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Progressive Taxation and Irreversible Investment Under Uncertainty

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Abstract

We analyze the impact of progressive taxation on irreversible investment under uncertainty. We show that if tax exemption is lower than sunk cost, higher tax rate will decelerate optimal investment by increasing the optimal investment threshold, while if tax exemption exceeds the sunk cost, tree different regimes arise. For "small" volatilities the optimal investment threshold is a positive function of volatility, but independent of tax rate. For "medium" volatilities it is independent of both tax rate and volatility. Finally, for "high" volatilities the optimal investment threshold depends positively on volatility, but negatively on tax rate so that we have "tax paradox"

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1 Introduction

We start from a realistic notion that most major investments are at least partially irreversible due to the fact that firms cannot disinvest without costs after having carried out their investment decision. This is because physical capital is not only industry-specific, but also firm-specific so that it may not be very usable for a different firm in the same industry. Even in the absence of firm-specific investments, partial irreversibility will quite likely be true due to the "lemons" problem meaning that the resale value is usually below the purchase cost. In the seminal book by Dixit and Pindyck (1994) various approaches and applications are excellently reviewed and extended (see also complementary surveys in terms of further research by Bertola (1998) and Caballero (1999)). In what follows we focus on an important issue of how taxation will affect investment behavior in the framework of irreversible investments under uncertainty. First we summarize briefly what has been done in this literature and after that we present a new research question concerning the potential role of progressive taxation in terms of optimal investing threshold, which we elaborate in this paper.

There are several recent studies in the framework of irreversible investment under uncertainty where justifications for the neutral tax system have been analyzed using real option theory. Niemann (1999) has shown that in the presence of uncertainty and irreversible investments the neutrality of both the cash flow tax and the Johansson-Samuelson tax system hold by extending the depreciation base. For further discussion about these tax schemes, see e.g. chapter 5 in Sinn (1987). Pennings (2000) has studied the impact of a subsidy to investment with a taxation of future profits on an irreversible investment and has shown that such a combination by raising a zero expected revenue will decrease the threshold value of investment, so that the expected investment goes up. Panteghini (2004) compares an ACE (Allowance for Corporate Equity) system with a CBIT (Comprehensive Business Income Tax) system in an open economy using a real option approach and suggests that preference for an ACE system is a realistic result.

One should also ask: what happens if tax rates are uncertain? Niemann (2004) demonstrates that both under risk neutrality and risk aversion higher tax rate uncertainty has an ambiguous investment effect depending among others on depreciation deductions. Moreover, he shows that the neutrality results under perfect foresight for the cash flow tax and the Johansson-Samuelson tax will also hold under tax rate uncertainty independent of whether investors are risk neutral or risk averse (see also Niemann and Sureth (2004)). Sureth (2002) investigates the impacts of taxes using a contingent claims analysis instead of a dynamic programming approach and shows that uncertainty and complete irreversibility do not violate the neutrality property of a Johansson-Samuelson tax. He obtains the same finding as Neumann (2004) that the neutrality of a cash flow and a Johansson-Samuelson tax system holds also in the case of risk averse investor behavior. Lund (1992) has also applied the contingent claims analysis to evaluate the impact of petroleum taxation under uncertainty on companies' behavior. His main focus is in a numerical approach.

What are the effects of corporate tax asymmetries on irreversible investments under a tax scheme, where tax base is given by the firm's return, net of an imputation rate? This issue has been studied in Panteghini (2001a), (2001b) and (2002) under various investment strategies. He has demonstrated that this asymmetric scheme may also be neutral under both income and capital uncertainty. Neutrality is an implication of Bernanke's (1983) Bad News Principle, according to which irreversible decisions are affected only by unfavorable events. Under the tax system proposed, the corporate tax is levied in the good states so that tax asymmetries exploit the asymmetric effect of uncertainty to guarantee neutrality.

Alvarez et. al (1998) have analyzed a more general issue in a dynamic stochastic adjustment model of firm behavior. In particular, they asked: what are the anticipatory effects of a corporate tax reform when the firms are realistically uncertain both about the timing and contents of the expected reform either in terms of tax cuts or in terms of tax base reductions? They show among others that future tax cut expectation causes the firms to accelerate optimal investment, while expected reduction in the tax base will have an opposite effect. In several OECD countries in the 1980s and 1990s a tax-cut plus base-broadening tax reform has been implemented and the authors show among others that under plausible assumptions this type of reform cannot be revenue-neutral. In Alvarez et. al (2000) it is shown that a corporate tax policy in a model with tax advantage to debt and expectations about a forthcoming tax reform may have significant incentive effects. In particular, under the assumptions made a tax cut plus base-broadening tax reform will cause a big short run investment spurt.

Hassett and Metcalf (1999) have studied the impact of tax policy uncertainty, associated for potential changes in investment tax credits, both on firm level and aggregate investment. Under geometric Brownian motion of value process higher uncertainty slows down investment despite the implicit subsidy arising from the variations in tax credit, but when tax policy is modelled as a stationary jump process higher tax policy uncertainty can have the opposite effect. In both model specifications higher tax policy uncertainty will imply a loss of tax revenues to the government (see also Metcalf and Hassett (1995)). Agliardi (2001) has assumed the possibility of investment scrapping so that investment can be considered as partially irreversible, and studied the impacts of tax policy including a corporate cash flow tax and a subsidy to asset values also in the case of tax policy uncertainty. He models the tax policy process in a way alternative to Hassett and Metcalf (1994) by assuming that the price of capital follows a different stochastic process. Naturally, he concludes that it may discourage investment and encourage the earlier shutdown of projects. Brennan and Schwartz (1986) have studied the case of partially reversible investment for the decision to open or close a mine.

