Irreversible Capital Accumulation and Non-Linear Tax Policy

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Abstract

We analyze the impact of tax progression on optimal investment policy and its value by first demonstrating that optimal investment policy maximizes the value of the firm and the rate at which it is growing as a function of operating capital stock. Then we show that three possible optimal regimes arise depending on the nature of tax policy. If the exogenously given progression threshold lies between the optimal capital stocks in the case of higher and lower marginal profit taxes, then optimal policy is independent of profit tax rates. Outside this corner solution optimal investment policy is conventional.

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1 Introduction

It is reasonable to argue that most major investments are at least partially irreversible due to the fact that firms cannot disinvest without costs after having carried out their investment decision. Physical capital is of course industry-specific, but also often firm-specific. In the recent literature using the framework of irreversible investment under uncertainty various justifications for the neutral tax system have been analyzed by using real option theory. In most studies corporate taxation has been assumed to be proportional meaning that the marginal and the average tax rate are constant and therefore equal. But in practice if for instance there is tax exemption this is not an appropriate assumption. Even though the marginal tax rate would be constant, the average tax rate would increases with the tax base so that in this case taxation is progressive. Alvarez and Koskela (2004) have studied the impact of tax progression - defined as the higher average tax rate in terms of the tax base - on irreversible investment under uncertainty. They have shown that if tax exemption is smaller than the sunk cost of investment, higher tax rate will decelerate optimal investment by raising the optimal investment threshold, while if tax exemption is greater than the sunk cost of investment, interestingly three different regimes in terms of optimal investment threshold arise. First, for a set of "sufficiently low" volatilities of underlying value dynamics, higher volatility decreases the optimal investment threshold, but is independent of tax rate. Second, for "medium" volatilities the optimal investment threshold does not depend either on tax rate or volatility. Finally, when volatility is "high enough", the optimal investment threshold will depend positively on volatility but negatively on tax rate so that in this case we have "tax paradox".

In the case of proportional taxation under certainty Samuelson (1964) and Johansson (1969) showed that under the uniform marginal tax rate the present value of the returns on investment project is not affected by the tax rate so that investment neutrality prevails if tax depreciation is equal to economic depreciation and debt interest is deductible. Since there might be difficulties to implement this because of true economic depreciation may not be observable, the cash flow tax emerges as an interesting alternative neutral tax. The idea is to write-off completely new investment expenditures in the year of acquisition but no later deduction for interest rate or depreciation (see Smith (1963)). Sandmo (1979) has shown that this is valid only under the assumption that the tax rate is constant over time (see also Boadway and Bruce (1984) and Sinn (1987), chapter 5).
The purpose of our paper is to abstract from uncertainty and study the impact of taxation on irreversible investment and capital accumulation in the situation where taxation is not proportional, but progressive.

First, we characterize a set of weak conditions under which a unique optimal investment policy exists and when Tobin’s q can be expressed in terms of the associated optimal timing problem. Then we assume that the marginal profit tax rate increases at an exogenously given progression threshold. We characterize the optimal investment policy and its value in the presence of progressive taxation and show that three possible optimal regimes arise depending on the nature of tax policy. More specifically, if the progression threshold lies between the optimal capital stocks in the case of higher and lower marginal profit tax rates, then optimal investment policy is independent of tax policy. But outside this corner solution, optimal capital accumulation policy is conventional, which has been analyzed in the literature.

We proceed as follows. In section 2 we present the optimal irreversible capital accumulation problem under certainty without taxation. Section 3 incorporates progressive tax system, where marginal profit tax rate increases after progression threshold and analyzes its impact on optimal capital accumulation and its value. In section 4 we illustrate explicitly our general findings concerning the effects of profit taxation both on the optimal capital accumulation policy and its value. Finally, there is a brief concluding section.

