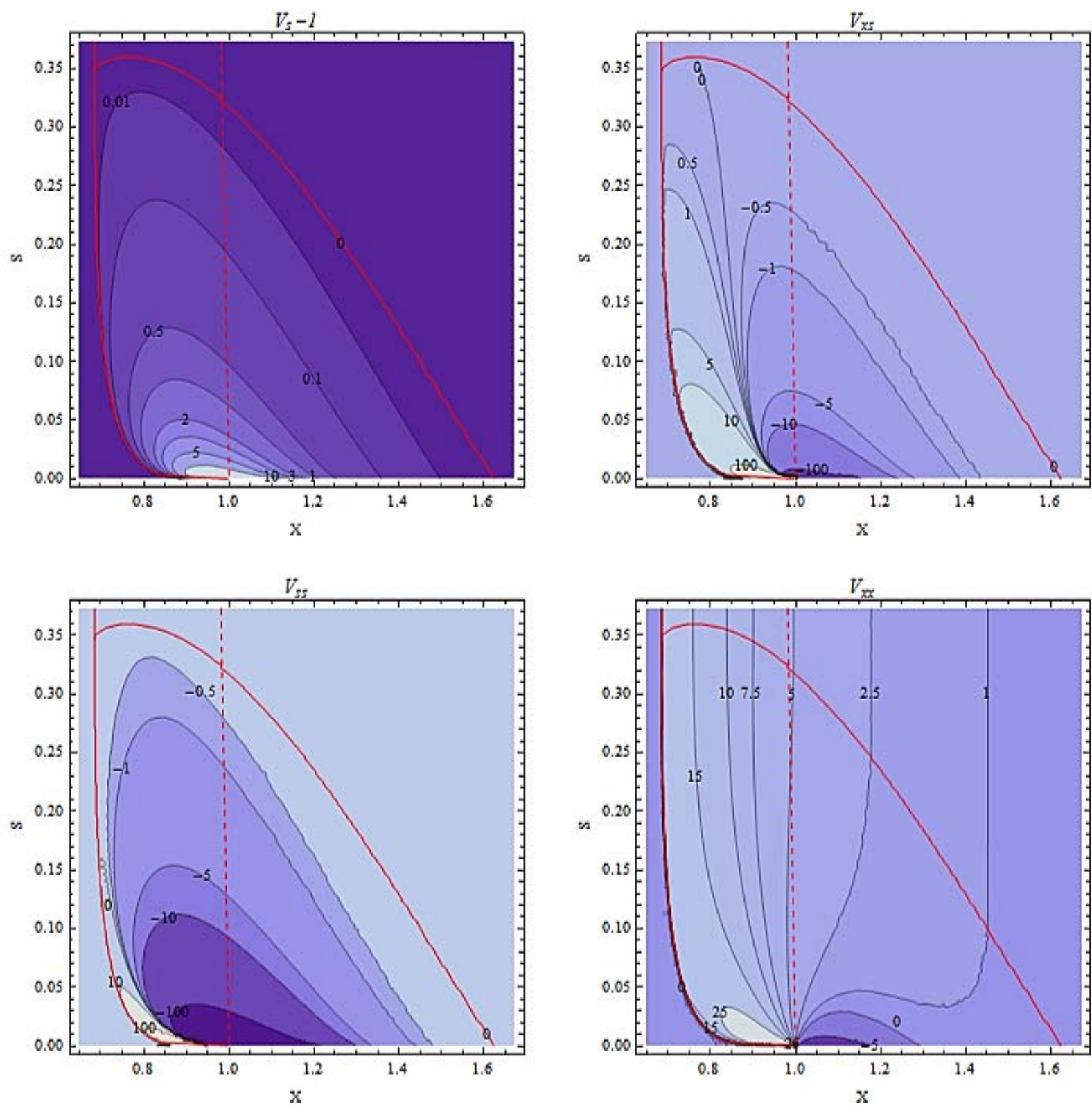
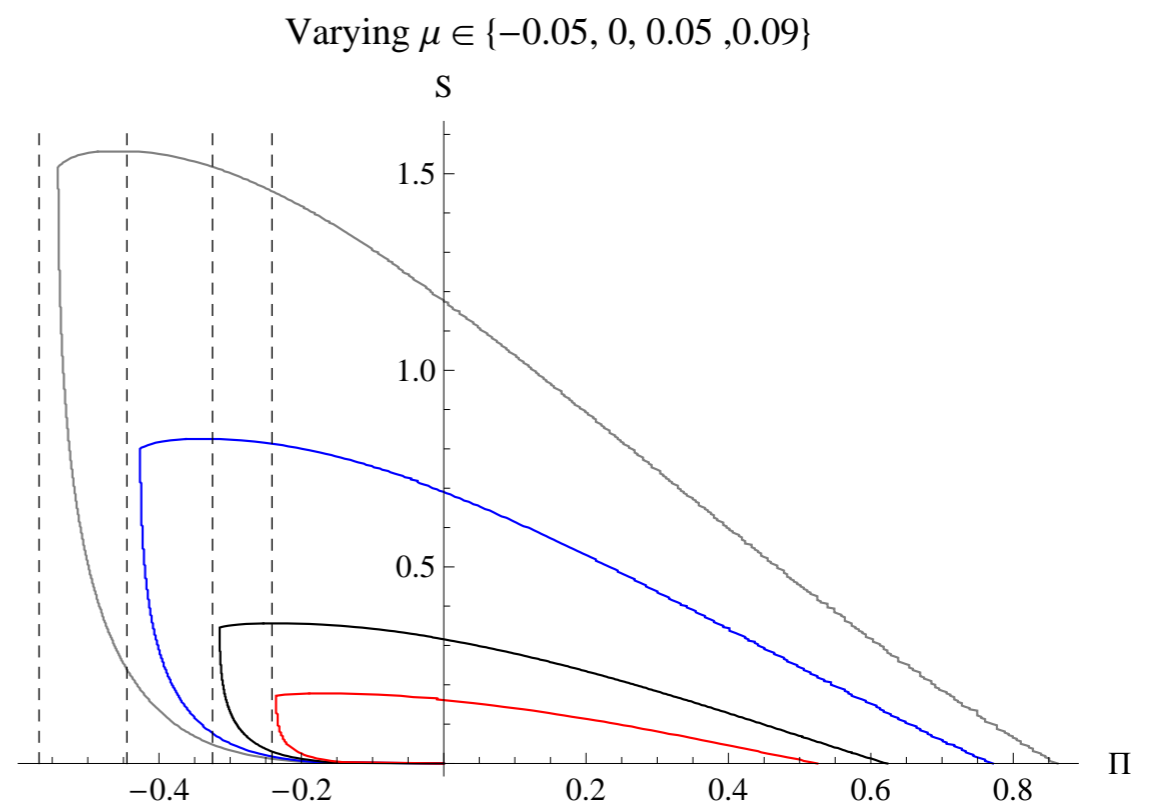
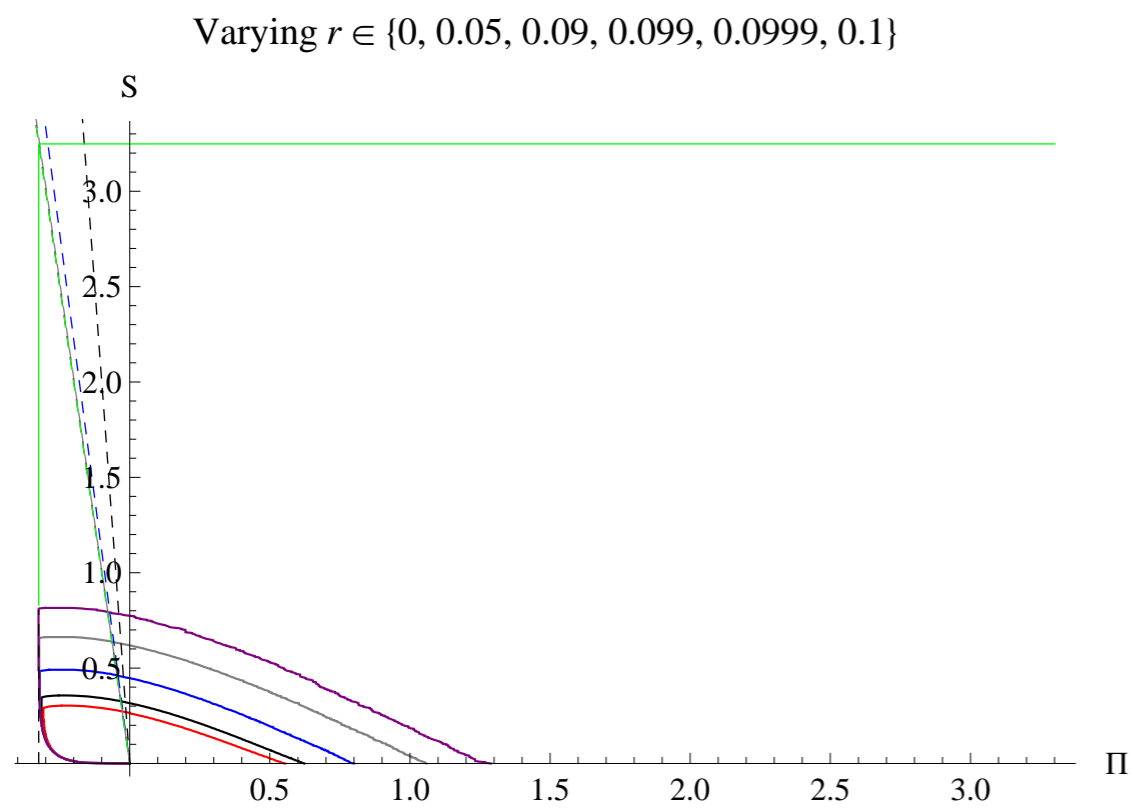
