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Affleck-Dine mechanism and Q-balls along SUSY flat directions

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Abstract

This thesis contains four research papers and an introduction, which provides the necessary backround and also contains a new analytical solution to the Q-ball equation of flat potentials in the Appendix.

At the classical level supersymmetric gauge theories have a large degeneracy of vacua of the scalar potential. The vacuum scalar field configurations are called flat directions or moduli fields. The degeneracy remains unbroken to arbitrary perturbative order. Degeneracy can be lifted by supersymmetry breaking effects or by adding suitable terms to the superpotential. Then the flat direction has a non-trivial potential that determines its dynamics.

The flat directions have a number of cosmological consequences. Within the MSSM flat directions that carry baryon and/or lepton number can provide a means for baryogenesis by forming coherently rotating condensates, Affleck-Dine condensates, which eventually decay into Standard Model fermions. If R-parity is conserved, then the decay products include the stable Lightest Scalar Particles, which can exist as dark matter today.

When one includes radiative corrections, the mass term grows slower than ϕ^2 for some flat directions. As a consequence the AD condensate fragments into non-topological solitons called Q-balls, which are minimum energy configurations with a conserved non-zero charge. Q-balls themselves can carry baryon number and protect it from the effects of the electroweak phase transition. Depending on the SUSY breaking mechanism the Q-balls can be stable and hence contribute to the dark matter. Q-balls formed out of flat direction condensates, that are not connected to SM fields, can act as self-interacting dark matter.

In the thesis we have studied the instabilities of the flat directions within MSSM; the formation of the AD condensates and their properties in both gravity and gauge mediated SUSY breaking scenario; the fragmentation of the Affleck-Dine condensate to Q-balls and the thermalization of the produced Q-ball distribution; and constraints for Q-balls being the self-interacting dark matter.

Contents

List of Papers

I K. Enqvist, A. Jokinen and J. McDonald,

Flat Direction Condensate Instabilities in the MSSM,

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II K. Enqvist, A. Jokinen, T. Multamäki and I. Vilja,

Numerical simulations of fragmentation of the Affleck-Dine condensate,

Phys. Rev. **D63** (2001) 083501.

III K. Enqvist, A. Jokinen, T. Multamäki and I. Vilja,

Constraints on self-interacting Q-ball dark matter,

Phys. Lett. **B526** (2002) 9.

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Analytical and numerical properties of Affleck-Dine condensate formation,

ArXiv: **hep-ph/0204086** (submitted to *Phys.Rev.D*).

1 Introduction

The long standing problem in the study of cosmology has been baryogenesis. Cosmological nucleosynthesis has given us strict bounds on the observed baryon-to-photon ratio $\eta = n_B/n_\gamma$, where n_B is the baryon number density and n_γ is the photon number density. This has been narrowed down to $1.2 \cdot 10^{-10} \leq \eta \leq 6.3 \cdot 10^{-10}$ [1]. It is a huge theoretical challenge to achieve a mechanism of baryogenesis that produces the prescribed η. The conditions, that are necessary for baryogenesis, were long ago given by Sakharov [2]: 1) Violation of baryon number, 2) violation of C and CP and 3) non-equilibrium conditions. In the Standard Model (SM) of particle physics baryogenesis is possible only through the electroweak anomaly. However, this has been ruled out by the lattice studies on the electroweak phase transition [3–6] and the LEP experiments on Higgs mass [7]. It is still possible that the electroweak mechanism would work in an extension of the SM such as the Minimal Supersymmetric Standard Model (MSSM), but this is on the edge of being ruled out, too [8, 9]. In [9] it was argued that if the Higgs mass is larger than 120 GeV, the electroweak baryogenesis within MSSM does not produce the observed baryon asymmetry. The current bound is $m_H \gtrsim 115$ GeV [10].

In 1985 Affleck and Dine [11] proposed a mechanism for baryogenesis based on scalar fields. The idea can be presented simply by considering a Lagrangian density of a complex scalar field, which has a global $U(1)$-symmetry (such as the baryon number or the lepton number)[1]

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - V(|\phi|). \tag{1}$$

From the usual definition of the Noether current related to Eq. (1) one obtains the charge density

$$q = i(\phi\dot{\phi}^* - \dot{\phi}\phi^*) = \dot{\theta}\varphi^2, \tag{2}$$

where we have written the complex scalar field $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$, where φ, θ are real scalar fields. The second form shows clearly that in order to have a non-zero charge density ϕ has to acquire a non-zero vacuum expectation value $| < \phi > | > 0$ and has to rotate

[1] We use the metric $\eta_{\mu\nu}$ =diag(1,-1,-1,-1), with μ, $\nu = 0, 1, 2, 3$.

around the origin with $\dot{\theta} \neq 0$. Naturally one has to include some kind of symmetry breaking into Eq. (1), for example a $U(1)$-violating term in the potential. Then one can calculate how the charge density evolves in an expanding universe with help of equations of motion related to Eq. (1), with the possible symmetry breaking terms taken into account, to obtain

$$\dot{q} + 3Hq + \frac{\partial V}{\partial \theta} = 0 \qquad \Leftrightarrow \qquad \frac{\partial (qR^3)}{\partial t} = -R^3 \frac{\partial V}{\partial \theta}, \tag{3}$$

where $H = \dot{R}/R$ is the Hubble parameter, R is the scale factor and qR^3 is the charge density in the co-moving volume.

Naturally one asks: what could the scalar field ϕ be? In the MSSM there are many scalar fields: squarks, sleptons and Higgses. Natural candidates are the squarks and sleptons, which give rise to baryogenesis and leptogenesis, respectively. In leptogenesis the lepton number is transformed into baryon number through sphaleron transitions at the electroweak phase transition. For this reason we do not make a difference between the baryon and the lepton number, but just consider a $U(1)$ charge in general.

There is also the issue of what kind charge violation can be included in the potential. At the renormalizable level B- and L-violating interactions are forbidden in the SM. They are also forbidden in the MSSM if R-parity is conserved. If R-parity is not conserved, it is possible to have interactions violating the baryon number. However, R-parity violating terms cause problems with proton decay unless the coupling constants are fine-tuned [12]. Therefore the violation has to be induced by a non-renormalizable operator. If a scalar field acquires a large expectation value, which it has to in order for the non-renormalizable operator to be effective, all the fields it interacts with gain a mass $g < \phi >$, where g is the corresponding coupling to the flat direction.

In supersymmetric gauge theories there is a rich vacuum degeneracy at the classical level. The scalar potential, which is a sum of the so-called F-terms and D-terms, vanishes identically for some field configurations *i.e.* along certain "flat directions". The space of all such flat directions is called the moduli space and the massless chiral superfields whose expectation values parameterize the flat directions are also known as moduli [13, 14]. The flat directions gain a mass comparable to the soft SUSY breaking mass and the non-renormalization theorems guarantee that these are the only radiative corrections they receive to arbitrary perturbative order [15–17]. However, non-perturbative effects do produce corrections to the flat directions, see *e.g.* [17]. Since only SUSY breaking and non-renormalizable terms produce an effective potential for the flat direction, it is possible for the flat direction to acquire a large vacuum expectation value (VEV) in the early universe.

If the flat direction contains fields that carry baryon or lepton number, then by introducing the B and/or L violation through non-renormalizable operators it is possible to have a consistent model for baryogenesis. In general a baryon-to-entropy ratio of $\mathcal{O}(1)$ can be produced [14]. Even if the flat directions were not carrying baryon number, as in various extensions of MSSM, the extra flat directions can have other cosmological consequences. For instance, they can be the dark matter. Therefore flat directions are interesting on their own right. One should also note that a non-zero VEV of a scalar field spontaneously breaks gauge symmetry.

The zero mode of a rotating scalar field forms a coherent condensate. The behaviour of the condensate is dependent on the mass term. If the mass term is the usual tree-level $m^2|\phi|^2$, on the average the condensate has zero pressure and eventually the condensate would decay through thermal scattering. However, if we take into consideration radiative corrections such as a logarithmic correction in case of gravity mediated SUSY breaking [18] or the almost flat potential of the gauge mediated SUSY breaking [19], a non-zero pressure is induced to the condensate. If the potential grows slower than ϕ^2, the pressure is negative [20]. Within MSSM with gravity mediated SUSY breaking some flat directions are unstable and some are not, when radiative corrections are included, as has been shown by solving renormalization group equations numerically in Paper I [21]. The pressure also depends on the orbit of the field. For pure oscillation the pressure reaches its maximum absolute value, whereas for circular orbit the pressure vanishes due to the fact that on the circular orbit kinetic and potential energy are equal, which was pointed out in Paper IV [22]. In an expanding universe the orbit cannot be strictly circular due to the dissipation caused by expansion. Thus, the pressure is always negative for some flat directions. Then the energy density of the condensate in the co-moving volume increases and makes the condensate unstable with respect to spatial perturbations [23]. Eventually these perturbations fragment the condensate.

For potentials that grow slower than ϕ^2 for some range of ϕ and which describe fields that are charged under a $U(1)$-symmetry, there exists a new kind minimum of energy. The minimum energy configuration in the sector of conserved charge is a non-topological soliton called the Q-ball [24]. Since, as it turns out, at the time of condensate fragmentation the flat direction potential is dominated by the $U(1)$-symmetric part, the charged condensate fragments will form Q-balls. Q-ball formation has been seen in several simulations [25–29] among them the one presented in the Paper II [28]. Actually even an oscillating condensate fragments into Q-balls, forming equal amounts of Q-balls and anti-Q-balls [27]. Because Q-balls are quite robust objects and do not decay easily through thermal scattering, the baryon number inside will be protected from sphaleron erasure until Q-balls eventually decay.

Although Q-balls are stable with respect to decay into their own quanta, there can exist decay channels to other scalars or to fermions. The fermion decay channel is interesting in that the Q-ball decays into fermions by evaporation from the surface of the Q-ball [31]. It is also possible that large Q-balls are completelely stable, in which case they might appear as dark matter (DM) [23]. This can happen in the gauge mediated SUSY breaking scenario where the Q-ball mass increases as $Q^{3/4}$ [32]. If Q-ball DM is baryonic, it seems that it can only form a small fraction of DM. However, in extensions of MSSM new flat directions can act as a source for Q-ball DM, but they can also be a source of problems [30]. Recently it has been suggested that Q-balls could act as self-interacting dark matter (SIDM) [33]. In Paper III [34] we have calculated constraints under which Q-balls can act as SIDM.

Recently the AD mechanism to produce Q-balls has been applied to inflation [35, 36]. In that approach the inflaton mass receives radiative corrections which give rise to a negative pressure of the inflaton condensate. Hence the inflaton fragments into Q-balls in the same manner as in the AD mechanism. Hence Q-balls decay into fermions through surface evaporation only, the reheating temperature after inflation decreases compared with the usual volume driven reheating.

Q-balls may also have other consequences relevant for cosmology. For more information about these see the recent review by Enqvist and Mazumdar [37] and the references cited therein.

The thesis is organized as follows: In Section 2 we present a short review of supersymmetry in order to have the relevant definitions used throughout the thesis. In Section 3 a quite thorough introduction to the flat directions is given. The results are applied to the MSSM. The lifting of the flat directions, *i.e.* generating a potential for the flat direction, is discussed at the end of the section. In Section 4 cosmological evolution of a flat direction is addressed, where it is shown how the field evolves during and after inflation. We then show how the negative pressure arises and discuss the growth of perturbations, which is caused by the negative pressure. In Section 5 general properties of Q-balls are presented and a new analytical approximation of the Q-ball profile in the gauge mediated scenario is derived. Details of this calculation can be found in the Appendix. We also consider the thermalization of the Q-ball distribution. In Section 6 we briefly describe cosmological applications of Q-balls such as baryogenesis and dark matter. In Section 7 we give our conclusions.

2 The basics of Supersymmetry

In this Section we give a brief review of the various aspects of supersymmetry relevant for the thesis. There exists excellent monographs and reviews on supersymmetry [12, 17]; Haber and Kane [38] deal specifically with the MSSM. Here we follow the notations of Nilles' review [12].

2.1 Superfields

In supersymmetric theories the fields are gathered into multiplets called superfields. The scalars, vectors and spinors are just different components of a superfield (in supergravity the graviton and gravitino are in the same superfield). There are practically two kinds of superfields: chiral and vector superfields (and a metric superfield in supergravity) [12, 17].