2 Irreversible Investment and Capital Accumulation

In this section we characterize the optimal irreversible capital accumulation problem subject to the standard capital accumulation rule in the deterministic case without taxation. More precisely, we proceed as follows: First we present the optimal control problem and show a relatively weak set of conditions under which a unique optimal investment policy exists. Second we characterize a set of conditions when the marginal value of capital and, therefore, Tobin’s q can be expressed in terms of an associated optimal timing problem which can be interpreted as an optimal exit problem familiar from the real options literature on optimal exit.
Consider the optimal control problem

\[ V(k) = \sup_{I \in \Lambda} \int_0^\infty e^{-rs}[\pi(K^t_s)ds - qdI_s] \]  

subject to standard capital accumulation rule

\[ dK^t_I = dI_t - \delta K^t_Idt \quad K_0 = k, \]  

stating that net investments are equal to gross investments less depreciation. In (2.1) \( q \) denotes the unit price of investment goods, \( r \) denotes the risk free discount rate, \( \pi(k) \) denotes the revenue flow accrued from operating with a capital stock \( k \), and \( I_t \) denotes the gross investment policy of the firm. We assume that the revenue flow \( \pi(k) \) is continuous and satisfies the condition \( \lim_{k \downarrow 0} \pi(k) = 0 \). We call a gross investment policy \( I_t \) admissible if it is non-negative, non-decreasing, and right-continuous, and denote the set of admissible investment policies as \( \Lambda \). It is worth emphasizing that the assumed monotonicity and non-negativity of the admissible investment policies imply that investment is irreversible. Hence, once the firm has invested the capital stock cannot be instantaneously adjusted to a lower capital stock by disinvestment.

Before introducing taxes into the general analysis of the considered class of irreversible investment problems, we first define the mapping \( \theta : \mathbb{R}_+ \mapsto \mathbb{R} \) according to the familiar characterization

\[ \theta(k) = \pi(k) - q(\delta + r)k, \]  

where \((\delta + r)qk\) denotes the user cost of capital (see e.g. Hall and Jorgenson (1967)). At this point it is worth noticing that \( \theta(k) \) can be interpreted as the growth rate of the excess returns accrued from following an optimal irreversible capital accumulation policy since

\[ V(k) \leq qk + \sup_{I \in \Lambda} \int_0^\infty e^{-rs}\theta(K^t_s)ds. \]  

We can now establish the following auxiliary result applied later in the general analysis of the optimal capital accumulation policy in the presence of a non-linear tax policy.

**Theorem 2.1.** Assume that \( \theta(k) \) attains a unique global maximum at \( \tilde{k} = \arg \max \{\theta(k)\} > 0 \) and that \( \theta(k) \) is non-decreasing for \( k < \tilde{k} \) and non-increasing for \( k > \tilde{k} \). Then the value of the rationally
managed firm reads as

$$V(k) = \begin{cases} 
qk + (R_r \theta)(k) + \frac{k^{-r/\delta}}{r} (R_r \theta)'(k) \tilde{k}^{1+r/\delta} k^{-r/\delta}, & k > \tilde{k} \\
qk + \theta(\tilde{k}), & k \leq \tilde{k} 
\end{cases}$$

(2.5)

where

$$(R_r \theta)(k) = \frac{k^{-r/\delta}}{\delta} \int_0^k y^{r/\delta - 1} \theta(y) dy$$

(2.6)

denotes the cumulative present value of the flow $\theta(k)$. Especially, the value $V(k)$ is continuously differentiable on $\mathbb{R}_+$ and satisfies the inequality $V'(k) \in (0, q]$. Moreover, if the revenue flow $\pi(k)$ is concave then $V(k)$ is concave as well.

**Proof.** See Appendix A.

Theorem 2.1 states a set of weak conditions under which a unique optimal investment policy exists and can be characterized in terms of a single capital stock threshold at which investing becomes optimal. It is worth emphasizing that the results of Theorem 2.1 are very general since it only requires that the auxiliary mapping $\theta(k)$ satisfies a set of weak monotonicity conditions and that the cumulative present value of the revenue flow exists.

An interesting implication of Theorem 2.1 stating a set of conditions under which the marginal value of capital and, therefore, Tobin’s q can be interpreted in terms of an associated optimal timing problem is now stated in the following.

**Theorem 2.2.** Assume that the conditions of Theorem 2.1 are satisfied. Then the marginal value of capital $V'(k)$ reads as

$$V'(k) = q + (R_r \theta)'(k) - H(k),$$

(2.7)

where

$$H(k) = \sup_{t \geq 0} \left[ e^{-(r+\delta)t} (R_r \theta)'(K_t) \right]$$

can be expressed as

$$H(k) = k^{-r/\delta - 1} \sup_{y \leq k} \left[ y^{r/\delta + 1} (R_r \theta)'(y) \right] = \begin{cases} 
(R_r \theta)'(\tilde{k}) \left( \frac{k}{\tilde{k}} \right)^{-r/\delta - 1}, & k > \tilde{k} \\
(R_r \theta)'(k), & k \leq \tilde{k}, 
\end{cases}$$

(2.8)

and $K_t$ denotes the capital stock in the absence of investment, that is, $K'_t = -\delta K_t$.  