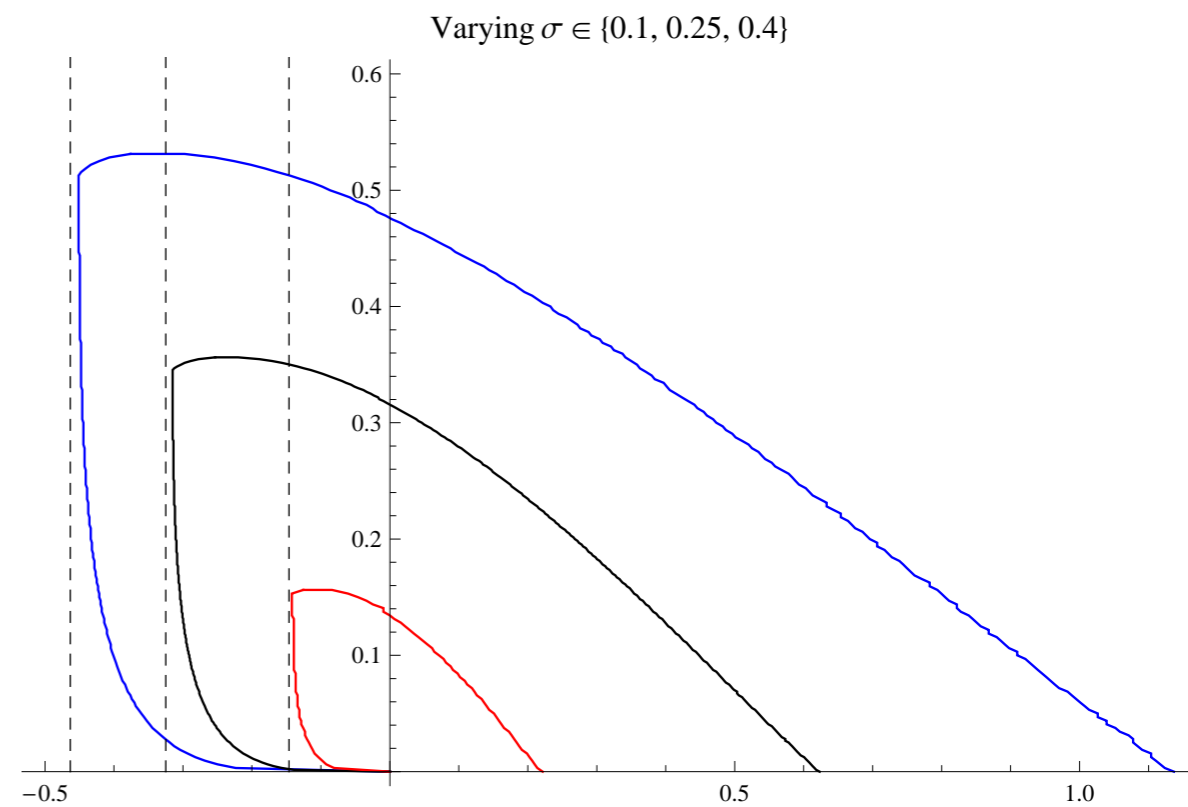
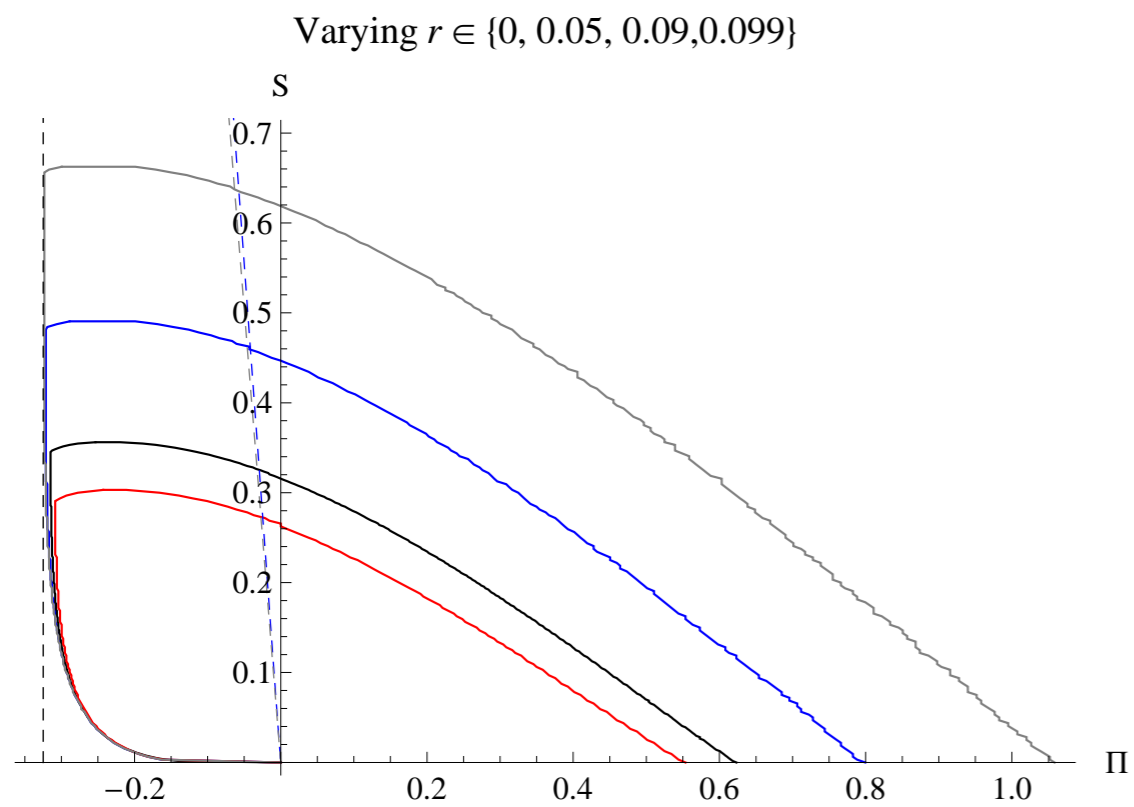