The covariant derivatives are defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\,\sigma^\mu_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\partial_\mu, \tag{4}$$

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\,\theta^\beta\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu, \tag{5}$$

where θ^α, $\bar\theta^{\dot\alpha}$ are Grassmann parameters.

With the help of the covariant derivatives, Eq. (4), we can define chiral superfields

$$\bar D_{\dot\alpha}\Phi_L = 0, \qquad D_\alpha\Phi_R = 0, \tag{6}$$

where Φ_L (Φ_R) is a left-handed (right-handed) chiral superfield. The left-handed chiral superfield can be written in the form

$$\Phi_L(y,\theta) = \phi(y) + \theta\psi(y) + \theta\theta F(y), \tag{7}$$

where ϕ, F are complex scalar fields, ψ is a Weyl spinor and $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$. The chiral superfield thus contains two spin-0 bosons and a spin-1/2 fermion. The auxiliary field F transforms as a total derivative under SUSY transformations. Hence the F term of a chiral superfield can be used for a supersymmetric action [12].

With the reality condition $V^\dagger = V$ one gets a vector superfield, whose components are

$$V(x,\theta,\bar\theta) = (1+\frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^\mu\partial_\mu)C + (i\theta+\frac{1}{2}\theta\theta\sigma^\mu\bar\theta\partial_\mu)\chi + \frac{i}{2}\theta\theta(M+iN) +$$
$$(-i\bar\theta+\frac{1}{2}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu)\bar\chi - \frac{i}{2}\bar\theta\bar\theta(M-iN) - \theta\sigma^\mu\bar\theta V_\mu + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D, \qquad (8)$$

where C, M, N, D are real scalar fields, χ, λ are Weyl spinors and V_μ is a real vector field. This multiplet contains spin-0, spin-1/2 and spin-1 fields. Now D transforms as a total derivative and can be used in the Lagrangian density. A simplification arises in the Wess-Zumino gauge, where $C = \chi = \bar\chi = M = N = 0$.

2.2 Supersymmetric gauge theories

Gauge invariant supersymmetric actions in the Wess-Zumino gauge are of the form [12]

$$S = \int d^4x \left[\mathcal{L}_D + \mathcal{L}_F + h.c.\right], \qquad (9)$$

where

$$\mathcal{L}_D = \int d^2\theta d^2\bar\theta \Phi^\dagger e^{gV}\Phi =$$
$$(\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}^\mu\phi) + i\psi\sigma^\mu\mathcal{D}^\dagger_\mu\bar\psi + F^\dagger F + i\sqrt{2}g(\phi^\dagger\lambda\psi - \bar\psi\bar\lambda\phi) + g\phi^\dagger D\phi \qquad (10)$$

with

$$\mathcal{D}_\mu\phi = \partial_\mu\phi + igV_\mu\phi \qquad (11)$$

and

$$\mathcal{L}_F = \int d^2\theta \left[W(\Phi) + \frac{1}{32g^2}W^{A\alpha}W^A_\alpha\right] \qquad (12)$$

where $W(\Phi)$ is the superpotential, which is at most tri-linear for a renormalizable theory, and W^A_α is the field strength tensor superfield; g is the gauge coupling. The field strength tensor is defined as

$$W_\alpha = \begin{cases} \bar D^2\, e^{-gV}\, D_\alpha\, e^{gV}, & (\mathrm{non-abelian}) \\ \bar D^2\, D_\alpha\, V, & (\mathrm{abelian}). \end{cases} \qquad (13)$$

Field strength is actually a chiral superfield. In the Wess-Zumino gauge it reads

$$\begin{aligned} W_\alpha^A(y,\theta) &= \bar{D}^2 D_\alpha V^A + igf^{ABC}\bar{D}(D_\alpha V^B)V^C \\ &= 4i\lambda_\alpha^A(y) - 4\theta_\alpha D^A(y) - 2i(\sigma^\mu\bar{\sigma}^\nu)_{\alpha\beta}\theta^\beta V_{\mu\nu}^A(y) + 4\theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{D}_\mu\bar{\lambda}^{A\dot{\alpha}}(y), \end{aligned} \tag{14}$$

where

$$\begin{aligned} V_{\mu\nu}^A &= \partial_\mu V_\nu^A - \partial_\nu V_\mu^A - gf^{ABC}V_\mu^B V_\nu^C \\ \mathcal{D}_\mu\bar{\lambda}^{A\dot{\alpha}} &= \partial_\mu\bar{\lambda}^{A\dot{\alpha}} - gf^{ABC}V_\mu^B\bar{\lambda}^{C\dot{\alpha}} \\ y^\mu &= x^\mu + i\theta\sigma^\mu\bar{\theta}. \end{aligned} \tag{15}$$

The field strength term leads to a Lagrangian

$$\mathcal{L}_V = \frac{1}{32g^2}\left[\int d^2\theta W^{A\alpha}W_\alpha^A + h.c.\right] = -\frac{1}{4}V_{\mu\nu}^A V^{A\mu\nu} - i\lambda^A\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^A + \frac{1}{2}D^A D^A, \tag{16}$$

where

$$V_{\mu\nu}^A = \partial_\mu V_\nu^A - \partial_\nu V_\mu^A + igf^{ABC}V_\mu^B V_\nu^C. \tag{17}$$

The scalar potential gained from the action Eq. (9) is

$$V = \sum_i F_i^\dagger F_i + \frac{1}{2}\sum_A D^A D^A, \tag{18}$$

where F and D terms are defined by [12]

$$F_i^\dagger = -\frac{\partial W(\phi)}{\partial\phi_i}, \qquad D^A = \sum_i g_A\phi_i^\dagger T^A\phi_i + \xi, \tag{19}$$

where ϕ_i are the scalar components of the corresponding chiral superfields Φ_i, g_A are the gauge couplings of the corresponding gauge group, and ξ is a Fayet-Iliopoulos term which vanishes for non-abelian symmetry but can be non-zero for an abelian symmetry group.

In principle different superfields can carry different representations of the same gauge group, in which case one has to use the corresponding form of T^A in Eq. (19) to the representation. For instance, the generators of the complex conjugate representation are the negatives of the transposed generators $-(T^A)^T$. The flat directions are the field configurations for which Eq. (18) vanishes.

2.3 Supergravity

The field configurations that are flat in globally supersymmetric limit are no longer flat when the SUSY is made local *i.e.* in supergravity. Since supergravity is a non-renormalizable theory, one has to include non-renormalizable terms in the superpotential, and kinetic terms, too. Then the most general Lagrangian that is globally supersymmetric and gauge invariant is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\, J\left(\Phi^\dagger e^{gV}, \Phi\right) + \int d^2\theta\, \left(W(\Phi) + f_{AB}(\Phi) W^{A\alpha} W^B_\alpha + h.c.\right), \tag{20}$$

where $f_{AB}(\Phi)$ is an arbitrary function of the chiral superfields. It transforms like a chiral superfield under SUSY and is a symmetric product of two adjoint representations with respect to the gauge group. If the theory is renormalizable, then $f_{AB}(\Phi) = \delta_{AB}$. J is real superfield and leads to a renormalizable theory only if $J = \Phi^\dagger e^{gV}\Phi$. The superpotential $W(\Phi)$ is a chiral superfield which is a polynomial of degree less or equal to three, if the theory is renormalizable.

The coupling of supergravity to matter is very complicated in the most general form. We give here only the bosonic part of the Lagrangian; for the rest see [12]:

$$\begin{aligned} e^{-1}\mathcal{L}_B &= M_p^4 e^{-G}\left(3 + G_{\bar{k}}(G^{-1})_l^{\bar{k}} G^l\right) - \frac{1}{2} M_p^4 (\mathrm{Re}\, f^{-1}_{AB})(G^i T_i^{Aj}\phi_j)(G^k T_k^{Bl}\phi_l) \\ &\quad - \frac{1}{4}(\mathrm{Re}\, f_{AB}) V^A_{\mu\nu} V^{B\mu\nu} + \frac{i}{4}(\mathrm{Im}\, f_{AB})\epsilon^{\mu\nu\sigma\xi} V^A_{\mu\nu} V^B_{\sigma\xi} \\ &\quad - M_p^2 G^i_{\bar{j}} (D_\mu\phi_i)(D^\mu\phi^{*\bar{j}}) - \frac{1}{2M_p^2} R, \end{aligned} \tag{21}$$

where $M_p = 2.4 \cdot 10^{18}$ GeV is the reduced Planck mass, R is the Ricci scalar, the covariant derivatives D_μ are covariant with respect to gravity and gauge group and $G_{\bar{k}} = \partial G/\partial\phi^{*\bar{k}}$, $G^k = \partial G/\partial\phi_k$ and $G^l_{\bar{k}} = \partial^2 G/\partial\phi_l\partial\phi^{*\bar{k}}$. The real function of the scalar fields $G(\phi^\dagger, \phi)$, usually called the Kähler potential, is defined by

$$G = 3\log\left(-\frac{J}{3}\right) - \log(|W|^2). \tag{22}$$

Usually one gives the Kähler potential in the form

$$G(\phi^\dagger, \phi) = -\frac{K(\phi^\dagger, \phi)}{M_p^2} - \log\left(\frac{|W|^2}{M_p^6}\right), \tag{23}$$

where K is also called a Kähler potential. Since we are only interested in scalar field dynamics we rewrite the scalar part of the Lagrangian Eq. (21) using Eq. (23) as

$$e^{-1}\mathcal{L} = \frac{1}{M_p^2}(D_\mu \phi_i) K^i_{\bar{j}} (D^\mu \phi^{\bar{j}*}) - V, \tag{24}$$

where

$$\begin{aligned} V &= e^{\frac{K}{M_p^2}} \left[(D_{\bar{i}} W)^* (K^{-1})^{\bar{i}}_j (D^j W) - \frac{3}{M_p^2} |W|^2 \right] + \frac{1}{2} (\mathrm{Re}\, f^{-1}_{AB}) D^A D^B, && (25) \\ D^j W &= W^j + \frac{K^j}{M_p^2} W, && (26) \\ D^A &= K^i T_i^{Aj} \phi_j + \xi, && (27) \end{aligned}$$

where ξ is the Fayet-Iliopoulos term. The choice of $K^i_{\bar{j}} = \delta^i_{\bar{j}}$ leads to minimal kinetic terms. However, with a non-renormalizable theory, such as supergravity, one should take into account the possibility of non-minimal kinetic terms.

2.4 Minimal Supersymmetric Standard Model: MSSM

MSSM is the minimal supersymmetric extension of the familiar Standard Model of particle physics. This means that there exist matter superfields, which are chiral, as follows: quark doublets $Q_i^{\alpha A}$ transforming as $(\mathbf{3}, \mathbf{2}, \frac{\mathbf{1}}{\mathbf{6}})$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$, quark singlets $\bar{u}_{iA}$ $(\bar{\mathbf{3}}, \mathbf{1}, -\frac{\mathbf{2}}{\mathbf{3}})$ and $\bar{d}_{iA}$ $(\bar{\mathbf{3}}, \mathbf{1}, \frac{\mathbf{1}}{\mathbf{3}})$, lepton doublets L_i^α $(\mathbf{1}, \mathbf{2}, -\frac{\mathbf{1}}{\mathbf{2}})$, lepton singlets $\bar{e}_i$ $(\mathbf{1}, \mathbf{1}, \mathbf{1})$, Higgs doublets H_u^α $(\mathbf{1}, \mathbf{2}, \frac{\mathbf{1}}{\mathbf{2}})$ and H_d^α $(\mathbf{1}, \mathbf{2}, -\frac{\mathbf{1}}{\mathbf{2}})$. Gauge fields are in vector superfields. There are three vector superfield multiplets V^A, A^α, B corresponding respectively to $SU(3)_C$ $SU(2)_L$ and $U(1)_Y$. The F part of the potential is obtained from the superpotential

$$W = \sum_{ij} \left(y_u^{ij} \epsilon_{\alpha\beta} Q_i^{\alpha A} H_u^\beta \bar{u}_{jA} + y_d^{ij} \epsilon_{\alpha\beta} Q_i^{\alpha A} H_d^\beta \bar{d}_{jA} + y_e^{ij} \epsilon_{\alpha\beta} L_i^\alpha H_d^\beta \bar{e}_j \right) + \mu \epsilon_{\alpha\beta} H_u^\alpha H_d^\beta, \tag{28}$$

where $y^{ij}_{u,d,e}$ are the Yukawa matrices and $\mu \sim M_W$. It is also possible to add the following terms to the superpotential Eq. (28)

$$\tilde{W} = \epsilon_{\alpha\beta} Q^{\alpha A} L^\beta \bar{d}_A + \epsilon_{\alpha\beta} L^\alpha L^\beta \bar{e} + \epsilon^{ABC} \bar{u}_A \bar{d}_B \bar{d}_C. \tag{29}$$

These terms are compatible with the $SU(3) \times SU(2) \times U(1)$ -symmetry, but violate explicitly the baryon and lepton number. Their couplings would have to be extremely small in order to be compatible with experimental limits. The terms in Eq. (29) are usually left out by imposing a discrete symmetry called R-parity, where $R = (-1)^{3B+L+2S}$. We usually ignore Eq. (29).