To conclude, in the existing recent literature, where corporation taxation issues have been studied in the irreversible investment framework under uncertainty by using real option theory, taxation has been assumed to be proportional. This means that the marginal tax rate and the average tax rate are constant and therefore equal. But in practice this is not an appropriate assumption even though the marginal tax rate would be constant if there are tax exemptions meaning that taxes have to be paid only after some exemption threshold. In this case taxation is not proportional, but progressive. Therefore, a practically important issue is to ask: what are the implications of progressive taxation in terms of investment behavior under uncertainty? Tax progression means either that the marginal tax rate is constant but, due to tax exemption, the average tax rate increases with the tax base. Another definition of tax progression is that in terms the tax base the marginal tax rate increases (see e.g. the seminal paper by Musgrave and Thin (1948) and a textbook analysis in Lambert (2001)).

The purpose of our paper is to analyze the following important issue, which to our best knowledge has not been previously studied in the literature: what is the impact of tax progression - defined as a higher average tax rate in terms of tax base - on irreversible investment under uncertainty? We provide several new and interesting findings about this practically realistic modelling of taxation: Under progressive taxation with positive tax rate and tax exemption, we demonstrate how the impact of the tax rate on the optimal investment threshold depends on the relative size between the tax exemption and sunk cost of investment. More precisely, if tax exemption threshold is below the sunk cost of investment, then higher tax rate will increase the optimal investment threshold and decrease the value of investment opportunity by decreasing the net-of-tax payoff. The negative relationship under these assumptions is natural; higher tax rate raises the size of tax deduction, and decreases the marginal revenue from investment project. Since the latter effect dominates whenever the tax exemption threshold is below the sunk cost of investment, we find that higher taxation slows down rational investments in that case.

However, when the tax exemption threshold exceeds the sunk cost of irreversible investment, then depending on the relationship between volatility and other parameters of the problem, there are three different regimes in terms of optimal investment threshold. First, for a set of sufficiently low volatilities, increased volatility decelerates investment by increasing the harvesting threshold, but the tax rate does not affect the optimal policy. Second, as volatility becomes larger, the optimal harvesting threshold coincides with the tax exemption threshold, and therefore becomes independent of both volatility and tax rate. Third, for a set of sufficiently high volatilities, the optimal investment threshold depends again positively on volatility, but interestingly, negatively on tax rate. Hence, under this latter condition there is "tax paradox", according to which higher tax rate accelerates rational investment by increasing the current investment incentives! It is worth noticing that these observations results also from the fact that in our framework government works as a risk-sharer via tax exemption.

We proceed as follows: In section 2 we present a framework to study the impact of progressive taxation under irreversible investment with stochastic value process and characterize new theoretical results. Section 3 illustrates our findings explicitly through numerical calculations. Finally, there is a concluding section.

2 Tax Exemption and Irreversible Investment

In this section we characterize the optimal irreversible investment problem under stochastic value process with progressive taxation, i.e. when both the tax rate and the tax exemption are positive so that the average tax rate increases with the tax base, ceteris paribus. More precisely, we proceed as follows: First, we specify the underlying value dynamics. Second, we demonstrate how the impact of the tax rate on the optimal investment threshold depends on the relative size between the tax exemption and the sunk cost of investment. Third, we state a set of weak conditions under which higher volatility will increase both the value and the exercise threshold of the optimal policy. Finally, we also illustrate the significance of progressive taxation as a risk-sharing device.

As usually, we assume that the random dynamics of the underlying value process are described by the Itô stochastic differential equation

$$dX_t = \mu(X_t)dt + \eta\sigma(X_t)dW_t, \quad X_0 = x \tag{2.1}$$

where both the drift $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are assumed to be sufficiently smooth (at least continuous) mappings for guaranteeing the existence of a solution for (2.1) and $\eta \geq 0$ is a known non-negative multiplier. The underlying value dynamics becomes deterministic as the volatility multiplier vanishes, that is, as $\eta \downarrow 0$. Moreover, the assumed positivity of the volatility coefficient $\sigma(x)$ implies that an increase in the value of the multiplier η can be interpreted as an increase in the overall volatility of the underlying value dynamics. In line with standard models applied in economics, we assume that the upper boundary ∞ is natural for X_t. Thus, although the underlying value may be expected to increase, it is never expected to become infinitely high in finite time. We also assume that the lower boundary is either natural, exit, or regular. In case it is regular, we assume then it is killing and, therefore, that the underlying value process ceases to exist whenever it hits the lower boundary of its state-space. As usually, we denote the differential operator associated to the controlled diffusion as

$$\mathcal{A}_\eta = \frac{1}{2}\eta^2\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

We also denote as $\psi_\eta(x)$ the increasing and as $\varphi_\eta(x)$ the decreasing fundamental solutions of the ordinary linear second order differential equation $(\mathcal{A}_\eta u)(x) = ru(x)$ (cf. Borodin and Salminen (2002), pp. 17–19). Moreover, the constant Wronskian of these solutions is denoted as $B = (\psi_\eta'(x)\varphi_\eta(x) - \varphi_\eta'(x)\psi_\eta(x))/S_\eta'(x)$, where

$$S_\eta'(x) = \exp\left(-\int \frac{2\mu(x)dx}{\eta^2\sigma^2(x)}\right)$$

denotes the density of the scale function of the underlying diffusion.