4
Proof. See Appendix B.

Theorem 2.2 demonstrates that the optimal investment policy does not only maximize the value of the firm, it also maximizes the rate at which this value is growing as a function of the operating capital stock (see e.g. Abel (1990) and Caballero (1999) who provide excellent surveys of the classical q-theory of investment and its various extensions). This finding is of interest, since it states an explicit connection between standard neoclassical investment theory and the q-theory of investment.

3 Progressive Taxation and Optimal Investment

After having characterized the optimal irreversible capital accumulation problem and a set of conditions for a unique solution and when Tobin’s q can be expressed in terms of the optimal timing problem, we now introduce a progressive profit tax system and analyze its impact on the optimal capital accumulation policy and its value. We proceed as follows: First, we characterize the non-proportional profit taxation, i.e. when the tax rate if not constant in terms of tax base. Progressivity can be characterized either (A) in terms of increasing average tax rate when the marginal tax rate is constant, but there is a tax exemption, which means that higher tax base will increase the average tax rate or (B) that the marginal tax rate will increases with the tax base. In Alvarez and Koskela (2004) the impact of tax progressivity - defined by (A) - has been analyzed in the framework of optimal investment problem under stochastic value process. In this paper we abstract from stochasticity and study the implications of progressivity when the marginal tax rate increases with the tax base (in terms of tax progressivity, see the seminal paper by Musgrave and Thin (1948) and a textbook analysis by Lambert (2001)). Second, we characterize the optimal investment policy and its value when the profit tax rate increases at a certain level of before tax revenue flow, i.e. definition (B). Finally, in section 4 we illustrate our new findings numerically.

We now plan to investigate the optimal investment problem (2.1), when the after tax revenue flow reads (cf. Kari (1999), Alvarez and Koskela (2004))

$$\pi(k) = (1 - \tau_1)\hat{\pi}(k) - (\tau_2 - \tau_1)(\hat{\pi}(k) - \hat{\pi}(k^*))^+ \tag{3.1}$$
where $\hat{\pi}(k)$ is the before tax revenue flow and $k^*$ is an exogenously determined constant progression threshold at which the profit tax rate increases from $\tau_1$ to $\tau_2$. In accordance with standard neoclassical studies of the firm, we assume that the short-run profit flow $\hat{\pi}(k)$ is continuously differentiable, increasing, strictly concave, and satisfies the standard *Inada-conditions*. Given these assumptions, the instantaneous yield $\theta(k)$ reads as

$$
\theta(k) = \begin{cases} 
\theta_2(k), & k > k^* \\
\theta_1(k), & k \leq k^*,
\end{cases}
$$

where

$$
\theta_1(k) = (1 - \tau_1)\hat{\pi}(k) - (r + \delta)qk
$$

and

$$
\theta_2(k) = (1 - \tau_2)\hat{\pi}(k) - (r + \delta)qk + (\tau_2 - \tau_1)\hat{\pi}(k^*).
$$

Denote now as $k_i$ the capital stock at which the yield $\theta_i(k)$ is maximized for $i = 1, 2$. Our assumptions imply that

$$
\theta_i'(k_i) = (1 - \tau_i)\pi'(k_i) - (\delta + r)q = 0
$$

for $i = 1, 2$. Especially, since $\tau_2 > \tau_1$ we find that

$$
\pi'(k_2) = \frac{(r + \delta)q}{1 - \tau_2} > \frac{(r + \delta)q}{1 - \tau_1} = \pi'(k_1)
$$

implying by the strict concavity of $\pi(k)$ that $k_2 < k_1$. Thus, we find that in the presence of the considered progressive tax system three possible cases may arise depending on the progression threshold $\tilde{k}$. Our main conclusions on the nature of the optimal investment policy and the value of the firm are now summarized in the following.