The field strengths related to the gauge fields are obtained from Eq. (14) as

$$\begin{aligned} W_\alpha^A &= \bar{D}^2 D_\alpha V^A + i g_3 f^{ABC} \bar{D}^2 (D_\alpha V^B) V^C, \\ W_\alpha^\beta &= \bar{D}^2 D_\alpha W^\beta + i g_2 \epsilon^{\beta\gamma\delta} \bar{D}^2 (D_\alpha W^\gamma) W^\delta, \\ B_\alpha &= \bar{D}^2 D_\alpha B. \end{aligned} \tag{30}$$

The pure gauge Lagrangian is formed as in Eq. (12) and reads

$$\mathcal{L}_V = \frac{1}{32}\left(\frac{1}{g_3^2} W^{A\alpha} W_\alpha^A + \frac{1}{g_2^2} W^{\beta\alpha} W_\alpha^\beta + \frac{1}{g_1^2} B^\alpha B_\alpha + h.c.\right)_F. \tag{31}$$

The interaction between gauge and matter multiplets is given by Eq. (16) so that

$$\begin{aligned} \mathcal{L}_\Phi &= \Bigg[\sum_i \Bigg(Q_i^\dagger \exp(i g_3 T^A V^A + i g_2 \tau^\alpha W^\alpha + \frac{1}{6} i g_1 B) Q_i \\ &+ \bar{u}_i^\dagger \exp(-i g_3 (T^A)^T V^A - \frac{2}{3} g_1 B) \bar{u}_i + \bar{d}_i^\dagger \exp(-i g_3 (T^A)^T V^A + \frac{1}{3} g_1 B) \bar{d}_i \\ &+ L_i^\dagger \exp(i g_2 \tau^\alpha W^\alpha - \frac{1}{2} i g_1 B) L_i + \bar{e}_i^\dagger \exp(i g_1 B) \bar{e}_i \Bigg) \\ &+ H_u^\dagger \exp(i g_2 \tau^\alpha W^\alpha + \frac{1}{2} i g_1 B) H_u + H_d^\dagger \exp(i g_2 \tau^\alpha W^\alpha - \frac{1}{2} i g_1 B) H_d \Bigg]_D . \end{aligned} \tag{32}$$

2.5 SUSY breaking

From experiments we know that supersymmetry is necessarily broken in our world. Therefore one needs to understand how SUSY is broken. One would also like to retain some of the properties of SUSY (such as the cancellation of quadratic divergences) while breaking it. This can be achieved by the soft SUSY breaking terms [39]

$$\begin{aligned} V_{\text{soft}} &= m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_L^2 |L|^2 + m_e^2 |e|^2 + m_Q^2 |Q|^2 + m_u^2 |\bar{u}|^2 + m_d^2 |\bar{d}|^2 \\ &+ \big(A_u m_{3/2} y_u^{ij} H_u^\alpha Q_{i\alpha}^a \bar{u}_{ja} + A_d m_{3/2} y_d^{ij} H_d^\alpha Q_{i\alpha}^a \bar{d}_{ja} + A_e m_{3/2} y_e^{ij} H_d^\alpha L_{i\alpha} e_j \\ &+ B m_{3/2} \mu H_u^\alpha H_{d\alpha} + h.c.\big) + \frac{1}{2} M_1 \bar{\lambda}_1 \lambda_1 + \frac{1}{2} M_2 \bar{\lambda}_2 \lambda_2 + \frac{1}{2} M_3 \bar{\lambda}_3 \lambda_3, \end{aligned} \tag{33}$$

where the fields are now the scalar components of the corresponding chiral superfields and λ_i are the gauginos. However, the origin of the breaking terms is left unexplained.

Another approach would be to break SUSY spontaneously. However, a non-supersymmetric vacuum has positive energy, which can be cosmologically problematic (although nowadays a positive cosmological constant is experimentally favourable). More problems arise from the supertrace relations [12]. For instance, a pure F-term breaking in renormalizable theories yields at tree level [40]

$$STr\mathcal{M}^2 = \sum J(-1)^{2J}(2J+1)Tr\mathcal{M}_J^2 = 0, \tag{34}$$

where $\mathcal{M}_J^2$ is the mass matrix of all particles with spin J. Because of Eq. (34) making the superpartners of the usual fermions heavy enough is very difficult. Therefore the only option seems to be the soft SUSY breaking terms.

The origin of the soft terms is usually assumed to be in a sector that is "hidden" to us. The hidden sector then couples to our "visible" sector via loops or non-renormalizable couplings thus avoiding the supertrace constraint. There are of course other constraints on the breaking terms in the Lagrangian; for example the flavour changing neutral currents have to be suppressed [7].

One hidden sector mechanism is based on the gravity mediation of SUSY breaking [12] from the hidden to the visible sector. A minimal Kähler potential for the hidden and visible sector fields gives rise to the required soft SUSY breaking parameters [12]. The typical mass scale of the superpartners is given by the gravitino mass $m_{3/2}$.

Another type of mechanism is based on gauge mediation [41]. In this case the supersymmetric partners of the SM receive the dominant part of their mass via gauge interactions with the hidden sector. For instance, a superpotential which includes a term

$$W = \lambda X\bar{\Psi}\Psi + \ldots, \tag{35}$$

where Ψ is a superfield with SM couplings but does not belong to the spectrum of MSSM, and X a SM singlet superfield in the hidden sector that gets a non-zero VEV in the D- and F-directions. The scalar VEV of X gives masses to the fermionic component of Ψ. The masses of the scalar components come from the VEVs of the scalar and F-components of X. The gauginos of MSSM then get mass corrections at the 1-loop level with $m \sim \Lambda \equiv < F_X > / < X >$. The scalars obtain masses $m \sim \Lambda$ at the 2-loop level. The tri-linear soft terms also arise at the 2-loop level.

3 Flat directions

Flat directions, also called the moduli space, are supersymmetric minima of the scalar potential. Since SUSY is spontaneously broken, supersymmetric flat directions correspond to field configurations whose D and F terms vanish, see Eqs. $(18, 19)$. In reality the flat directions are lifted by supersymmetry breaking effects. This means that the degeneracy along a flat direction is broken and a potential for the flat direction is generated. Another mechanism is to add new non-renormalizable terms into the superpotential. The coupling to supergravity induces SUSY breaking in the Early Universe, which provides its own contribution to the effective potential of the flat direction.

3.1 Flat directions in general

We start with general considerations. Let us take N chiral superfields X_i, which transform under some gauge group G as a (in general reducible) representation in which the generators of the Lie gauge algebra are matrices T^A. In principle all the flat directions of the model can be found by solving directly the constraints (ignoring now the Fayet-Iliopoulos term)

$$D^A = \sum_{ij} x_i^* T_{ij}^A x_j = 0, \tag{36}$$

$$F_{x_i}^* = -\frac{\partial W(x)}{\partial x_i} = 0 \tag{37}$$

where x_i are the scalar components of the superfields X_i. We first describe an example of a direct solution to Eq. (36).

Example 3.1 *$SU(N)$ gauge theory with squarks, q_i^A with $i = 1, \ldots, m$ and $A = 1, \ldots, N$, in* $\mathbf{N}$ *and anti-squarks, $\bar{q}_{j\bar{A}}$ with $j = 1, \ldots, n$ and $\bar{A} = 1, \ldots, N$, in* $\bar{\mathbf{N}}$ *[13].*

One can solve the D-term condition Eq. (36) by introducing a basis for the Lie algebra of $SU(N)$. It consists of traceless Hermitean matrices, and we write

$$(T_{\bar{B}}^A)_D^{\bar{C}} = \delta_D^A \delta_{\bar{B}}^{\bar{C}} - \frac{1}{N} \delta_{\bar{B}}^A \delta_D^{\bar{C}}, \qquad A,\, \bar{B},\, \bar{C},\, D = 1, \ldots, N. \tag{38}$$

The complex conjugate representation is generated by $-(T^A_{\bar{B}})^* = -(T^A_{\bar{B}})^T$. Using the definition of generators Eq. (38) in Eq. (36) one obtains after some manipulation (see [13] for details)

$$\sum_{i=1}^{m} q^*_{i\bar{B}} q^A_i - \sum_{j=1}^{n} \bar{q}^{*A}_j \bar{q}_{j\bar{B}} = \frac{1}{N}\delta^A_{\bar{B}} \sum_{C=1}^{N}\left[\sum_{i=1}^{m} |q^C_i|^2 - \sum_{j=1}^{n} |\bar{q}_{jC}|^2\right] \equiv k\delta^A_{\bar{B}}, \qquad (39)$$

where k is a constant. Eq. (39) is an orthogonality constraint with respect to the $U(m,n)$-metric of N vectors of the form $(q^A_1, \ldots, q^A_m, \bar{q}^{*A}_1, \ldots, \bar{q}^{*A}_n)$ with $A = 1, \ldots, N$. Here the N vectors all have the same norm.

3.2 Flat directions in terms of gauge invariant polynomials

Solving Eq. (36) directly is quite simple in the case of $SU(N)$ symmetry in the defining representation. However, one always has to give the form of the generators before one can start to solve Eq. (36). For representations other than the fundamental one the form of the generators is more complicated, not to mention other gauge groups. For example in $SU(5)$ GUT the particles are in representations $\mathbf{10}$ and $\bar{\mathbf{5}}$. When discussing the AD mechanism, we would have to introduce a non-renormalizable operator that lifts the flat direction. One has to add some polynomial to the superpotential, whose F-term lifts the degeneracy of the solution of Eqs. (36, 37). A direct solution of Eq. (36) does not yield such a polynomial which has to be found separately.

Another way of parameterizing the flat directions relies on the correspondence of the D flat directions and gauge invariant holomorphic polynomials of the chiral superfields X_i [13, 42–47]. The smaller moduli space of D and F flat directions is parameterized by the same basis of monomials and redundancy constraints as the D flat directions alone and subjected to the additional constraints following from $F_{x_i} = 0$. One should note that this discussion applies only for gauge groups without Fayet-Iliopoulos terms. The case with non-zero Fayet-Iliopoulos term should be considered separately.

We give simple arguments why gauge invariant polynomials produce D flat directions, without going into the technical details which can be found in [43–47]. Let $I(X)$

be a gauge invariant polynomial. Then under an infinitesimal gauge transformation the scalar component transforms in the following way:

$$\delta I(x) = \sum_i \frac{\partial I(x)}{\partial x_i}\delta x_i = \sum_{ij} \frac{\partial I(x)}{\partial x_i}\epsilon^A T^A_{ij} x_j = 0, \tag{40}$$

where ϵ^a are infinitesimal parameters. Eq. (40) vanishes by virtue of the gauge invariance of the polynomial. Note that Eq. (40) resembles the D flatness constraint Eq. (36). As first noted in [13], Eq. (40) is equivalent to Eq. (36), if

$$\frac{\partial I(x)}{\partial x_i} = C(x_i)^* \tag{41}$$

for some $C \neq 0$. Therefore if there is a gauge invariant polynomial satisfying Eq. (40), then the D-term, Eq. (36), automatically vanishes. The reverse of this statement, *i.e.* that all the D-flat directions can be parameterized by gauge invariant polynomials, was first conjectured in [13]. The proof of this conjecture requires a considerable amount of mathematics. We just outline the method of the proof.