Having characterized the underlying stochastic value dynamics, we now plan to consider the optimal timing problem of an irreversible investment opportunity in the presence of both value uncertainty and progressive taxation with positive tax rate and tax exemption. More precisely, we now plan to consider the optimal timing problem

$$V_\eta(x) = \sup_{\tau<\tau_0} \mathbf{E}_x\left[e^{-r\tau}\max(\pi(X_\tau),0)\right], \tag{2.2}$$

where τ is an arbitrary stopping time, $\tau_0 = \inf\{t \geq 0 : X_t \leq 0\} \leq \infty$ denotes the potentially infinite date at which the underlying value dynamics vanish (and can, therefore, be interpreted as a liquidation date), and

$$\pi(x) = x - c - t(x-\bar{x})^+,$$

where $\bar{x} \in \mathbb{R}_+$ is a known exogenously given exemption threshold satisfying the condition $t\bar{x} < c$. Our first results, characterizing the optimal investment rule in the case where exercising the investment opportunity at states below the exemption threshold $\bar{x}$ is suboptimal, are now summarized in

Lemma 2.1. *Assume that $\bar{x} \leq c$ and that for all $t \in [0,1)$ there is a unique $\hat{x}_t$ such that*

$$(1-t)(\mu(x) - r(x-c)) \gtreqqless rt(\bar{x}-c), \quad x \lesseqqgtr \hat{x}_t. \tag{2.3}$$

Then, the value of the optimal policy reads as

$$V_\eta(x) = \psi_\eta(x)\sup_{y\geq x}\left[\frac{(1-t)y-(c-t\bar{x})}{\psi_\eta(y)}\right] = \begin{cases}(1-t)x-(c-t\bar{x}) & x \geq x_t^*(\eta)\\ (1-t)\frac{\psi_\eta(x)}{\psi_\eta'(x_t^*(\eta))} & x < x_t^*(\eta)\end{cases} \tag{2.4}$$

where the optimal investment threshold $x_t^*(\eta) \in ((c-t\bar{x})/(1-t),\infty)$ is the unique root of the ordinary first order condition $(1-t)[\psi_\eta(x_t^*(\eta)) - \psi_\eta'(x_t^*(\eta))(x_t^*(\eta)-c)] = \psi_\eta'(x_t^*(\eta))t(\bar{x}-c)$. Moreover, the optimal investment threshold is an increasing function and the value of the investment opportunity is a decreasing function of the tax rate t.

Proof. See Appendix A. □

Lemma 2.1 states a set of weak conditions under which a unique investment threshold at which the irreversible investment opportunity should be exercised exists and is above the Marshallian trigger $(c-t\bar{x})/(1-t)$ at which the net present value of the project becomes positive. According to Lemma 2.1 increased tax rate decelerates investment and decreases the value of the investment opportunity whenever the tax exemption threshold is below the sunk cost of investment. The reason for this observation is due to the fact that although higher tax rate increases the size of the tax deduction $t\bar{x}$ it simultaneously decreases the after tax net revenues $(1-t)x$ of the firm. Since the latter effect dominates whenever the tax exemption threshold is below the sunk cost of investment we find that the overall impact of increased taxation on the optimal investment policy and its value is negative.

The optimal investment rule in the case where $\bar{x} > c$ is now summarized in

Theorem 2.2. *Assume that $c \in (t\bar{x}, \bar{x})$ and that for all $t \in [0,1)$ there is a unique $\hat{x}_t$ such that inequality (2.3) is satisfied. Then, the value of the optimal policy reads as*

$$V_\eta(x) = \psi_\eta(x) \sup_{y \geq x} \left[\frac{y - c - t(y-\bar{x})^+}{\psi_\eta(y)} \right]. \tag{2.5}$$

(A) If $\bar{x} \geq x_0^*(\eta) > x_t^*(\eta)$ then the investment opportunity is exercised at the investment threshold $x_0^*(\eta)$ and the value of the optimal policy reads as

$$V_\eta(x) = \begin{cases} x - c - t(x-\bar{x})^+ & x \geq x_0^*(\eta) \\ (x_0^*(\eta) - c)\frac{\psi_\eta(x)}{\psi_\eta(x_0^*(\eta))} & x < x_0^*(\eta), \end{cases} \tag{2.6}$$

where the optimal investment threshold $x_0^*(\eta) > c$ is the unique root of the first order condition $\psi_\eta(x_0^*(\eta)) = \psi'_\eta(x_0^*(\eta))(x_0^*(\eta) - c)$.
(B) If $x_0^*(\eta) > \bar{x} > x_t^*(\eta)$ then the investment opportunity is exercised at the exemption threshold $\bar{x}$ and the value of the optimal policy reads as

$$V_\eta(x) = \begin{cases} (1-t)(x-c) + t(\bar{x}-c) & x \geq \bar{x} \\ (\bar{x} - c)\frac{\psi_\eta(x)}{\psi_\eta(\bar{x})} & x < \bar{x} \end{cases} \tag{2.7}$$

(C) If $x_0^*(\eta) > x_t^*(\eta) \geq \bar{x}$ then the investment opportunity is exercised at the investment threshold $x_t^*(\eta)$ and the value of the optimal policy reads as

$$V_\eta(x) = \begin{cases} (1-t)(x-c) + t(\bar{x}-c) & x \geq x_t^*(\eta) \\ ((1-t)(x_t^*(\eta) - c) + t(\bar{x}-c))\frac{\psi_\eta(x)}{\psi_\eta(x_t^*(\eta))} & x < x_t^*(\eta), \end{cases} \tag{2.8}$$

where the optimal investment threshold $x_t^*(\eta) > (c - t\bar{x})/(1-t)$ is the unique root of the first order condition $(1-t)[\psi_\eta(x_t^*(\eta)) - \psi'_\eta(x_t^*(\eta))(x_t^*(\eta) - c)] = \psi'_\eta(x_t^*(\eta))t(\bar{x} - c)$. Moreover, increased taxation accelerates investment dy decreasing the optimal exercise threshold. That is, $dx_t^*(\eta)/dt < 0$.