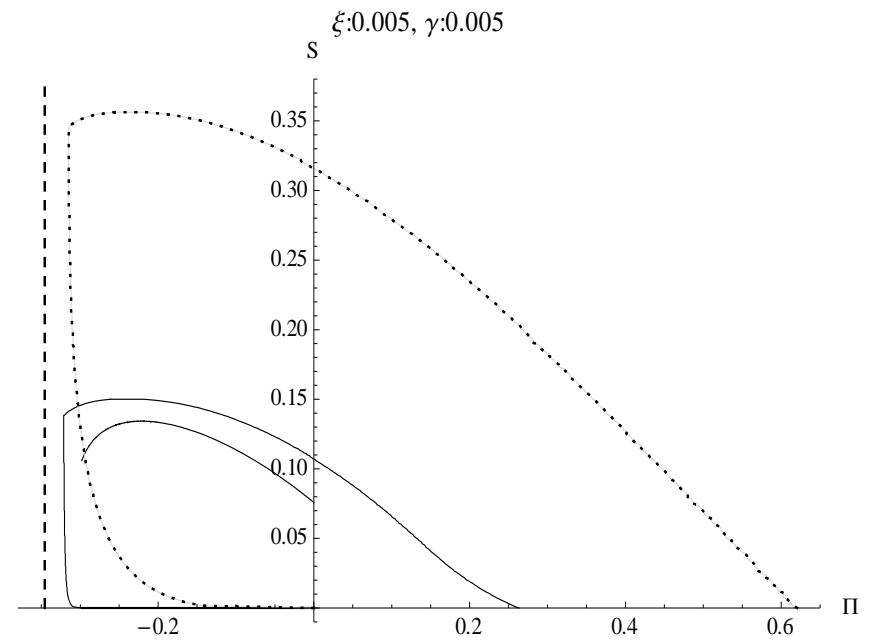
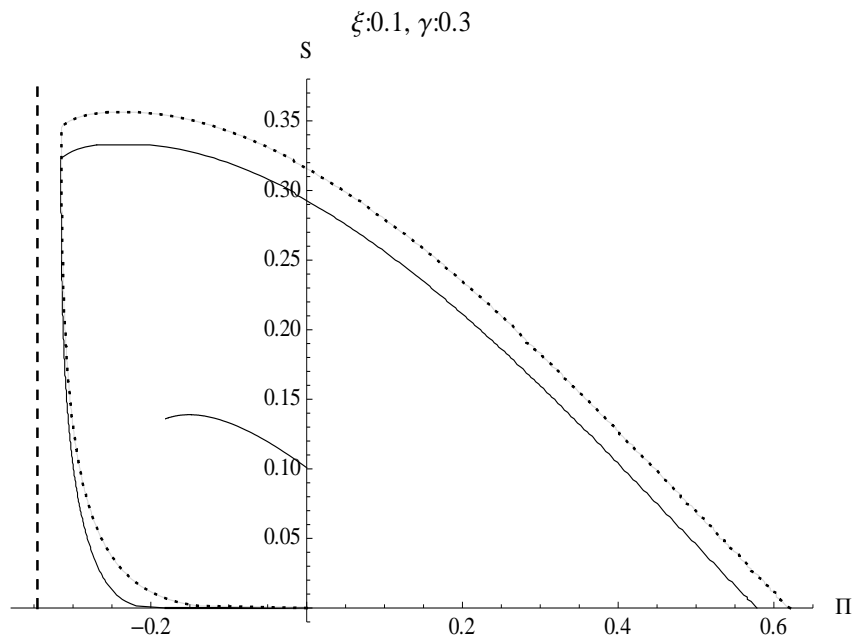
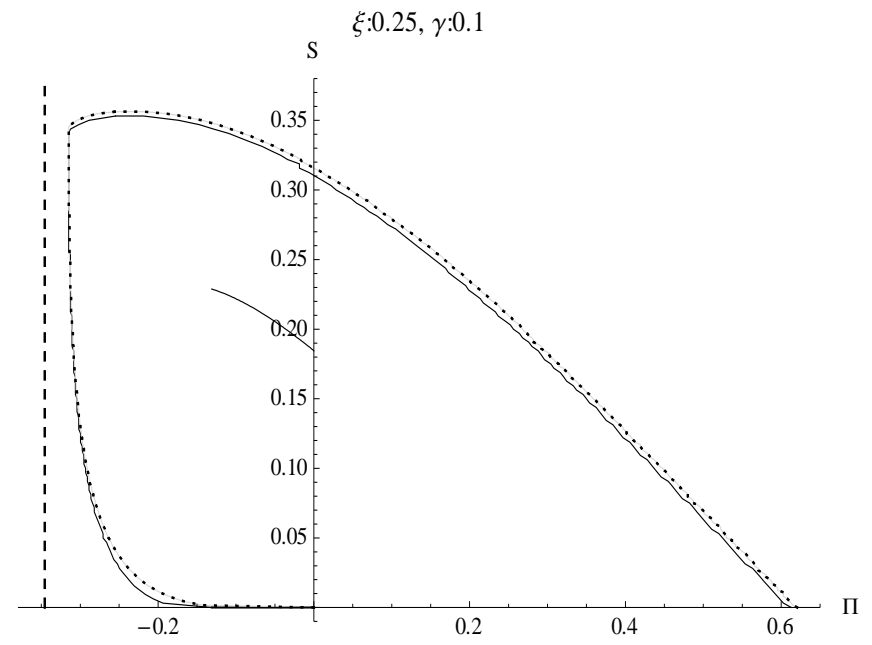