"Proof of the conjecture"

As noted in Eq. (39), the D-flatness constraint is actually an orthogonality condition with respect to the complex scalar product $< x, y >= \sum_i x_i^* y_i$. In this respect the vector x is orthogonal to $T^A x$ at x if the D-term vanishes. T^A are generators of the Lie algebra, which from the point of view of differential geometry are tangents of the curves in the Lie group, of which all the elements in the Lie group can be generated. The orbit of x, $\{gx | g \in G_{\mathbb{C}}\}$, under the action of the complexified Lie group forms a surface in the representation space.[2] So x is now orthogonal to the orbit of x at the points where the D-term vanishes. Surfaces generated by an action of the complexified Lie group can be parameterized as $I(x) = C$, where I is a polynomial and C is a constant, meaning that the surfaces are algebraic [48]. The complex conjugate of the

[2]The complexified Lie algebra is generated by T^A and iT^A, where T^A are generators of Lie group, and the vanishing D-term vanishes with generators of the complexified Lie algebra if it vanishes with Lie algebra. The complexified Lie group is acquired by exponentiating the complexified Lie algebra. The need for the complexification arises because for example with $SU(N)$ one obtains $|x| = |g \cdot x|$ for all $g \in SU(N)$ and $x \in \mathbb{C}^N$. From this follows that $< x, T^A x >= 0$ for all $x \in \mathbb{C}^N$ and $T^A \in Lie(SU(N))$. Complexification fixes this ambiguity.

gradient of the function, $\nabla I(x) = \hat{e}_i \partial_i I(x)$, is always orthogonal to the level surface $I(x) = C$. So the points where $(\nabla I(x))^*$ is parallel to x are the points where the D-term vanishes. In summary, the points where D-term vanishes are the same as the points x, which are orthogonal to their orbit under the complexified Lie group, which in turn are the same points where the gradient is parallel to the complex conjugate of the field, $\nabla I(x) = Cx^*$ with $C \neq 0$. Hence the conjecture is "proven".[3]

We next give a few examples of how to apply Eq. (41) in the case of an $SU(N)$ group before giving the complete catalogue of the basis of gauge invariant monomials of MSSM in Section 3.3. The only thing one needs in order to apply the method of [13] is the list of the gauge invariant tensors of the Lie group. These can be generated as sums, products and contractions of the so-called primitive tensors of the group, and they are listed in [49].

Example 3.2 *Abelian $U(1)$ gauge theory with two charged superfields Φ_+ and Φ_-.*

This is a rather trivial example, but it is useful as a first example. The D flat directions in this case are obtained simply by directly solving Eq. (36)

$$D = |\phi_+|^2 - |\phi_-|^2 = 0 \Rightarrow |\phi_+| = |\phi_-|, \tag{42}$$

where $\phi_\pm$ is the scalar component of $\Phi_\pm$. Now we show how to use gauge invariant polynomials. All the $U(1)$ gauge invariant polynomials of this example are generated by the product $\Phi_+\Phi_-$. All the powers of this product actually parameterize the same flat direction, so for simplicity we take $I(\Phi_+, \Phi_-) \equiv I(\Phi_+\Phi_-) = \Phi_+\Phi_-$. Now the conditions of Eq. (41) become

$$\begin{aligned} \frac{\partial I}{\partial \phi_+} &= \phi_- = C\phi_+^* \\ \frac{\partial I}{\partial \phi_-} &= \phi_+ = C\phi_-^*. \end{aligned} \tag{43}$$

One should now note that it is essential in Eq. (41) that C is the same for all derivatives. Eq. (43) is easy to solve by multiplying the first equation with ϕ_+ and the

[3]Several mathematical details have been omitted in the "proof". The most important one is why all the surfaces can be parameterized as $I(x) = C$, which can be found at [48].

second equation by ϕ_-, thus resulting in Eq. (42). If one adds to the superpotential some power of the polynomial I, then the F-term constraint Eq. (37) produces a term proportional to the left hand side of Eq. (43) and thus causes the right hand side to vanish. This is a generic feature with gauge invariant polynomials.

Example 3.3 *Non-abelian $SU(N)$ gauge theory with matter superfields Φ_i^A and $\bar{\Phi}_{j\bar{A}}$ transforming according to the defining N representation and complex conjugate $\bar{N}$ representation, respectively.*

This was already solved in Example 3.1, but here we demonstrate the use of gauge invariant tensors. The primitive tensors of $SU(N)$ are the Kronecker delta $\delta_{\bar{B}}^{A}$ and the Levi-Civita tensor $\epsilon_{A_1\cdots A_N}$, $\epsilon^{\bar{A}_1\cdots\bar{A}_N}$ in N dimensions [49]. An extra simplification is provided by the fact that the product of Levi-Civita tensors produces a generalized Kronecker delta

$$\epsilon_{A_1\cdots A_N}\epsilon^{\bar{A}_1\cdots\bar{A}_N} = \delta_{A_1\cdots A_N}^{\bar{A}_1\cdots\bar{A}_N} = \delta_{A_1}^{\bar{A}_1}\cdots\delta_{A_N}^{\bar{A}_N} \pm (\text{permutations}). \tag{44}$$

Let us assume that there are m matter superfields Φ_i^A, $i = 1, \ldots, m$ and n anti-matter superfields $\bar{\Phi}_{j\bar{A}}$, $j = 1, \ldots, n$, which we contract in all possible ways with the primitive tensors. Thus the generating monomials are

$$\begin{aligned} (\Phi_i\bar{\Phi}_j) &\equiv \Phi_i^A\delta_A^{\bar{A}}\bar{\Phi}_{j\bar{A}}, \\ (\Phi_{i_1}\cdots\Phi_{i_N}) &\equiv \epsilon_{A_1\cdots A_N}\Phi_{i_1}^{A_1}\cdots\Phi_{i_N}^{A_N}, \\ (\bar{\Phi}_{i_1}\cdots\bar{\Phi}_{i_N}) &\equiv \epsilon^{\bar{A}_1\cdots\bar{A}_N}\bar{\Phi}_{i_1\bar{A}_1}\cdots\bar{\Phi}_{i_N\bar{A}_N}. \end{aligned} \tag{45}$$

Let us first study the gauge invariant monomial $(\Phi_i\bar{\Phi}_j)$. The D flatness constraint Eq. (36) now reads

$$\begin{aligned} \phi_i^A &= C\bar{\phi}_j^{*A} \\ \bar{\phi}_{jA} &= C\phi_{iA}^{*}, \qquad A = 1, \ldots, N, \end{aligned} \tag{46}$$

where ϕ_i^A ($\bar{\phi}_{j\bar{A}}$) is the scalar component of Φ_i^A ($\bar{\Phi}_{j\bar{A}}$). By solving C from both equations one can see that Eq. (46) is the same as Eq. (39) with a zero norm with respect to

$U(1,1)$. By substituting $\bar{\phi}_{jA}$ from the second equation to the first one obtains $|C| = 1$. Now the flat direction can be parameterized by N complex scalar fields ϕ_i^A and a real phase δ_c. We can further reduce the number of independent fields by choosing a gauge. With an $SU(N)$ transformation one can choose $N-1$ of the components of ϕ_i to vanish *i.e.* $\phi_i^A = 0$ for $A = 2, \ldots, N$. What is left is essentially a complex scalar field $\phi_i^1 \equiv \phi$, a real phase δ_C and $\bar{\phi}_{j1} = e^{i\delta_C}\phi^*$. One can still transform the phase δ_C away in such a way that $\bar{\phi}_{j1} = \phi_i^1 = \phi$. So finally, the flat direction $(\Phi_i \bar{\Phi}_j)$ is parameterized by one complex scalar field ϕ up to gauge transformations.

Let us next take $I(\Phi_{i_1}, \ldots, \Phi_{i_N}) = (\Phi_{i_1} \cdots \Phi_{i_N})$. All the flavour indices i_j have to be different, *i.e.* $i_j \neq i_k$, for $j \neq k$ because of the anti-symmetrization in Eq. (45). Because of this the number of flavours, m, has to be larger than or equal to the number of colours, N. Now the D flatness constraint Eq. (41) reads

$$\epsilon_{A_1 \cdots A_N} \phi_{i_1}^{A_1} \cdots \phi_{i_{k-1}}^{A_{k-1}} \phi_{i_{k+1}}^{A_{k+1}} \cdots \phi_{i_N}^{A_N} = C \phi_{i_k}^{*A_k}, \qquad k = 1, \ldots, N. \tag{47}$$

By multiplying Eq. (47) with $\phi_{i_k}^{A_k}$ and summing over A_k one obtains

$$(\phi_{i_1} \cdots \phi_{i_N}) = C |\phi_{i_j}|^2, \qquad \text{for all } i_j = 1, \ldots, m. \tag{48}$$

Hence one sees that all the vectors ϕ_{i_j} have the same length, $|\phi_{i_j}| \equiv |\phi|$. Let us now multiply Eq. (47) with $\phi_{i_j}^{A_k}$ and sum over A_k, where $k \neq j$. One then obtains

$$C < \phi_{i_j}, \phi_{i_k} >= \epsilon_{A_1 \cdots A_N} \phi_{i_1}^{A_1} \cdots \phi_{i_{k-1}}^{A_{k-1}} \phi_{i_j}^{A_k} \phi_{i_{k+1}}^{A_{k+1}} \cdots \phi_{i_j}^{A_j} \cdots \phi_{i_N}^{A_N} = 0, \tag{49}$$

because A_j indices are antisymmetrized. Thus all the fields have the same length and are orthogonal. There is one more constraint, which follows when one takes the absolute value squared of Eq. (47) and sums over A_k. One obtains

$$\begin{aligned} |C|^2 |\phi_{i_k}|^2 &= \sum_{A,B} \epsilon_{A_1 \cdots A_{k-1} A_k A_{k+1} \cdots A_N} \phi_{i_1}^{A_1} \cdots \phi_{i_{k-1}}^{A_{k-1}} \phi_{i_{k+1}}^{A_{k+1}} \cdots \phi_{i_N}^{A_N} \times \\ &\quad \epsilon_{B_1 \cdots B_{k-1} A_k B_{k+1} \cdots B_N} \phi_{i_1}^{B_1 *} \cdots \phi_{i_{k-1}}^{B_{k-1}} \phi_{i_{k+1}}^{B_{k+1}} \cdots \phi_{i_N}^{B_N} \\ &= \delta_{A_1 \cdots A_{k-1} A_{k+1} \cdots A_N}^{B_1 \cdots B_{k-1} B_{k+1} \cdots B_N} \phi_{i_1}^{A_1} \cdots \phi_{i_{k-1}}^{A_{k-1}} \phi_{i_{k+1}}^{A_{k+1}} \cdots \phi_{i_N}^{A_N} \phi_{i_1}^{B_1 *} \cdots \phi_{i_{k-1}}^{B_{k-1}} \phi_{i_{k+1}}^{B_{k+1}} \cdots \phi_{i_N}^{B_N} \\ &= |\phi_{i_1}|^2 \cdots |\phi_{i_{k-1}}|^2 |\phi_{i_{k+1}}|^2 \cdots |\phi_{i_N}|^2 = |\phi|^{2(N-1)}, \end{aligned} \tag{50}$$

from which follows that $|C| = |\phi|^{N-2}$. In the third line only the combination with $A_1 = B_1, \ldots, A_N = B_N$ survives and all the other combinations vanish because of Eq. (49). Altogether the flat direction is parameterized by N complex scalar fields which are orthogonal and have the same normalization $< \phi_{i_j}, \phi_{i_k} >= |\phi|^2 \delta_{jk}$. Choosing a gauge one can take

$$\phi_{i_j}^T = (0 \cdots 0\, |\phi| e^{i\delta_j}\, 0 \cdots 0) \tag{51}$$

and fix $N-1$ of the phases δ_j, $j = 2, \ldots, N$ to be equal to $\delta_1 \equiv \arg \phi$. The phase of C is determined from Eq. (47) by inserting Eq. (51). One then obtains $\delta_C = \sum \delta_j$. Hence again the flat direction is parameterized by one complex scalar field ϕ such that $\phi_{i_j}^k = \phi \delta_j^k$. The flat direction $(\bar{\Phi}_{i_1} \cdots \bar{\Phi}_{i_N})$ is similar. These flat directions can again be lifted by adding a suitable monomial to the superpotential.