Proof. See Appendix B. □

Theorem 2.2 demonstrates that if the tax exemption threshold $\bar{x}$ is greater than the sunk cost c then there are three different cases which may arise depending on the relative sizes of the parameters of the problem. Interestingly, we find that under the conditions of Theorem 2.2 a higher tax rate has an positive impact on rational investment. This accelerating effect of increased taxation on the optimal investment policy is based on its negative effect on the after tax costs of investment. More precisely, it is now clear that for all $x \geq \bar{x}$ the after tax net investment costs are $c - t\bar{x}$. This, in turn, means that the investor can deduct $t\bar{x}$ from the tax base. Since this deduction is an increasing function of the tax rate, we find that although a higher tax rate decreases the profitability of an irreversible investment project, it simultaneously decreases the net investment costs. Since the latter effect dominates the former when $t\bar{x} < c < \bar{x}$, we find that the net effect of increased taxation on optimal investment is unambiguously positive. It is also worth noticing that Theorem 2.2 also demonstrates that among the potential regimes there are two interesting special cases resulting into non-standard optimal investment behavior. First, in both regimes (A) and (B), the optimal investment rule is independent of the tax rate and, therefore, on those regimes marginal changes in the tax policy do not affect investment behavior. Second, in regime (B) the investment opportunity is exercised at the exemption threshold which is a constant. Consequently, the optimal investment rule characterized in regime (B) is also independent of volatility. This observation is important since it characterizes the significance of the exceptional incentive effects of tax exemption on optimal investment in two ways (cost deduction and risk-sharing).

Our main results on the comparative static properties of the optimal policy and its value are now summarized in the following.

Theorem 2.3. *Assume that the net appreciation rate $\mu(x) - rx$ is non-increasing and that $\mu(0) \leq 0$ whenever 0 is attainable for the underlying value process. Assume also that for all $t \in [0, 1)$ there is a unique $\hat{x}_t$ such that inequality (2.3) is satisfied. Then, increased volatility increases the value of the investment opportunity and decelerates rational exercise by increasing or leaving unchanged the optimal investment threshold at which the opportunity should be exercised. More precisely, if $\hat{\eta} \geq \eta$ then $x_t^*(\hat{\eta}) \geq x_t^*(\eta)$ and $V_{\hat{\eta}}(x) \geq V_\eta(x)$ for all $t \in [0, 1)$.*

Proof. See Appendix C. □

Theorem 2.3 states a set of weak conditions under which increased volatility unambiguously increases both the value and the exercise threshold of the optimal policy. This observation is of interest since as Theorem 2.2 clearly indicates, increased volatility may result into a transition from the regime (A) towards regime (B) and then further into regime (C). Moreover, since increased volatility does not affect the optimal investment threshold in regime (B), we observe that the set where investment is volatility-independent may be relatively large. In order to characterize such a situation explicitly, we now prove the following key result illustrating the significance of progressive taxation as a risk-sharing device.

Theorem 2.4. *Assume that the conditions of Theorem 2.3 are satisfied, that $t\bar{x} < c < \bar{x}$, that $\mu(\bar{x}) < r(\bar{x} - c)$, and that $\lim_{\eta\to\infty} \psi_\eta(\bar{x})/\psi_\eta'(\bar{x}) > (\bar{x} - c)/(1 - t)$. Then there are two critical volatility multipliers $\eta_1^* < \eta_2^*$ satisfying the conditions*

$$\psi_{\eta_1^*}(\bar{x}) = \psi_{\eta_1^*}'(\bar{x})(\bar{x} - c) \tag{2.9}$$

and

$$(1 - t)\psi_{\eta_2^*}(\bar{x}) = \psi_{\eta_2^*}'(\bar{x})(\bar{x} - c). \tag{2.10}$$

Moreover, if $\eta \leq \eta_1^*$ then the conditions of part (A) of Theorem 2.2 are satisfied and the value reads as in (2.6), if $\eta_1^* < \eta < \eta_2^*$ then the conditions of part (B) of Theorem 2.2 are satisfied and the value reads as in (2.7), and if $\eta \geq \eta_2^*$ then the conditions of part (C) of Theorem 2.2 are satisfied and the value reads as in (2.8).

Proof. See Appendix D. □

Essentially Theorem 2.4 states a set of conditions under which the the optimal threshold can be characterized as a non-decreasing function of the underlying volatility according to the definition

$$x^* = \begin{cases} x_t^*(\eta) & \eta \geq \eta_2^* \\ \bar{x} & \eta_1^* < \eta < \eta_2^* \\ x_0^*(\eta) & \eta \leq \eta_1^* \end{cases}$$

This result is important since it is valid for a broad class of processes modelling the underlying stochastic revenue dynamics. Especially, it is worth emphasizing that the condition that $\mu(\bar{x}) < r(\bar{x} - c)$ is sufficient for the existence of a critical volatility multiplier η_1^*. Thus, the transition from from regime (A) to regime (B) as volatility increases is always guaranteed as long as the exemption threshold dominates the optimal investment threshold in the absence of uncertainty and taxation. Hence, a regime where investment is independent of both volatility and taxation always exists as long as the exemption threshold dominates the optimal investment threshold in the absence of uncertainty and taxation. If $\mu(\bar{x}) \geq r(\bar{x} - c)$ then a critical volatility multiplier η_1^* satisfying (2.9) does not exist and in that case the only potential regimes are either (B) or (C).