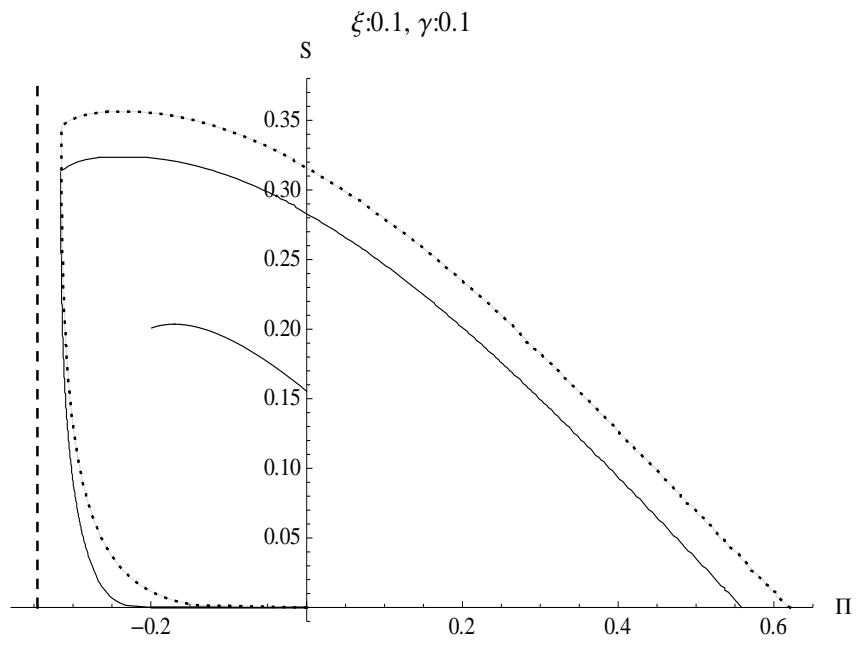


Figure 3. Partial derivatives of the value function.

Figure 4. Optimal policy of a liquidity constrained firm: Comparative statics





**Figure 5.** Comparative statics of the optimal policy with respect to fixed ( $\xi$ ) and marginal ( $\gamma$ ) cost of raising new capital.