3.3 Flat directions of the MSSM

In the MSSM there are many flat directions at the renormalizable level. However, these are only approximately flat, since supersymmetry breaking lifts the flat directions. There are also other effects that lift the flat directions, as will be discussed in the Section 3.4. In the present Section we just catalogue the basis of flat directions of the MSSM in the globally supersymmetric limit.

The field content of MSSM was discussed in Section 2.4. Here we apply the method of Section 3.2 to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry group of MSSM. We form the basis of gauge invariant monomials by first forming $SU(3)$ invariant combinations, then out of these $SU(2)$ invariant combinations and finally from the $SU(3) \times SU(2)$ invariant monomials the $U(1)$ invariant combinations. Then by applying the F flatness constraints one obtains the basis of gauge invariant monomials of the MSSM. This analysis was carried out by Gherghetta, Kolda and Martin [50]. Here we only review the main points. The final basis of flat directions can be found in Table 1 at the end of this Section.

Under $SU(3)_C$ the chiral superfields transform as singlets ($\bar{e}_i$, L_i, H_u, H_d), triplets (Q_i) or antitriplets ($\bar{u}_i$, $\bar{d}_i$). As mentioned in Section 3.1, the gauge invariant tensors of $SU(3)$ are ϵ_{ABC}, ϵ^{ABC} and δ^B_A. Contracting these with the quark superfields one obtains

$$\begin{aligned}
(Q_i\bar{q}_j) &\equiv Q_i^A \bar{q}_{jA}(Q_i\bar{q}_j), &(52)\\
(Q_iQ_jQ_k) &\equiv \epsilon_{ABC}Q_i^AQ_j^BQ_k^C, &(53)\\
(\bar{q}_i\bar{q}_j\bar{q}_k) &\equiv \epsilon^{ABC}\bar{q}_{iA}\bar{q}_{jB}\bar{q}_{kC}, &(54)
\end{aligned}$$

where $\bar{q}_i = \bar{u}_i, \bar{d}_i$. These $SU(3)$ -invariant products of chiral superfields in Eq. (52) generate a reducible representation of $SU(2)$ in general. The doublet representation can be reduced into irreducible parts. The product of $(Q_i\bar{q}_j)$ naturally gives a doublet, since $\bar{q}_j$ is a singlet under $SU(2)$. The product $(\bar{q}_i\bar{q}_j\bar{q}_k)$ is also a singlet under $SU(2)$ and thus invariant with respect to it. The product $(Q_iQ_jQ_k)$ reduces as

$$\mathbf{2}\times\mathbf{2}\times\mathbf{2} = \mathbf{2}+\mathbf{2}+\mathbf{4}. \tag{55}$$

Here $\mathbf{4}$ is a symmetric representation under $SU(2)$ and thus $(Q_iQ_jQ_k)_\mathbf{4}$ has to be antisymmetric in its family indices. Therefore there is a unique $SU(3)_C$-singlet made out of three Q's, which is a $\mathbf{4}$ under $SU(2)_L$, namely

$$(Q_iQ_jQ_k)_4^{(\alpha\beta\gamma)} \equiv Q_i^{A\alpha}Q_j^{B\beta}Q_k^{C\gamma}\epsilon_{ABC}\epsilon^{ijk}. \tag{56}$$

The remaining combinations are $SU(2)_L$ doublets, where one pair of Q's has been antisymmetrized, *i.e.* contracted with $\epsilon_{\alpha\beta}$, and can be written in the form

$$(Q_iQ_jQ_k)^\alpha \equiv Q_i^{A\beta}Q_j^{B\gamma}Q_k^{C\alpha}\epsilon_{ABC}\epsilon_{\beta\gamma}, \tag{57}$$

which is subject to the constraint that not all three of the family indices are allowed to be the same due to the anti-symmetrization. In Eq. (57) there are two other possibilities to contract with $\epsilon_{\alpha\beta}$ to produce a $\mathbf{2}$ under $SU(2)$. With the antiquark fields $\bar{q}$'s the situation is simpler, since they are $SU(2)$-singlets, *i.e.* invariant under $SU(2)_L$. The rest of the superfields are either doublets or singlets under $SU(2)$.

Now the $SU(3)_C\times SU(2)_L$ flat directions can be formed by combining $(Q_iQ_jQ_k)_4^{\alpha\beta\gamma}$, $(Q_iQ_jQ_k)^\alpha$, $(Q_i\bar{q}_j)^\alpha$, L^α, H_u^α and H_d^α into $SU(2)$ -singlets. This is achieved by contracting all the terms with $\epsilon_{\alpha\beta}$, for example $\epsilon_{\alpha\beta}(Q_iQ_jQ_k)^\alpha L_m^\beta$. Then it is already trivial to form the $U(1)_Y$ invariant combinations out of the $SU(3)\times SU(2)$ -invariant combinations, for example $\epsilon_{\alpha\beta}H_d^\alpha L_i^\beta \bar{e}_j$. For details see [50].

Let us now consider a couple of examples of the field configurations of the MSSM D flat directions such as $\epsilon_{\alpha\beta}Q_i^{A\alpha}L_j^\beta \bar{d}_{kA}$ and $\epsilon_{\alpha\beta}H_u^\alpha L_i^\beta$, which lead to the field configurations (up to a gauge transformation)

$$Q_i^1=\begin{pmatrix}\frac{1}{\sqrt{3}}\phi\\ 0\end{pmatrix},\qquad L_j=\begin{pmatrix}0\\ \frac{1}{\sqrt{3}}\phi\end{pmatrix},\qquad \bar{d}_k^1=\frac{1}{\sqrt{3}}\phi,$$
$$H_u=\begin{pmatrix}\frac{1}{\sqrt{2}}\phi\\ 0\end{pmatrix},\qquad L_i=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\phi\end{pmatrix}. \tag{58}$$

We now have to apply the constraints coming from F-terms, Eq. (37), using the superpotential of the MSSM given in Eq. (28), to the D flat monomials. We require

$$F_{H_u}^\alpha = \mu H_d^\alpha + y_u^{ij}Q_i^\alpha \bar{u}_j = 0, \tag{59}$$
$$F_{H_d}^\alpha = -\mu H_u^\alpha + y_d^{ij}Q_i^\alpha \bar{d}_j + y_e^{ij}L_i^\alpha \bar{e}_j = 0, \tag{60}$$
$$F_{Q_i}^{a\alpha} = y_u^{ij}H_u^\alpha \bar{u}_j^a + y_d^{ij}H_d^\alpha \bar{d}_j^a = 0, \tag{61}$$
$$F_{L_i}^\alpha = y_e^{ij}H_d^\alpha \bar{e}_j = 0, \tag{62}$$
$$F_{\bar{u}_i}^a = y_u^{ji}H_u Q_j^a = 0, \tag{63}$$
$$F_{\bar{d}_i}^a = y_d^{ji}H_d Q_j^a = 0, \tag{64}$$
$$F_{e_i} = y_e^{ji}H_d L_j = 0. \tag{65}$$

Assuming that the Yukawa matrices, $y_{u,d,e}^{ij}$, are non-singular we can cancel them from the constraints Eqs. (62)-(65). When contracting the constraints (59)-(65) to form gauge-invariant combinations one can see that some of the D-flat directions are constrained to vanish. In that case we say that the flat direction is lifted. Eqs. (62)-(65) show that all the flat directions, where H_d is contracted with any other field except H_u, vanish. From the example Eq. (58) one can see that $Q_iL_j\bar{d}_k$ is F flat, too, but $L_iH_d\bar{e}_j$ vanishes by Eqs. (62) and (65) as it contains H_d.

One should note that in general $\mu\sim M_W$, *i.e.* it is of the same order as the soft SUSY breaking masses. Therefore we regard a direction flat, if it is flat up to the μ

term. Essentially this amounts to leaving the μ-term out of the Eqs. (59) and (60) when classifying the flat directions of the MSSM and including it in the terms lifting the flat directions.

One could also include the R-parity violating terms given in Eq. (29) to the superpotential. In that case Eqs. (59)-(65) would become more complicated and the structure of flat directions would change. However, R-parity violating terms break explicitly the baryon and lepton number conservation at the renormalizable level and would lead to a rapid proton decay unless the couplings are fine-tuned.

Table 1: Basis of flat directions of the MSSM

Flat dir.	$B-L$	Flat dir.	$B-L$	Flat dir.	$B-L$
LH_u	-1	$QQQL$	0	$\bar{u}\bar{u}\bar{u}\bar{e}\bar{e}$	1
H_uH_d	0	$Q\bar{u}Q\bar{d}$	0	$Q\bar{u}Q\bar{u}\bar{e}$	1
$\bar{u}\bar{d}\bar{d}$	-1	$Q\bar{u}L\bar{e}$	0	$QQQQ\bar{u}$	1
$LL\bar{e}$	-1	$\bar{u}\bar{u}\bar{d}\bar{e}$	0	$(QQQ)_4LLL\bar{e}$	-1
$Q\bar{d}L$	-1	$\bar{d}\bar{d}\bar{d}LL$	-3	$\bar{u}\bar{u}\bar{d}Q\bar{d}Q\bar{d}$	-1

3.4 Lifting the flat directions

So far we have described the method of how to find all the flat directions given a gauge group and the matter fields. Since the flat directions correspond to continuous degeneracy of vacua, one gets an infinite number of possibilities for gauge symmetry breaking. Supersymmetry breaking effects and non-renormalizable terms lift this degeneracy.

As pointed out before, soft SUSY breaking terms lift the flat directions. However, the F-term constraints Eqs. (59)-(65) make the A-terms in the soft SUSY breaking potential to vanish in the flat direction. Therefore only the mass terms and the B-term can lift the flat directions with

$$\begin{aligned} V_{soft} &= m_{H_u}^2|H_u|^2 + m_{H_d}^2|H_d|^2 + m_L^2|L|^2 + m_e^2|e|^2 + m_Q^2|Q|^2 + m_u^2|\bar{u}|^2 + m_d^2|\bar{d}|^2 \\ &\quad +(Bm_{3/2}\mu\epsilon_{\alpha\beta}H_u^\alpha H_d^\beta + h.c.) + \mu^2|H_u|^2 + \mu^2|H_d|^2. \end{aligned} \tag{66}$$

The gaugino mass terms are ignored, since we are only interested in the scalar part. The B-term is important only for the H_uH_d flat direction. The mass terms coming from the μ-term of the superpotential have been included here, since they were ignored in section 3.3. However, this is important only for the flat directions H_uL and H_uH_d. The soft terms in Eq. (66) generally produce a mass term for the flat direction

$$V(\phi) = m_\phi^2|\phi|^2, \qquad \text{where} \qquad m_\phi^2 = \sum_{i=1}^{N} a_i^2 m_i^2, \tag{67}$$

when there are N fields gaining a VEV along a flat direction. Normalization is such that $\phi_i = \sqrt{a_i}\phi$ and $\sum a_i^2 = 1$.

The easiest way to lift a flat direction is to add an operator, which is formed of the gauge invariant polynomial describing the flat direction. In that case the polynomial will become of the form $I(\Phi) \sim \Phi^n$, if I is composed of a monomial of degree n. If the flat direction is described by a renormalizable operator, then one has to raise it to some power to obtain a non-renormalizable operator, since the form of renormalizable operators in the superpotential is restricted. Of course it is possible that another kind of operator of lower degree can accomplish the lifting, too. In general one obtains a superpotential term of the form

$$W = \frac{\lambda}{dM^{d-3}}\,\Phi^d \tag{68}$$

where $d = nk$, and M is some large mass scale such as the GUT or Planck scale. We generically identify M with the Planck scale $M_p = 2.4 \cdot 10^{18}$ GeV.

There is also another type of operator, which lifts the flat direction. It consists of a field not in the flat direction and some number of fields which make up the flat direction:

$$W = \frac{\lambda}{M^{d-3}}\,\Psi\Phi^{d-1}. \tag{69}$$

For a superpotential of this form F_Ψ is non-zero along the flat direction. In the flat space limit, with minimal kinetic terms, the lowest order contributions of either type of superpotential term, *i.e.* Eqs. (68) and (69), give rise to a potential

$$V(\phi) = \left|\frac{\partial W(\phi)}{\partial \phi}\right|^2 = \frac{|\lambda|^2}{M^{2(d-3)}}\,|\phi|^{2(d-1)}, \tag{70}$$

where ϕ is the scalar component of Φ. For sufficiently large field values these terms dominate the potential.