3 Explicit Illustration

After having characterized our new theoretical results concerning the relationship between the tax rate and the investment threshold as well as the relationship between the investment threshold and the volatility of value process, we now illustrate our results explicitly within a frequently applied setting. More specifically, we show how the relationship between the investment threshold, tax rate and volatility depends on the relative size of the tax exemption and the sunk cost. In particular, under progressive taxation when tax exemption exceeds sunk cost of investment there is a non-empty set of volatilities where the optimal investing threshold does not depend on the volatility of value process at all.

Let us assume that the underlying value dynamics evolve according to an ordinary geometric Brownian motion characterized by the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x,$$

where $\mu \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$ are exogenously given constants. It is now a standard exercise to demonstrate that in this case the fundamental solutions read as $\psi_\sigma(x) = x^{\alpha_\sigma}$ and $\varphi_\sigma(x) = x^{\beta_\sigma}$, where

$$\alpha_\sigma = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the positive and

$$\beta_\sigma = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

denotes the negative root of the quadratic characteristic equation $\sigma^2 a(a-1) - 2\mu a - 2r = 0$. Given these observations we find that our Lemma 2.1 can be re-expressed in this particular example as

Lemma 3.1. *Assume that $\bar{x} \leq c$ and that the absence of speculative bubbles condition $r > \mu$, guaranteeing the finiteness of the value of the optimal policy, is satisfied. Then, the value of the* *optimal policy reads as*

$$V_\sigma(x) = \begin{cases} (1-t)x - (c - t\bar{x}) & x > x_t^*(\sigma) \\ \frac{(1-t)}{\alpha_\sigma} x^{\alpha_\sigma} x_t^*(\sigma)^{1-\alpha_\sigma} & x < x_t^*(\sigma) \end{cases} \tag{3.1}$$

where

$$x_t^*(\sigma) = \frac{\alpha_\sigma (c - t\bar{x})}{(\alpha_\sigma - 1)(1-t)} = \left(1 - \frac{1}{\beta_\sigma}\right) \frac{r(c - t\bar{x})}{(r-\mu)(1-t)}$$

is the optimal investment threshold at which the irreversible investment opportunity should be exercised. The optimal investment threshold satisfies the inequalities

$$\begin{aligned} \frac{\partial x_t^*(\sigma)}{\partial t} &= \frac{\alpha_\sigma (c - \bar{x})}{(\alpha_\sigma - 1)(1-t)^2} \geq 0 \\ \frac{\partial x_t^*(\sigma)}{\partial \sigma} &= \frac{2\alpha_\sigma (c - t\bar{x})}{(\alpha_\sigma - 1)(1-t)\sigma(\alpha_\sigma - \beta_\sigma)} > 0. \end{aligned}$$

Thus, in the present example both increased tax rate and increased volatility raises the optimal investment threshold and, therefore, postpone the rational exercise of the investment opportunity. Especially, if $\bar{x} < c$ then along the iso-incentive curve where investment incentives remain unchanged the following holds

$$\left.\frac{dt}{d\sigma}\right|_{dx_t^*(\sigma)=0} = -\frac{2(1-t)(c - t\bar{x})}{\sigma(c - \bar{x})(\alpha_\sigma - \beta_\sigma)} < 0.$$

An important implication of Lemma 3.1 is that

$$x_t^*(\sigma) > \frac{r(c - t\bar{x})}{(r-\mu)(1-t)} > \frac{(c - t\bar{x})}{(1-t)}.$$

This means that the optimal investment threshold dominates the certainty trigger characterizing the optimal policy in the absence of volatility. This trigger, in turn, dominates the Marshallian threshold at which the net present value of the project becomes positive.

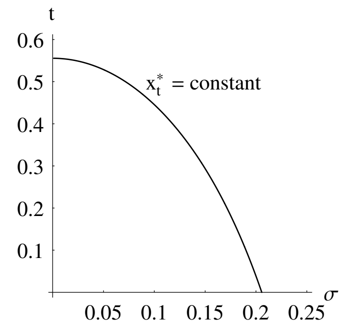

Figure 1: The iso-incentive curve

In order to characterize the case where the tax exemption rule is beneficial for the investment opportunity (i.e., when $t\bar{x} < c < \bar{x}$) we first observe that if $r > \mu$ then in the absence of taxation the optimal investment threshold satisfies the conditions

$$\lim_{\sigma \downarrow 0} x_0^*(\sigma) = \frac{rc}{(r-\mu)}$$

and

$$\frac{\partial x_0^*(\sigma)}{\partial \sigma} = \frac{2\alpha_\sigma c}{(\alpha_\sigma - 1)\sigma(\alpha_\sigma - \beta_\sigma)} > 0.$$

Thus, we find that the results of our main Theorem 2.2 can now be expressed as follows.