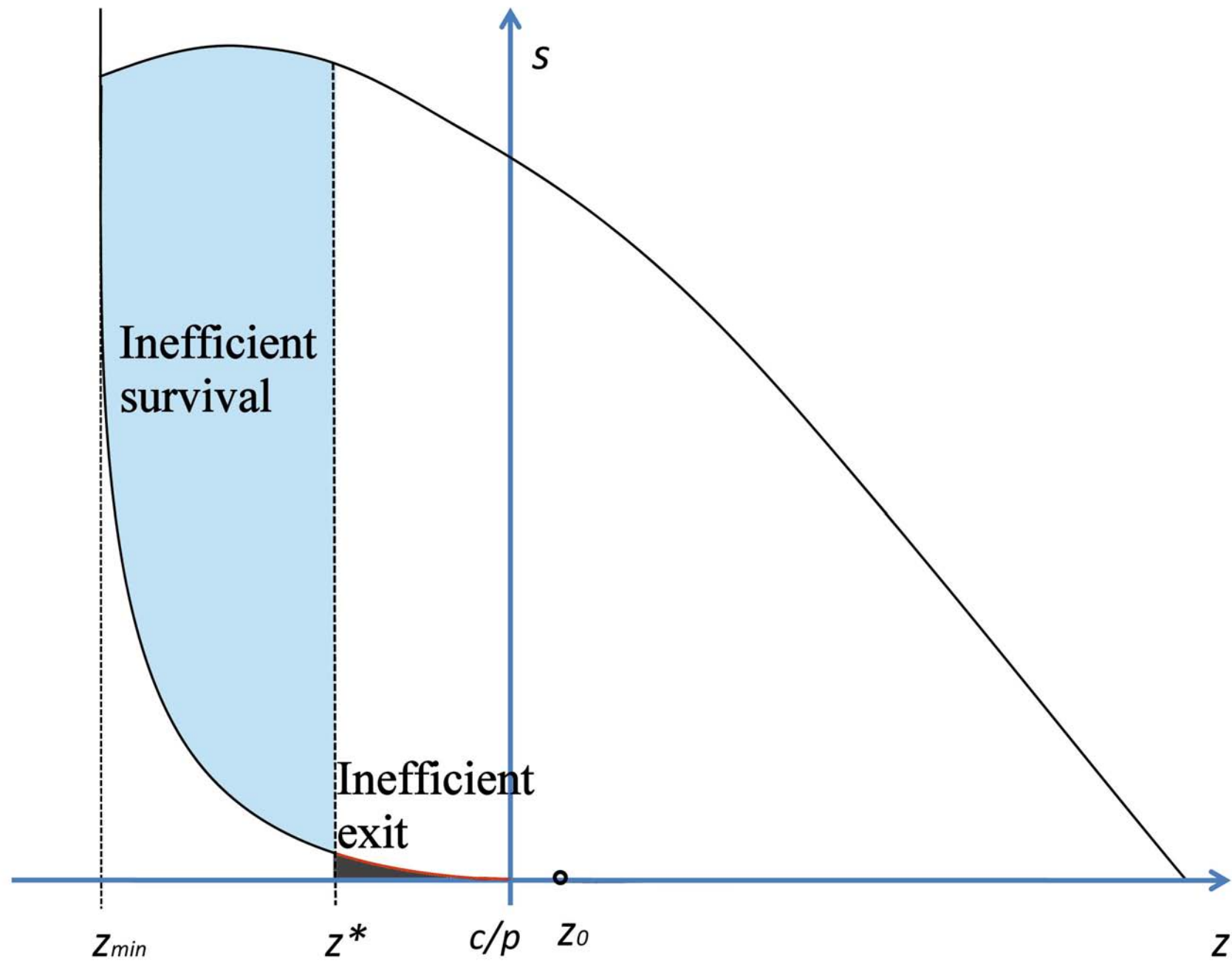


Figure 6. Liquidity Constraint and Average Productivity