We already considered SUSY breaking. However, if we consider the general form of the hidden sector SUSY breaking, one has A-terms of the form [12, 14, 51]

$$V(\phi) = Am_{3/2}W(\phi) = \frac{\lambda A m_{3/2}}{M_p^{d-3}}\phi^d, \tag{71}$$

where the superpotential is of the form Eq. (68).

In the early universe there are effects, which produce terms to the scalar potential, that are dominating over the soft SUSY breaking terms. In a radiation dominated era the relevant scale of excitations is the temperature of the universe. On the other hand, during the inflationary era the scale of quantum deSitter fluctuations is given by the Hubble constant. When a flat direction acquires a large VEV, the non-flat directions gain a mass, $m_\perp \sim y_{u,d,e}| < \phi > |$, from Yukawa and gauge couplings in the superpotential Eq. (28). Also the gauge particles gain masses, $m_g \sim g| < \phi > |$ where g is the gauge coupling, through super-Higgs mechanism, since the gauge symmetry is broken along the flat direction. For large VEVs the masses of the non-flat modes and gauge particles become heavier than the excitation scales. This causes the non-flat directions to quickly settle on to their minimum values, and they effectively decouple. Therefore all the dynamics takes place in the moduli space, when the fields are large. In general one flat direction may not give a mass to all other directions. In that case there can be other directions gaining a VEV, too. In terms of gauge invariant polynomials this means that new terms are added to the polynomial.

The most important source of SUSY breaking in the Early Universe is the finite energy density of the Universe [14, 51]. If the energy density has a non-zero expectation value, it implies that the supercharge does not annihilate the vacuum state and supersymmetry is thus broken. During the inflationary epoch the vacuum energy is positive by definition and SUSY is broken. During the post-inflationary epoch the energy density is dominated by inflaton oscillations so that the vacuum energy averaged over time is non-vanishing. In a radiation dominated era SUSY is also broken, since the occupation numbers of bosons and fermions in the thermal backround are different. A similar quantum mechanical effect also exists during inflation due to the deSitter fluctuations, which give different occupation numbers for bosons and fermions. However, the thermal and quantum effects are in general less important at large field values than the SUSY breaking due to finite energy, although this may not always be the case, especially for the thermal effects.

A simple way of generating a mass term for the flat direction in the global supersymmetric limit is to add to the Kähler potential, J of Eq. (20), a contribution of the form (we ignore the vector superfields here)

$$\delta J = \frac{1}{M_p^2} \int d^2\theta \, d^2\bar{\theta} \, I^\dagger I \Phi^\dagger \Phi, \tag{72}$$

where I is a field that dominates the energy density of the universe and Φ is the canonically normalized flat direction. If I dominates the energy density, then $\rho \sim \int d^2\theta d^2\bar{\theta} \, I^\dagger I$. Using the Friedmann equation $\rho = 3M_p^2 H^2$ one can see that Eq. (72) generates a mass term for the scalar ϕ proportional to $\sim -H^2$. So, for $H > m_\phi$ the finite energy SUSY breaking is more important than soft SUSY breaking.

Since Planck scale operators are discussed, supergravity interactions should be included. The general scalar potential for supergravity is given in Eq. (25). The general study of inducing a potential for the flat directions was done in [14] for F-term inflation and in [52] for D-term inflation (F and D term inflation in general have been discussed in [55]. Here we only give a simple example leading to the simplest form of potential in the flat directions. We add to the minimal Kähler potential an interaction that produces a mass of order Hubble scale:

$$K(\phi, I) = I^\dagger I + \phi^\dagger \phi + \frac{\zeta}{M_p^2} I^\dagger I \phi^\dagger \phi, \tag{73}$$

where $|\zeta| \sim 1$ and I, ϕ are scalar parts of the superfields. We also assume that the superpotential can be split into two parts $W = W(I) + W(\Phi)$ related to the inflaton and the flat direction, respectively.

Assuming $I, \phi \ll M_p$ we expand the F-term part of the potential Eq. (25) to obtain

$$\begin{aligned} V_F &= |W_I|^2 - \frac{3}{M_p^2}|W(I)|^2 + \frac{I^\dagger I}{M_p^2}|W_I|^2 + |W_\phi|^2 \\ &\quad + \frac{1-\zeta}{M_p^2}|W_I|^2 \phi^\dagger \phi + \frac{1}{M_p^2}[W(I)^*(\phi W_\phi - 3W(\phi)) + h.c.]. \end{aligned} \tag{74}$$

The first three terms on the first row give the F-term inflaton potential. If this term dominates the inflaton potential one obtains from the Friedmann equation $W_I \sim M_p H$ and $W(I) \sim M_p^2 H$. The third term in that case gives a Hubble-squared correction to the inflaton mass, which causes problems for slow-roll conditions. The fourth term gives the F-term part for the potential of the flat direction. The renormalizable part of the superpotential Eq. (28) does not contribute, because it vanishes along the flat direction. The first term on the second row gives the Hubble squared correction to the mass of the flat direction. The second term produces A-type terms, which are of order Hubble. Altogether one obtains the following potential for the flat direction, where the soft SUSY breaking terms Eqs. (66, 71) have also been taken into account:

$$V_F(\phi) = m_\phi^2 |\phi|^2 + c_H H^2 |\phi|^2 + \left[\frac{A m_{3/2} + aH}{d M_p^{d-3}} \lambda \phi^d + h.c.\right] + \frac{|\lambda|^2}{M_p^{2(d-3)}} |\phi|^{2(d-1)}, \qquad (75)$$

where A, a, $\lambda \sim 1$ are in general complex constants, $c_H \sim 1$ is real and its sign depends on ζ. In particular, the induced Hubble squared mass is negative if $\zeta > 1$.

The low-energy expansion of the D-term part of the potential Eq. (25) with the Kähler potential Eq. (73) is done in the same way. First we find the potential to order M_p^{-2},

$$\begin{aligned} V_D &= \frac{1}{2} \mathrm{Re}\, f_{AB}^{-1} (I^\dagger T^A I + \xi)(I^\dagger T^B I + \xi) + \\ & \frac{\zeta}{M_p^2}\, \phi^\dagger \phi\, \left[\xi (I^\dagger T^A I + I^\dagger T^B I) + 2 (I^\dagger T^A I)(I^\dagger T^B I)\right] + \mathcal{O}(M_p^{-4}). \end{aligned} \qquad (76)$$

The Fayet-Iliopoulos contribution, ξ, is included here and it is understood that it is non-zero only if the symmetry group is $U(1)$, which is the typical scenario in the D-term inflation [52]. For a $U(1)$ symmetry, the generators, T^A, are just the charges of the fields. The D-terms of ϕ do not contribute, because they vanish by definition, see Eq. (36). If the inflaton potential is dominated by the D-term and in particular by its Fayet-Iliopoulos term, the flat direction receives a negligible correction to its mass. If, on the other hand, the potential is dominated by the terms that do not depend on ξ, which is the case after inflation, then the flat direction receives an order Hubble correction to its mass. Another difference with respect to F-term inflation is that there are no A-terms in the D-term inflation case. However, the superpotential contributions from the F-term for the flat direction are still relevant, as are the soft SUSY breaking terms. So finally with D term inflation the potential reads

$$V_D(\phi) = m_\phi^2|\phi|^2 + c_H H^2|\phi|^2 + \left[\frac{Am_{3/2}}{dM_p^{d-3}}\lambda\phi^d + h.c.\right] + \frac{|\lambda|^2}{M_p^{2(d-3)}}|\phi|^{2(d-1)}, \qquad (77)$$

where c_H vanishes during inflation and $c_H \sim 1$ after inflation [52] with $c_H < 0$ if $\zeta < 0$.

4 Cosmological evolution of the SUSY flat directions

In this Section we study the dynamical evolution of the flat direction in the Early Universe using the potential derived in Section 3.4 Eqs. (75, 77). The evolution is quite generic and does not depend on the particular values of the coupling constants. Once the AD condensate has formed the mass term dominates the potential and defines the future evolution of the flat direction. The dominant contribution to the mass is given by the soft terms, whose explicit forms depends on radiative corrections. If the mass term grows slower than ϕ^2, the pressure is negative [20, 22, 53], resulting in an instability of the AD condensate with respect to spatial perturbations [18, 23].

We note here that the potentials Eqs. (75, 77) have a built-in baryon number violation, since the potential does not depend solely on $|\phi|$. C and CP violation is produced if the constants A and a are complex, since the C transformation effectively corresponds to $\phi \to \phi^*$. However, if A is real and $a = 0$, CP violation is still produced by the initial conditions. The non-equilibrium conditions are provided by inflation. Hence all the Sakharov conditions are fulfilled and baryogenesis is possible [2].

4.1 Formation of the AD condensate

The dynamical evolution of the flat directions was solved by Dine, Randall and Thomas [14] using the potentials (see Eqs. (75, 77))

$$V(\phi) = m_\phi^2|\phi|^2 - c_H H^2|\phi|^2 + \left[\frac{Am_{3/2} + aH}{dM_p^{d-3}}\lambda\phi^d + h.c.\right] + \frac{|\lambda|^2}{M_p^{2(d-3)}}|\phi|^{2(d-1)}, \qquad (78)$$

where $a = 0$ ($a \sim 1$) for the D-term (F-term) inflation, $c_H = 0$ during the D-term inflation and $c_H \sim 1$ after inflation, for F-term inflation $c_H \sim 1$ both during and after inflation. Models where $c_H \leq 0$ have also been considered. Under certain conditions sufficient amount of baryogenesis can be produced [14, 54]. The dynamics of the flat direction is governed by the homogeneous mode, until fragmentation occurs. Before the fragmentation the equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi^*} = 0. \qquad (79)$$

This is just a Newtonian equation of motion for a potential $V(\phi)$ with a damping term. It is usually easier to study the Eq. (79) by using the parameterization $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$, which gives

$$\ddot{\varphi} + 3H\dot{\varphi} - \dot{\theta}^2\varphi + \frac{\partial V}{\partial \varphi} = 0, \qquad (80)$$

$$\ddot{\theta} + \left(3H + \frac{2\dot{\varphi}}{\varphi}\right)\dot{\theta} + \frac{1}{\varphi^2}\frac{\partial V}{\partial \theta} = 0. \qquad (81)$$

For numerical purposes the parameterization $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ is better.

During inflation the Hubble parameter is approximately constant, $H = H_I \sim 10^{13}$ GeV [55], so that the potential is static. Then the AD field evolves to a minimum of the potential, which is actually d-fold degenerate if $c_H > 0$ (for definiteness we take $c_H = 1$). The AD field evolves to a minimum typically in the order of $10\,e$-folds. The two inflationary models result into a slightly different evolution in the following way. In the F-term case both the mass of the radial mode, φ, and the phase, θ, are of order H, so that the damping is critical [14] (damping and potential gradient are of equal magnitude), where $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$. In the D-term case the phase has a small mass $\sim \sqrt{m_{3/2}H}$, so that the phase is overdamped and thus the classical evolution is very slow and the phase effectively freezes.[4] Since the phase is effectively massless compared with the Hubble scale, it has a scale invariant spectrum of fluctuations $< |\varphi\delta\theta|^2 >= H^2/(2\pi)^2$ [55]. The fluctuations are coherent on the Hubble scale. During the inflation the same happens to the radial mode φ, too. However, at the end of the inflation the vacuum energy related to the Fayet-Iliopoulos term is converted to kinetic energy and potential energy of the mass term. Then the flat direction gets a Hubble order correction to its mass. Thus, one would expect that the radial mode has evolved close to the minimum when inflation ends. This sets the initial conditions of the evolution of the flat direction after inflation. In Fig. 1 we have shown an example

[4]The slow evolution of the phase to a minimum takes about $H_I/m_{3/2}$ e-folds.