Theorem 3.2. *Assume that $t\bar{x} < c < \bar{x}$, $r > \mu$, and $rc < (r-\mu)\bar{x}$. Then there are two critical volatility coefficients $\sigma_1^* < \sigma_2^*$ satisfying the conditions*

$$\bar{x} = \alpha_{\sigma_1^*}(\bar{x} - c) \tag{3.2}$$

and

$$(1-t)\bar{x} = \alpha_{\sigma_2^*}(\bar{x} - c). \tag{3.3}$$

(A) If $\sigma \leq \sigma_1^*$ then the investment opportunity is exercised at the investment threshold $x_0^*(\sigma)$ and the value of the optimal policy reads as

$$V_\sigma(x) = \begin{cases} x - c - t(x-\bar{x})^+ & x > x_0^*(\sigma) \\ (x_0^*(\sigma) - c)(x/x_0^*(\sigma))^{\alpha_\sigma} & x < x_0^*(\sigma). \end{cases} \tag{3.4}$$

(B) If $\sigma_1^* < \sigma < \sigma_2^*$ then the investment opportunity is exercised at the exemption threshold $\bar{x}$ and the value of the optimal policy reads as

$$V_\sigma(x) = \begin{cases} (1-t)(x-c) + t(\bar{x} - c) & x > \bar{x} \\ (\bar{x} - c)(x/\bar{x})^{\alpha_\sigma} & x < \bar{x}. \end{cases} \tag{3.5}$$

(C) If $\sigma \geq \sigma_2^*$ then the investment opportunity is exercised at the investment threshold $x_t^*(\sigma)$ and the value of the optimal policy reads as

$$V_\sigma(x) = \begin{cases} (1-t)(x-c) + t(\bar{x} - c) & x > x_t^*(\sigma) \\ ((1-t)(x_t^*(\sigma) - c) + t(\bar{x} - c))(x/x_t^*(\sigma))^{\alpha_\sigma} & x < x_t^*(\sigma) \end{cases} \tag{3.6}$$

Theorem 3.2 states a set of conditions under which the optimal investment rule is determined by the volatility of the underlying value process. Interestingly, as we already argued in the general analysis of the optimal investment problem, we find that if the conditions of Theorem 3.2 are satisfied, then there is a non-empty set of volatilities (the set (σ_1^*, σ_2^*)) where the optimal investment rule is completely independent of the volatility of the value process. A second important consequence of Theorem 3.2 is that depending on the volatility of the underlying value dynamics, the optimal investment rule may be independent of the tax rate (on the set $(0, \sigma_2^*)$). Consequently, as was already indicated by our Theorem 2.4, there is a regime where the optimal investment policy is independent of both the tax rate and the volatility of the underlying value process. Under such circumstances neither the tax rate nor the volatility will have no effect on the investment threshold even while they do affect the value of the investment opportunity. These observations are explicitly illustrated in Figure 2. Finally, as was already found in our Theorem 2.4, for sufficiently high volatilities an increased tax rate will accelerate rational investment by decreasing the optimal investment threshold while increased volatility will have the opposite effect by decreasing the investment incentives. This means that under such circumstances the tax authorities may weaken the negative impact of

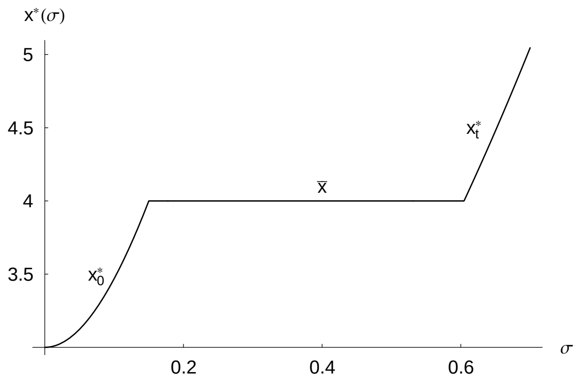

Figure 2: The optimal exercise threshold as a function of the volatility coefficient σ

volatility on rational investment by raising the tax rate. Hence our model emphasizes the significance of an active tax policy as a stabilizing mechanism in the presence of progressive taxation, irreversibility, and uncertainty (cf. Dixit and Pindyck (1994), p. 14).

It is, however, worth noticing that if $rc > (r-\mu)\bar{x}$ then equation (3.2) does not have an interior root and, therefore, in that case the only possible regimes are either (B) or (C). Finally, since

$$\lim_{\sigma \downarrow 0} x_t^*(\sigma) = \frac{r(c - t\bar{x})}{(r-\mu)(1-t)} > \frac{rc}{(r-\mu)} = \lim_{\sigma \downarrow 0} x_0^*(\sigma)$$

we observe that if $(1-t)\mu\bar{x} > r(\bar{x} - c)$ then neither equation (3.2) nor equation (3.3) can have an interior root and, therefore, in that case the only possible regime is (C).

4 Conclusions

In this paper we have analyzed the following issue: what is the impact of tax progression - defined as a higher average tax rate in terms of tax base - on irreversible investment under uncertainty? We have demonstrated several new and interesting findings about this practically realistic modelling of taxation. Under progressive taxation with tax rate and tax exemption, we have shown how the impact of the tax rate on the optimal investment threshold depends on the relative size between the tax exemption and the sunk cost of investment. If the tax exemption threshold is below the sunk cost of investment, then higher tax rate will increase the optimal investment threshold and decrease the value of investment opportunity by decreasing the net-of-tax payoff. The negative relationship under these assumptions is natural; higher tax rate raises the size of tax deduction, and decreases the marginal revenue from investment project. Since the latter effect dominates whenever the tax exemption threshold is below the sunk cost of investment, we find that a higher tax rate decelerates optimal investments.

But when the tax exemption threshold exceeds the sunk cost of irreversible investment, then depending on the relationship between volatility and other parameters of the problem, there are three different regimes in terms of optimal investment threshold. First, for a set of sufficiently low volatilities, increased volatility decelerates investment by increasing the harvesting threshold, but tax rate does not affect the optimal policy. Second, as volatility becomes larger, the optimal harvesting threshold coincides with the tax exemption, and therefore becomes independent of both volatility and tax rate. Third, for a set of sufficiently high volatilities, the optimal investment threshold depends again positively on volatility, but interestingly, negatively on tax rate. Under this latter condition there is ”tax paradox”, according to which higher tax rate accelerates rational investment by increasing the current investment incentives! It is worth noticing that these observations results also from the fact that in our framework government works as a risk-sharer via tax exemption.