**Theorem 3.1.** The value of the rationally managed firm reads as

$$
V(k) = \begin{cases} 
qk + (R_r\theta)(k) + \frac{\delta}{r}(R_r\theta)'(\tilde{k})\tilde{k}^{1+r/\delta}k^{-r/\delta}, & k > \tilde{k} \\
qk + \frac{\theta(\tilde{k})}{r}, & k \leq \tilde{k}
\end{cases}
$$

(3.2)
where the optimal investment threshold

\[ \tilde{k} = \arg \max \{ \theta(k) \} = \begin{cases} k_2, & \text{if } k^* < k_2 < k_1 \\ k^*, & \text{if } k_2 < k^* < k_1 \\ k_1, & \text{if } k_2 < k_1 < k^*. \end{cases} \quad (3.3) \]

Proof. The result is an straightforward consequence of Theorem 2.1.

Theorem 3.1 characterizes the optimal investment policy and its value in the presence of progressive taxation. As intuitively is clear, we find that in the presence of progressive taxation three possible regimes arise depending on the progression threshold. Interestingly, in the case where \(k^* \in (k_2,k_1)\) the optimal investment policy is independent of the tax policy, while outside this region the optimal capital accumulation policy is conventional.

4 Illustration

In order to illustrate explicitly our general findings on the impact of progressive profit taxation on the optimal capital accumulation policy and its value, we assume that the revenue flow \(\hat{\pi}(k)\) now reads as \(\hat{\pi}(k) = ak^b\), where \(a > 0\) and \(b \in (0,1)\) are exogenously given constants. It is now a simple exercise in nonlinear programming to establish that if \(\tau_1 < \tau_2\) then in the present example

\[ k_1 = \left( \frac{(1 - \tau_1)ab}{(r + \delta)q} \right)^{1/(1-b)} > \left( \frac{(1 - \tau_2)ab}{(r + \delta)q} \right)^{1/(1-b)} = k_2 \]

implying for \(i = 1,2\) the familiar comparative statics

\[ \frac{\partial k_i}{\partial \tau_i} = -\frac{k_i}{(1-b)(1-\tau_i)} < 0 \]

stating that higher profit taxation should decelerate rational investment demand by decreasing the optimal capital stock threshold at which reinvestment becomes optimal. Especially, Theorem 3.1 now implies that there are three different optimal regimes depending on the precise magnitude of the exogenously determined threshold \(k^*\). Moreover, the value of the optimal policy now reads as in (3.2), where in our explicit case the cumulative present value of the flow \(\theta(k)\) can be expressed as

\[ (R,\theta)(k) = \begin{cases} (1 - \tau_2)\frac{ak^k}{r + \delta} - (\tau_1 - \tau_2) \frac{ak^*k}{r} \left( 1 - \frac{\delta k}{r + \delta} \left( \frac{k}{k^*} \right)^{-r/\delta} \right) - qk & k \geq k^* \\ (1 - \tau_1) \frac{ak^k}{r + \delta} - qk & k < k^* \end{cases} \quad (4.1) \]
so that

\[(R_t \theta)'(k) = \begin{cases} 
(1 - \tau_2) \frac{abk^{b-1}}{r + \delta b} - (\tau_1 - \tau_2) \frac{abk^{b-1}}{r + \delta b} (\frac{k}{k^*})^{-r/\delta - 1} - q & k \geq k^* \\
(1 - \tau_1) \frac{abk^{b-1}}{r + \delta b} - q & k < k^*. 
\end{cases} \quad (4.2)\]

The growth rate of the excess returns accrued from following an optimal irreversible capital accumulation policy and the optimal investment thresholds are illustrated in Figure 1 (under the assumption that \(a = 1, b = 0.75, \tau_1 = 0.15, \tau_2 = 0.3, q = 0.45\)). As was stated in our Theorem

3.1, there are three possible optimal regimes depending on the progression threshold \(k^*\). As our numerical illustration indicates, the optimal investment threshold now reads as

\[\tilde{k} = \begin{cases} 
1.85, & \text{if } k^* < 1.85 \\
k^*, & \text{if } 1.85 \leq k^* \leq 4.03 \\
4.03, & \text{if } k^* > 4.03. 
\end{cases} \]

Thus, the optimal capital accumulation policy is independent of the profit tax rate as long as the progression threshold satisfies the inequality \(1.85 \leq k^* \leq 4.03\). Outside this corner solution the optimal capital accumulation policy is a conventional one.
5 Conclusions

In this paper we have analyzed the following issue: what is the impact of profit tax progression – defined as the increase of the marginal profit tax rate at an exogenously given progression threshold – on the optimal investment policy and its value. First, we provided a set of weak conditions under which a unique optimal investment policy exists and second, we demonstrated that the optimal investment policy does not only maximize the value of the firm but also maximizes the rate at which this value is growing as a function of the operating capital stock. Finally, and importantly, we showed that three possible optimal regimes arise depending on the progression threshold. More precisely, if the progression threshold lies between the optimal capital stocks in the case of higher and lower marginal profit taxes, then optimal investment policy is independent of the profit tax rate. Outside this corner solution the optimal capital accumulation policy is conventional.