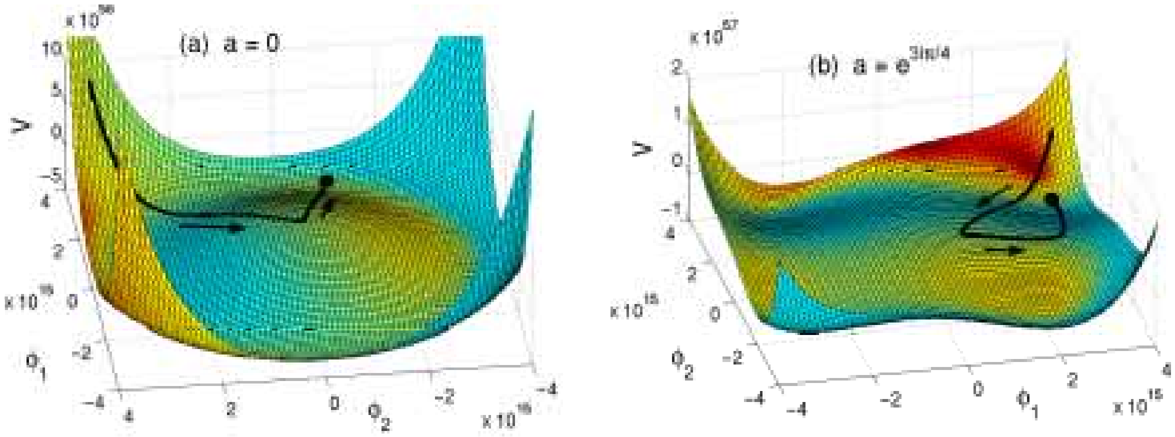

Figure 1: The evolution of the AD field during inflation for $A = -1$, $M_p = 10^{18}$ GeV, $m_S = m_{3/2} = 100$ GeV, $H = 10^{13}$ GeV, $\lambda = c_H = 1$ and (a) $a = 0$ and (b) $|a| = 1$, $\theta_a = 3\pi/4$ with initial conditions $\varphi_i = 1.5\varphi_{\min}$ and $\theta_i = \pi/4$. The AD field has been parameterized by $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. The evolution takes approximately (a) 10 and (b) 30 e-folds. In case (a) the final phase is approximately -0.3 with minima at 0, $\pi/2$, π, $3\pi/2$ and in (b) $\pi/16$ which is also a minimum.

of the evolution during inflation for both D- and F-term inflation, where in both cases we have chosen $c_H = 1$ even if this is not strictly true for the D-term inflation.

When the inflation has ended, the Universe is briefly matter dominated by inflaton oscillations in the minimum of its potential. Thus the Hubble parameter becomes time-dependent and is given by $H = 2/(3t)$. The potential minimum shifts towards lower energy scale and at the same time increases. The AD field tracks down the evolution of the minimum acquiring a fixed point value of Eq. (79) [14], $\varphi_{fix} = k \cdot \varphi_{\min}$ where $k \gtrsim 1$ and $\varphi_{\min}$ is defined at Eq. (82), see Fig. 2 a. The phase stays approximately constant. When $H \sim m_\phi$ the potential behaves as in a first order phase transition with the Hubble parameter playing the role of temperature, as can be seen in Fig. 2. However, in different phase directions the "phase transition" happens at sligthly different moments of time because of the A-term in the potential, so that there appears a pit, in which the AD field starts to rotate. The AD field has only time to rotate a short while in the pit before the pit vanishes completely making the potential approximately symmetric again.[5] When the potential becomes symmetric again, the AD field continues to rotate

[5]This also depends on c_H. In [14] $c_H = 9/4$ was used and the result was that for certain initial phases the AD field rotated further in the pit and eventually rotated to another direction producing

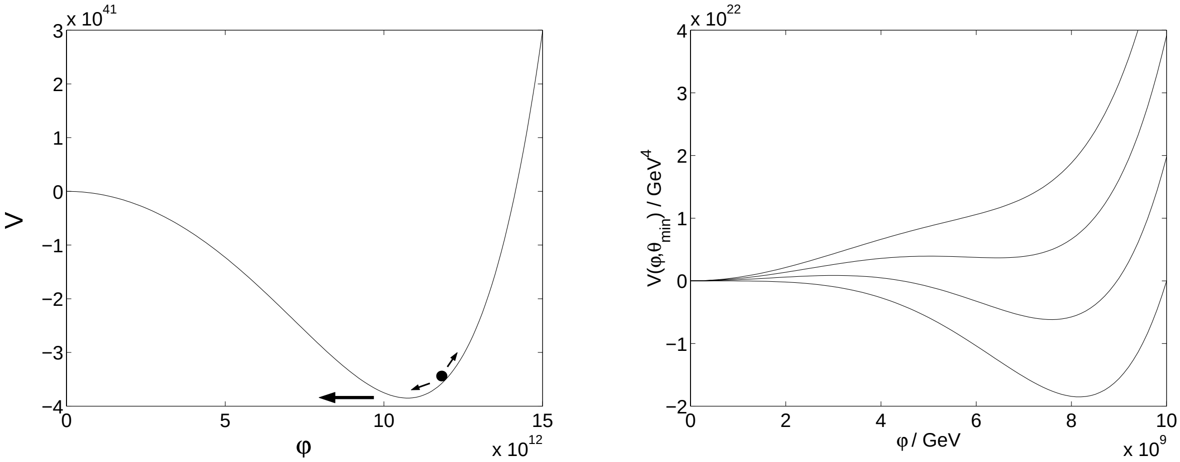

Figure 2: (a) The AD field at the fixed point, the black dot, in the potential when $H = 10^8$ GeV. The large arrow represents the changing of the minimum and the small arrows the oscillation of the AD field around the fixed point. (b) The potential for $H = m$, $0.98m$, $0.96m$, $0.94m$ (from bottom to top), demonstrating the potential behaves in a manner typical for first order phase transitions. In both figures the parameter values $m = 100$ GeV, $A = -1$, $m_{3/2} = m$, $d = 4$, $\theta = \theta_{\min}$, $|\lambda| = 1$ and $M_p = 10^{18}$ GeV have been used.

on an elliptically spiraling orbit towards the minimum of the potential. The sequence of events is shown in Fig. 3 in the D-term case to which the discussion of this paragraph applies. For the F-term case the potential is asymmetric all the time. In order to produce charge the phases of the parameters A and a have to different, so that at $H \sim m_{3/2}$ a phase transition takes place in the θ direction, too, which is continuous. If $m_{3/2} \sim m_\phi$, as in the gravity mediated SUSY case, both phase transitions happen at around same time and the AD field rises up in the potential to the phase direction in the same manner as the radial mode did and the end result is the same as one gets from the D-term model. However, if $m_{3/2} \ll m_\phi$ the phase transition in the θ direction happens later than the φ transition. Then there is no pit to enhance charge generation and much less charge is produced. The case $m_{3/2} \gg m_\phi$ is not interesting because then the vacuum stays at $\varphi > 0$ and no charge is generated.

The produced charge density can be estimated as follows. The minimum of the

different sign for the charge.

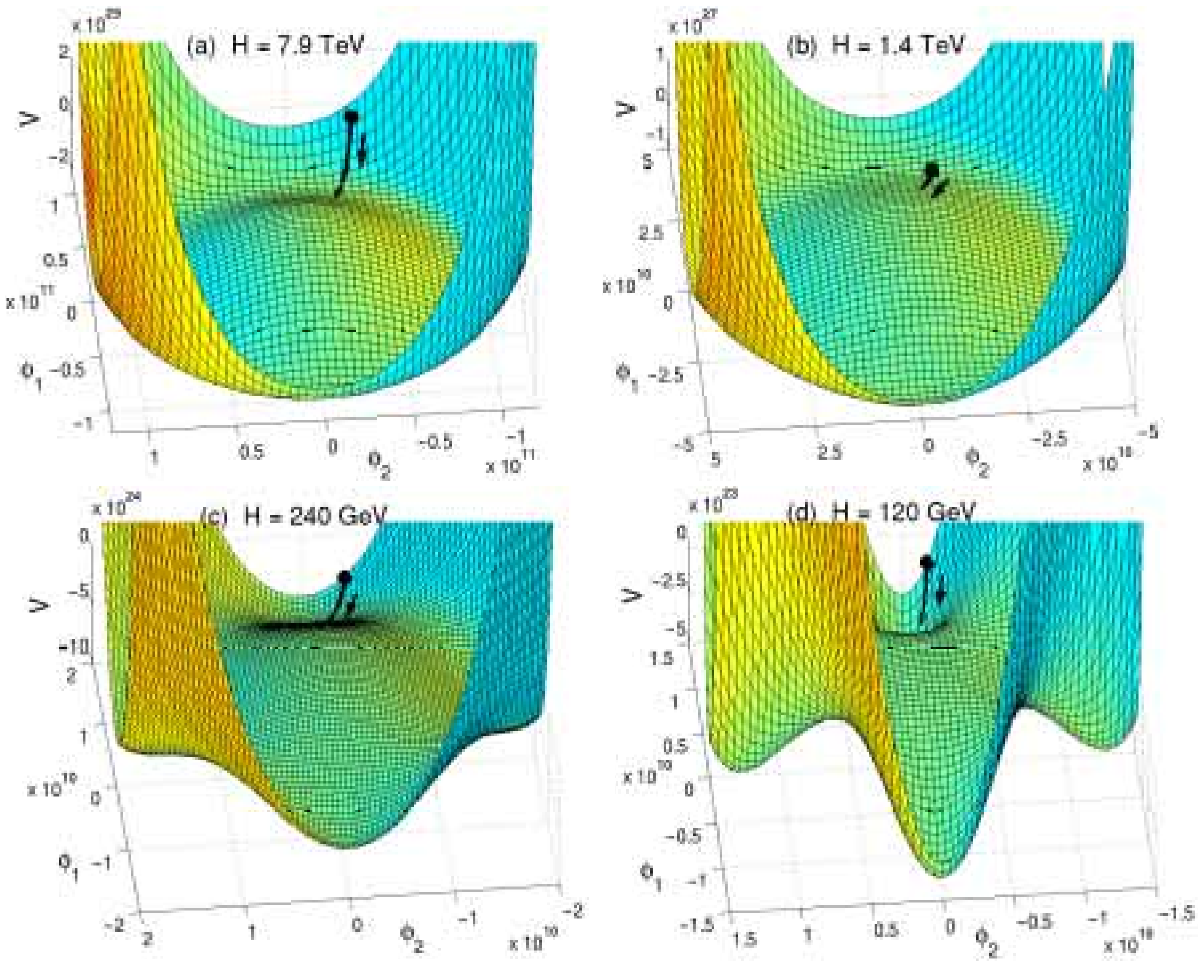

Figure 3: The ball represents the AD field at the time indicated in the figure. The line in front of it represents the direction of evolution of the AD field in case the potential were static. The length of the line is chosen arbitrarily for (a) and (b), but for (c) and (d) the length is such that at the end of the line the AD field is at the next figure. Continued on the next page.

potential can be solved from Eq. (78) to be

$$\varphi_{\min} = \sqrt{2}\left(\frac{M_p^{d-3}H}{|\lambda|}\right)^{1/(d-2)} \cdot c, \tag{82}$$

where $c \sim 1$ and this result is applicable for $H \gtrsim m_\phi$ only. Roughly speaking the rotation starts when $H \sim 0.1 m_\phi$, at which time $\phi \sim 0.1\phi_{\min}$ $(H = m_\phi)$. For circular orbit one can easily solve $\dot{\theta}$ by using classical mechanics, *i.e.*, on a circular orbit kinetic energy and potential energy are equal. At the time when the rotation starts only the soft mass term is important, so one obtains $\dot{\theta} = m_\phi$. Hence in the end the maximum