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Appendix

A Proof of Lemma 2.1

Proof. The proof of (2.3) and (2.4) is analogous with the proof of Lemma 2.1 in Alvarez and Koskela (2004). It is now clear that the first order optimality condition characterizing the optimal investment threshold can be expressed as

$$\frac{\psi_\eta(x_t^*(\eta))}{\psi_\eta'(x_t^*(\eta))} - (x_t^*(\eta) - c) = \frac{t}{1-t}(\bar{x} - c).$$

Implicit differentiation now yields

$$\frac{dx_t^*(\eta)}{dt} = \frac{{\psi_\eta'}^2(x_t^*(\eta))(c-\bar{x})}{(1-t)^2\psi_\eta(x_t^*(\eta))\psi_\eta''(x_t^*(\eta))} \geq 0 \tag{A.1}$$

by the inequality $c \geq \bar{x}$ and the local convexity of the increasing fundamental solution $\psi_\eta(x)$ at the optimal boundary. Finally, since the exercise payoff $x - c - t(x-\bar{x})^+$ is a decreasing mapping of the tax rate t on the set $(\bar{x}, \infty)$ we find that increased taxation decreases the value of the investment opportunity. □

B Proof of Theorem 2.2

Proof. In order to prove the results of our theorem, we first consider the mapping

$$F_t(x) = (1-t)\left[\frac{\psi_\eta(x)}{S_\eta'(x)} - \frac{\psi_\eta'(x)}{S_\eta'(x)}(x-c)\right] - \frac{\psi_\eta'(x)}{S_\eta'(x)}t(\bar{x}-c) = \frac{\psi_\eta^2(x)}{S_\eta'(x)}\frac{d}{dx}\left[\frac{(1-t)x-(c-t\bar{x})}{\psi_\eta(x)}\right].$$

It is now clear from this definition that

$$F_t(x) = (1-t)F_0(x) - \frac{\psi_\eta'(x)}{S_\eta'(x)}t(\bar{x}-c)$$

implying that

$$F_t'(x) = (1-t)F_0'(x) - r\psi_\eta(x)m_\eta'(x)t(\bar{x}-c) = ((1-t)(\mu(x)-r(x-c)) - rt(\bar{x}-c))\psi_\eta(x)m_\eta'(x).$$

Hence, our assumption (2.3) presented in the text implies that for all $t \in [0,1)$ the mapping $F_t(x)$ satisfies the monotonicity condition

$$F_t'(x) \gtreqless 0 \quad x \lesseqgtr \hat{x}_t.$$

Moreover, since Lemma 2.1 guarantees the existence of a unique threshold $x_t^*(\eta) \in ((c-t\bar{x})/(1-t), \infty)$ satisfying the condition $F_t(x_t^*(\eta)) = 0$ we find that $(1-t)F_0(x_t^*(\eta)) = \psi_\eta'(x_t^*(\eta))t(\bar{x}-c)/S_\eta'(x_t^*(\eta)) > 0$ proving that $x_0^*(\eta) > x_t^*(\eta)$. We also find that $F_t(x)$ is decreasing on the set where $F_0(x)$ is decreasing.

Given these observations, assume first that $\bar{x} > x_0^*(\eta) > x_t^*(\eta)$. Then, the mapping $(x-c)/\psi_\eta(x)$ attains its global maximum on the set $(0, \bar{x})$. Since $F_t(x)$ is decreasing on the set where $F_0(x)$ is decreasing we find that $x_0^*(\eta) = \operatorname{argmax}\{(x-c-t(x-\bar{x})^+)/\psi_\eta(x)\}$ and, therefore, that the proposed value function dominates the exercise payoff $(x - c - t(x - \bar{x})^+)$ for all $x \in \mathbb{R}_+$. Consider now the mappings

$$L_{\psi_\eta}(x) = \frac{V_\eta'(x)}{S_\eta'(x)}\psi_\eta(x) - \frac{\psi_\eta'(x)}{S_\eta'(x)}V_\eta(x)$$

and

$$L_{\varphi_\eta}(x) = \frac{V_\eta'(x)}{S_\eta'(x)}\varphi_\eta(x) - \frac{\varphi_\eta'(x)}{S_\eta'(x)}V_\eta(x)$$

It is now a straightforward exercise in ordinary differentiation to demonstrate that $L_{\psi_\eta}'(x) = L_{\varphi_\eta}'(x) = 0$ for all $x \in (0, x_0^*(\eta))$, $L_{\psi_\eta}'(x) = (\mu(x) - r(x-c))\psi_\eta(x)m_\eta'(x) < 0$ and $L_{\varphi_\eta}'(x) = (\mu(x) - r(x-c))\varphi_\eta(x)m_\eta'(x) < 0$ for all $(x_0^*(\eta), \bar{x})$, $L_{\psi_\eta}'(x) = ((1-t)(\mu(x)-r(x-c))+rt(c-\bar{x}))\psi_\eta(x)m_\eta'(x) < 0$ and $L_{\varphi_\eta}'(x) = ((1-t)(\mu(x) - r(x-c)) + rt(c - \bar{x}))\varphi_\eta(x)m_\eta'(x) < 0$ for all $(\bar{x}, \infty)$. Moreover, since $L_{\psi_\eta}(x)$ is non-positive and $L_{\varphi_\eta}(x)$ is non-negative for all $x \in \mathbb{R}_+$ we observe that the proposed value function satisfies the conditions of Proposition 3.3 in Salminen (1985) and, therefore, constitutes a r-excessive majorant of the exercise payoff $x - c - t(x - \bar{x})^+$. Since the value is the smallest of these majorants, we find that the proposed value is indeed the value of the optimal policy thus completing the proof of part (A) of our theorem. Establishing part (B) and part (C) is analogous. □