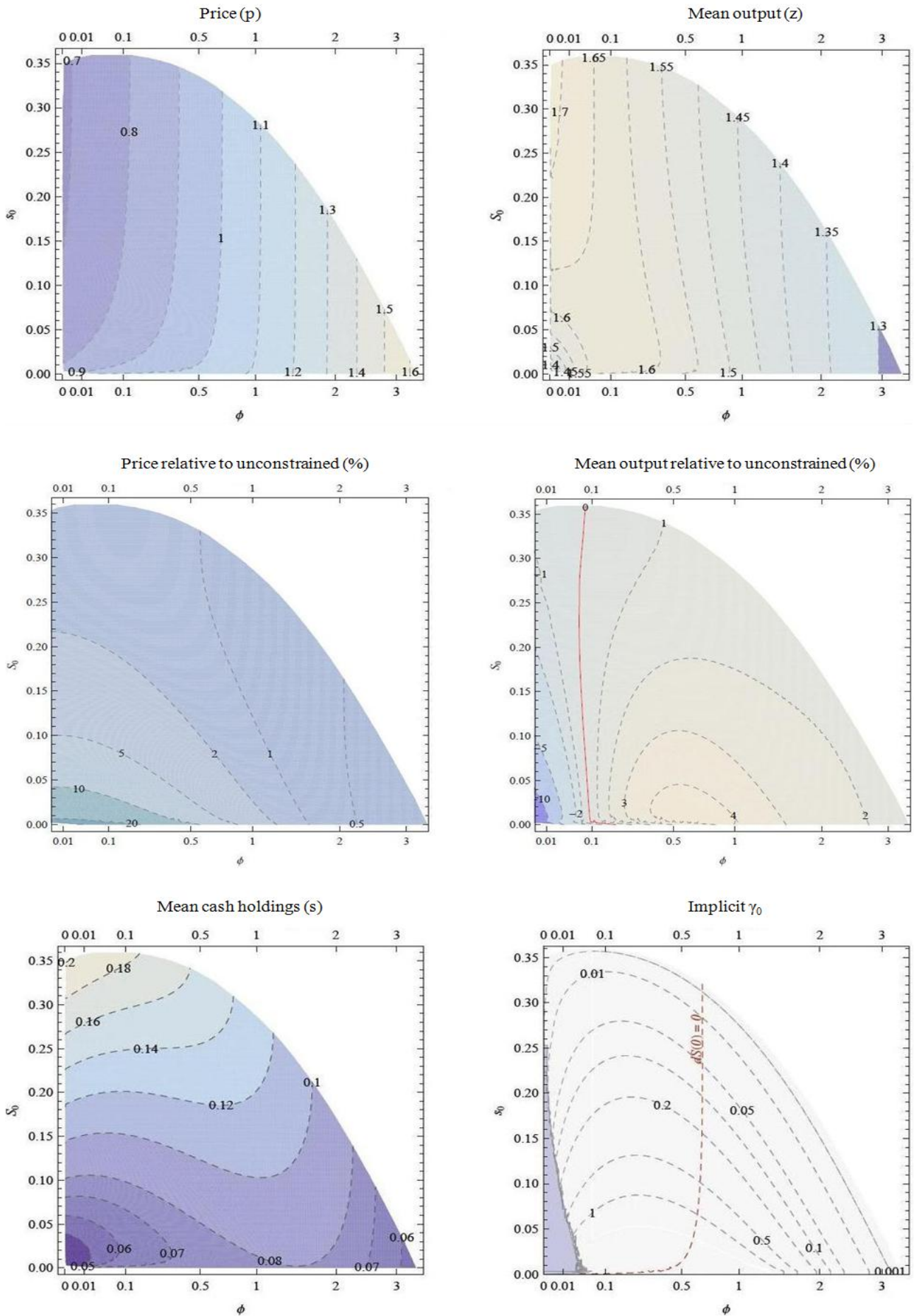


Figure 7. Impact of liquidity constraint on industry equilibrium.