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References


A Proof of Theorem 2.1

Proof. Consider first the cumulative present value of the flow $\theta(k)$. It is now clear that

$$ (R_r \theta)(k) = \int_0^\infty e^{-rs} \theta(ke^{-\delta s}) ds. $$

Making the change of variable $y = ke^{-\delta s}$ implies that $e^{-rs} = (y/k)^{r/\delta}$ and, therefore, that the cumulative present value $(R_r \theta)(k)$ can be expressed as in (2.6). Given this observation, consider the functional

$$ J(k) = \frac{\delta}{r} k^{r/\delta + 1} (R_r \theta)'(k) = \frac{1}{r} k^{r/\delta} \theta(k) - \frac{1}{\delta} \int_0^k y^{r/\delta - 1} \theta(y) dy. $$

If $0 < k < z < \tilde{k}$, then

$$ J(z) - J(k) = \frac{1}{r} \left[ z^{r/\delta} \theta(z) - k^{r/\delta} \theta(k) \right] - \frac{1}{\delta} \int_k^z y^{r/\delta - 1} \theta(y) dy $$

$$ \geq \frac{1}{r} k^{r/\delta} [\theta(z) - \theta(k)] \geq 0. $$

Similarly, if $\tilde{k} < k < z < \infty$, then

$$ J(z) - J(k) = \frac{1}{r} \left[ z^{r/\delta} \theta(z) - k^{r/\delta} \theta(k) \right] - \frac{1}{\delta} \int_k^z y^{r/\delta - 1} \theta(y) dy $$

$$ \leq \frac{1}{r} k^{r/\delta} [\theta(z) - \theta(k)] \leq 0. $$

Hence, the mapping $J(k)$ is increasing on $(0, \tilde{k})$, decreasing on $(\tilde{k}, \infty)$, and attains a unique global maximum at $\tilde{k}$.

Denote now the proposed value function as $V_p(k)$. Since $V_p(k)$ is attained by applying the admissible singular investment strategy defined as $I_0 = (\tilde{k} - k)^+$ and

$$ I_t = \delta \tilde{k}, \; t \geq \inf \{ t \geq 0 : K^t_I = \tilde{k} \} $$

we observe that $V(k) \geq V_p(k)$. In order to prove the opposite inequality we first observe that the proposed value function is non-negative and continuously differentiable. Since $-\delta k V'_p(k) - r V_p(k) + \pi(k) = 0$ on $(\tilde{k}, \infty)$ and

$$ -\delta k V'_p(k) - r V_p(k) + \pi(k) = \theta(k) - \theta(\tilde{k}) < 0 $$

by the assumed monotonicity of $\theta(k)$ we find that $-\delta k V'_p(k) - r V_p(k) + \pi(k) \leq 0$ for all $k \in \mathbb{R}_+$. Moreover, standard differentiation implies that for all $k \in (\tilde{k}, \infty)$ it holds that

$$ V_p'(k) = q + \frac{r}{\delta} k^{r-\delta-1} \left( J(k) - J(\tilde{k}) \right) \leq q, $$

12
since $\tilde{k} = \text{argmax}\{J(k)\}$. Consequently, $V_p'(k) \leq q$ for all $k \in \mathbb{R}_+$. Hence, the proposed value function satisfies the sufficient variational inequalities and, therefore, $V_p(k) \geq V(k)$.

It remains to prove that $V'(k)$ is decreasing and, therefore, that $V(k)$ is concave whenever the cash flow $\pi(k)$ is concave. To accomplish this task we first observe that since the value satisfies the condition $-\delta kV''(k) = rV(k) - \pi(k)$ on $(\tilde{k}, \infty)$ we find by ordinary differentiation that $-\delta kV''(k) = (r + \delta)V'(k) - \pi'(k)$ on the set where $\pi(k)$ is differentiable. Thus, in order to prove the alleged concavity of the value function it is sufficient to demonstrate that $(r + \delta)V'(k) \geq \pi'(k)$ on $(\tilde{k}, \infty)$.