C Proof of Theorem 2.3

Proof. Denote as $\psi_{\hat{\eta}}(x)$ the increasing fundamental solution of the ordinary linear second order differential equation $\hat{\eta}^2\sigma^2(x)u''(x)/2 + \mu(x)u'(x) - ru(x) = 0$, where $\hat{\eta}$ satisfies the inequality $\hat{\eta} \geq \eta$. As was established in Alvarez (2004b), the assumed monotonicity of the net appreciation rate $\mu(x) - rx$ and the boundary condition requiring that $\mu(0) \leq 0$ whenever 0 is attainable for X_t imply that the increasing fundamental solutions $\psi_\eta(x)$ and $\psi_{\hat{\eta}}(x)$ are strictly convex and satisfy for all $x \in \mathbb{R}_+$ and $y \in [x, \infty)$ the inequalities

$$\frac{\psi_\eta(x)}{\psi_\eta(y)} \leq \frac{\psi_{\hat{\eta}}(x)}{\psi_{\hat{\eta}}(y)} \quad \text{and} \quad \frac{\psi_\eta'(x)}{\psi_\eta(x)} \geq \frac{\psi_{\hat{\eta}}'(x)}{\psi_{\hat{\eta}}(x)}.$$

Given the representation (2.5), presented in the text, we immediately observe that

$$V_\eta(x) = \sup_{y \geq x}\left[(y - c - t(y - \bar{x})^+)\frac{\psi_\eta(x)}{\psi_\eta(y)}\right] \leq \sup_{y \geq x}\left[(y - c - t(y - \bar{x})^+)\frac{\psi_{\hat{\eta}}(x)}{\psi_{\hat{\eta}}(y)}\right] = V_\eta(x)$$

which proves that increased volatility increases the value of the investment opportunity. On the other hand, if $g(x)$ is non-decreasing, continuous, and continuously differentiable outside on $\mathbb{R}_+\backslash\mathcal{D}$, where $\mathcal{D}$ is a countable set of points in $\mathbb{R}_+$, then for all $x \in \mathbb{R}_+\backslash\mathcal{D}$ it holds that

$$\frac{\psi_\eta^2(x)}{\psi_\eta'(x)}\frac{d}{dx}\left[\frac{g(x)}{\psi_\eta(x)}\right] = g'(x)\frac{\psi_\eta(x)}{\psi_\eta'(x)} - g(x) \leq g'(x)\frac{\psi_{\hat{\eta}}'(x)}{\psi_{\hat{\eta}}(x)} - g(x) = \frac{\psi_{\hat{\eta}}^2(x)}{\psi_{\hat{\eta}}'(x)}\frac{d}{dx}\left[\frac{g(x)}{\psi_{\hat{\eta}}(x)}\right]$$

which shows that $g(x)/\psi_{\hat{\eta}}(x)$ is non-decreasing at any extreme point of $g(x)/\psi_\eta(x)$. Since the exercise payoff $x - c - t(x - \bar{x})^+$ satisfies these conditions and the maximum of the mapping $(x - c - t(x - \bar{x})^+)/\psi_\eta(x)$ is unique, we find that increased volatility increases the optimal investment threshold, that is, that $x_t^*(\hat{\eta}) \geq x_t^*(\eta)$. □

D Proof of Theorem 2.4

Proof. As was established in Lemma 2.1, the equation

$$(1-t)\psi_\eta(x) = \psi'_\eta(x)((1-t)x - (c - t\bar{x}))$$

has a unique root $x_t^*(\eta)$ for all $t \in [0,1)$. According to (A.1) the assumption $t\bar{x} < c < \bar{x}$ implies that the optimal threshold is a decreasing mapping of the tax rate t and, therefore, that $x_t^*(\eta) \leq x_0^*(\eta)$ for all $t \in (0,1)$. However, as was in turn established in Theorem 2.3, increased volatility increases the optimal investment threshold and, therefore, we find that $x_0^*(\eta) \geq x_t^*(\eta) \geq \lim_{\eta\downarrow 0} x_t^*(\eta)$ for all tax rates $t \in (0,1)$ and all $\eta > 0$. Since the optimal investment threshold satisfies in the absence of uncertainty and taxation (i.e. when $t = 0$ and $\eta = 0$) the ordinary first order condition $\mu(x_0^*(0)) = r(x_0^*(0) - c)$ we find that the assumed monotonicity of the net appreciation rate $\mu(x) - rx$ and the assumption $\mu(\bar{x}) < r(\bar{x} - c)$ imply that $\bar{x} > x_0^*(0)$ and, therefore, that $x_0^*(\eta) \leq \bar{x}$ as long as $\eta \leq \eta_1^*$. On the other hand, since $\psi_\eta(x)/\psi'_\eta(x)$ is increasing as a function of the volatility multiplier η, the condition $\lim_{\eta\to\infty} \psi_\eta(\bar{x})/\psi'_\eta(\bar{x}) > (\bar{x} - c)/(1-t)$ implies that $(1-t)\psi_\eta(\bar{x}) > \psi'_\eta(\bar{x})(\bar{x} - c)$ for a sufficiently great volatility multiplier η and, therefore, that there is critical volatility multiplier η_2^* such that $x_t^*(\eta) \geq \bar{x}$ as long as $\eta \geq \eta_2^*$. The rest of the alleged results follow directly from Theorem 2.1.

□