For all $k \in (\tilde{k}, \infty)$ it holds

$$V'(k) - \frac{\pi'(k)}{(r + \delta)} = \frac{r}{\delta} k^{r/\delta - 1} (J(k) - J(\tilde{k})) - \frac{\theta'(k)}{(r + \delta)}.$$

The assumed concavity of $\pi(k)$ now implies that $\theta(k)$ is concave as well. Combining this observation with the monotonicity of $\theta(k)$ on $(\tilde{k}, \infty)$ then yields

$$\frac{r}{\delta} k^{r/\delta - 1} (J(k) - J(\tilde{k})) = \frac{1}{\delta} k^{r/\delta - 1} \int_\tilde{k}^k y^{r/\delta} \theta'(y) dy \geq \frac{\theta'(k)}{(r + \delta)} \left( 1 - \left( \frac{k}{\tilde{k}} \right)^{r/\delta + 1} \right) \geq \frac{\theta'(k)}{(r + \delta)},$$

which completes the proof of our Theorem.

\[\square\]

B Proof of Theorem 2.2

Proof. Consider first the optimal stopping problem

$$H(k) = \sup_{t \geq 0} \left[ e^{-(r+\delta)t} (R_r \theta)'(K_t) \right].$$

We claim that the optimal stopping time is $t^* = \inf\{ t \geq 0 : K_t \leq \tilde{k} \} = \max(0, \ln(k/\tilde{k})/\delta)$ and, therefore, that the value of the optimal stopping policy reads as

$$H^*(k) = \begin{cases} (R_r \theta)'(\tilde{k}) \left( \frac{k}{\tilde{k}} \right)^{-r/\delta - 1}, & k > \tilde{k} \\ (R_r \theta)'(k), & k \leq \tilde{k}. \end{cases}$$

It is clear that since the stopping time $t$ in the stopping problem (2.8) is arbitrary, we have that $H(k) \geq H^*(k)$ for all $k \in \mathbb{R}_+$. To prove the opposite inequality, we observe that since $\tilde{k} = \text{argmax}\{J(k)\} = \text{argmax}\{\theta(k)\}$ we have the inequality

$$\frac{\delta}{r} k^{r/\delta + 1} (R_r \theta)'(k) \geq \frac{\delta}{r} \tilde{k}^{r/\delta + 1} (R_r \theta)'(\tilde{k})$$
implying that
\[(R_r \theta)'(k) k^{r/\delta + 1} \leq \tilde{k}^{r/\delta + 1} (R_r \theta)'(\tilde{k})\]
for all \( k \in \mathbb{R}_+ \). Moreover, since \( k^{-r/\delta - 1} \) is the fundamental solution of the ordinary first order differential equation \(-\delta k u'(k) - (r + \delta) u(k) = 0\) we find that \( e^{-(r+\delta)t} K_t^{-r/\delta - 1} = k^{-r/\delta - 1} \) for all \((t, k) \in \mathbb{R}^2_+\). Thus, if \( k > \tilde{k} \) then
\[
H(k) = \sup_{t \geq 0} \left[ e^{-(r+\delta)t} \frac{(R_r \theta)'(K_t)}{K_t^{-r/\delta - 1}} \right] \leq (R_r \theta)'(\tilde{k}) \left( \frac{k}{\tilde{k}} \right)^{-r/\delta - 1} = H^*(k).
\]
On the other hand, \( K_t = ke^{-\delta t} \leq k \) for all \( t \geq 0 \) and
\[
K_t^{r/\delta + 1} (R_r \theta)'(K_t) \leq (R_r \theta)'(k) k^{r/\delta + 1}
\]
for all \( k \in (0, \tilde{k}] \) since \((R_r \theta)'(k) k^{r/\delta + 1}\) is non-decreasing on \((0, \tilde{k}]\). Thus, we find that if \( k \leq \tilde{k} \) then
\[
H(k) \leq \sup_{t \geq 0} \left[ e^{-(r+\delta)t} (R_r \theta)'(k) k^{r/\delta + 1} K_t^{-r/\delta - 1} \right] = (R_r \theta)'(k) = H^*(k),
\]
proving that \( H(k) = H^*(k) \) for all \( k \in \mathbb{R}_+ \). Combining this result with (2.5) then proves (2.7). \( \square \)