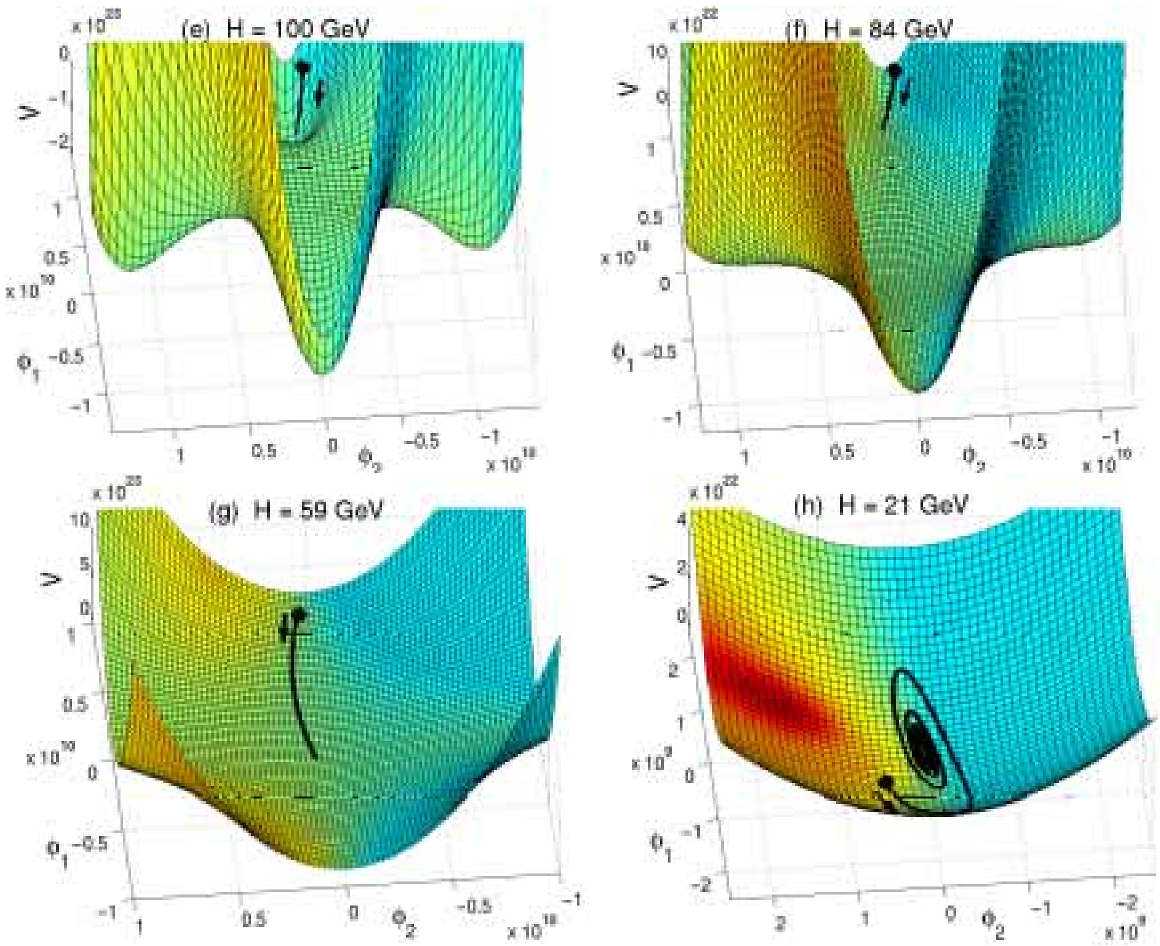

Figure 4: Fig. 3 cont. from the previous page. The length of the line is chosen such that at the end of the line the AD field is at the next figure.

charge density produced is

$$q_{\rm max} = \dot\theta\varphi^2 \sim 0.01 m_\phi \left(\frac{M_p^{d-3} m_\phi}{|\lambda|}\right)^{2/(d-2)} . \tag{83}$$

In reality the charge density is $q = \epsilon q_{\rm max}$, where $0 \le \epsilon \le 1$ with $\epsilon = 0$ for oscillation and $\epsilon = 1$ for circular orbit. One can find the charge density by integrating its equation of motion Eq. (3), giving [56]

$$\begin{aligned} q &\sim H^{-1}\frac{|A| m_{3/2}\sin(d\theta+\theta_A+\theta_\lambda) + |a| H \sin(d\theta+\theta_a+\theta_\lambda)}{2^{(d-2)/2} M_p^{d-3}} \varphi_{\rm min}^{\frac{d}{d-2}} \\ &\sim (|A| m_{3/2}\sin\delta_A + |a| m_\phi \sin\delta_a)\varphi_{\rm min}^2 , \end{aligned} \tag{84}$$

where the calculation has been done at the time rotation starts $H \sim m_\phi$ and the scale of the field is given by the minimum Eq. (82). The CP violating effective phase is given by $\sin\delta_A$, $\sin\delta_a \sim 1$.

4.2 Radiative corrections

During this era the potential is dominated by the mass term and the charge density in the co-moving volume freezes to a constant value. The end result is a rotating coherent condensate. The behaviour, when $H \ll m_\phi$, is dependent on the detailed form of the mass term. The simplest choice is the usual tree-level mass term for which $m_\phi = const.$ as was done in [14]. Then the condensate eventually decays into baryons and/or leptons through thermal scattering. If the reheating temperature of the Universe is larger than the electroweak scale, $T_W \sim 100$ GeV, the baryon- and lepton-number violating sphaleron transitions are in equilibrium and only the flat directions with non-zero $B - L$ can produce a non-vanishing net baryon number.

Let us consider a different possibility: The mass term with radiative corrections included. Then the AD condensate evolves differently. Let us first consider different mass terms that are possible. In the gravity mediated case the mass term [18] is

$$V(\phi) = m_S^2 \left[1 + K\log\left(\frac{|\phi|^2}{M^2}\right)\right]|\phi|^2, \tag{85}$$

where M is a large mass scale and ϕ is defined to have a mass m_S, which is of the order of the mass of SUSY breaking scale that is typically $m_{3/2}$, at the scale M. K is defined by [18, 21]

$$K = \frac{1}{q^2}\frac{\partial m_\phi^2}{\partial t} \tag{86}$$

where $t = \log(M_X/q)$, $m_\phi^2 = \sum_i a_i^2 m_i^2$ with the flat direction superfield defined by $\Phi = \sum_i a_i \Phi_i$ and $\sum_i a_i^2 = 1$. The scale, at which K is evaluated, is defined to be the scale $q = |\phi_{\rm min}|$ evaluated at the time the charge is generated when $H \sim m_\phi$. The various MSSM flat directions were studied numerically in Paper I [21] where it was found that in general K can be positive or negative depending on the parameter values. The typical value seems to be $K \sim -0.05$ for $m_\phi \sim m_{3/2}$. If $K > 0$, the AD scalar is stable and there is no practical difference to the case $K = 0$, unless $K \sim 1$. For $K < 0$ the AD condensate becomes unstable and decays in the end to Q-balls, which are discussed in the Section 5.

The gauge mediated scenario gives a different kind of correction to the mass. For the low-energy scales the mass term is the usual quadratic one, but above the so-called messenger scale the mass term becomes logarithmic, *i.e.*, almost flat. The mass term in this case is [19]

$$V(\phi) = \begin{cases} m_\phi^2 |\Phi|^2, & |\phi| \ll M_S \\ M_F^4 \left[\log\left(\frac{|\phi|^2}{M_S^2}\right)\right]^2, & |\phi| \gg M_S. \end{cases} \tag{87}$$

Usually the two terms are joined to [23]

$$V(\phi) = m_\phi^4 \log\left(1 + \frac{|\phi|^2}{m_\phi^2}\right), \tag{88}$$

which has the same behaviour at high- and low-energy scale as Eq. (87).

After inflation there is a dilute plasma of inflaton decay products, which has a temperature $T \sim (HT_R M_p^2)^{1/4}$ [57]. If the AD field couples to a field ψ, which is in the thermal bath of inflaton decay products, at one-loop level the mass of the AD field acquires a thermal correction of the form [58]

$$V(\phi) = y^2 T^2 |\phi|^2, \tag{89}$$

where y is a Yukawa coupling or gauge coupling between the AD field and the thermal particles. For $d \geq 6$ the mass produced to non-flat directions is larger than temperature, so that they are not thermalized and thus do not produce a correction like Eq. (89). In this case the thermal correction has to be calculated in a different manner and results in [59]

$$V(\phi) = c_T^2 T^4 \log\left(\frac{|\phi|^2}{T^2}\right), \tag{90}$$

where c_T is the Yukawa or gauge coupling. The numerical study of AD mechanism with thermal mass Eqs. (89, 90) was done in [60].

The amplitude of rotation decays. For a potential $V \sim |\phi|^n$ the amplitude decays as $R^{-6/(n+2)}$ [20, 22], where R is the scale factor of the universe. For the gravity mediated case this becomes $R^{-3/(2+2K)}$, where the potential Eq. (85) can be approximated by $V \sim |\phi|^{2+2K}$ near $|\phi| \lesssim M$. This result is valid also in the gauge mediated case with

$n = 0$ to obtain R^{-3}. For the thermal mass Eq. (89) the calculation done in [22] is not valid, since the temperature has a time dependence of its own. The correct result in this case is $R^{-21/16}$ [56].

4.3 Instability of the AD condensate

Quantum fluctuations of the scalar fields get magnified outside horizon, where they freeze to a constant value during inflation. Once the inflation has ended these fluctuations come back inside the horizon making the spatially homogeneous fields slightly inhomogeneous. Eventually the inhomogeneities in dominating components of energy density act as seeds for galaxy formation. The perturbations of the AD condensate act as seeds that fragment the condensate.

Let us first consider the effect of negative pressure. The multifluid evolution equations of perturbations are [61]

$$\ddot{\delta}_{\mathbf{k}i} + 2H\dot{\delta}_{\mathbf{k}i} + \left(\frac{k}{R}\right)^2 \frac{\delta p_{\mathbf{k}i}}{\rho_i} = 4\pi G\rho\delta_{\mathbf{k}}, \tag{91}$$

where $\delta_i = \delta\rho_i/\rho_i$ with ρ_i the energy density of the component i, $\delta\rho_i$ its perturbation and ρ the total energy density. Inside the horizon the gradient energy dominates, so one can neglect the right-hand side of Eq. (91). The flat direction has the equation of state $p = w\rho$, where w is approximately constant. Then Eq. (91) becomes, with the assumption of matter dominated universe,

$$\ddot{\delta}_{\mathbf{k}\phi} + \frac{4}{3t}\dot{\delta}_{\mathbf{k}\phi} + \left(\frac{k}{R_\phi}\right)^2 \left(\frac{t_\phi}{t}\right)^{\frac{4}{3}} w\,\delta_{\mathbf{k}\phi} = 0, \tag{92}$$

where t_ϕ is the time when $H = m_\phi$ [18]. Assuming that the pressure is negative $w < 0$, one can solve Eq. (92) exactly to obtain

$$\delta_{\mathbf{k}\phi} = C_1 t^{-\frac{1}{6}} I_{-1/2}(3Dt^{\frac{1}{3}}) + C_2 t^{-\frac{1}{6}} I_{1/2}(3Dt^{\frac{1}{3}}), \tag{93}$$

where $D = \sqrt{|w| t_\phi^{4/3} k/R_\phi}$ and I_ν is the modified Bessel function. The Bessel functions behave asymptotically as $I_\nu(z) \to e^z/\sqrt{2\pi z}$ for $z \gg 1$. So the negative pressure leads to an exponential growth of density perturbations of the flat direction. If one assumes that the orbit of the AD field is circular, one can approximate $\delta_{\mathbf{k}\phi} = \delta\varphi_{\mathbf{k}}/\varphi$ to obtain an exponential growth of perturbations [18].[6]

The pressure and energy density of the homogeneous scalar field are

$$p = |\dot{\phi}|^2 - V(\phi), \tag{94}$$

$$\rho = |\dot{\phi}|^2 + V(\phi). \tag{95}$$

The equation of state can be approximately solved by taking a time average over the oscillation or rotation cycle provided the oscillation is rapid compared with the expansion rate, so that the energy density over a cycle is approximately constant. One has at the time of fragmentation $\dot{\theta} \sim m_\phi$ whereas $H \ll m_\phi$, so that the assumption is acceptable. So we wish to calculate

$$w = \frac{\int\limits_0^T dt\,\frac{p}{\rho}}{\int\limits_0^T dt}. \tag{96}$$

Eq. (96) can be simplified by the change of variables $dt = d|\phi|/|\dot{\phi}|$ and $|\dot{\phi}| = \sqrt{\rho - V}$, leading to

$$w = 2\frac{\int_c d|\phi| \left(1 - \frac{V}{\rho}\right)^{1/2}}{\int_c d|\phi| \left(1 - \frac{V}{\rho}\right)^{-1/2}} - 1. \tag{97}$$

Since the total energy density ρ is assumed to be conserved, it can be determined from the maximum and minimum values of the potential as $\rho = V_{\max} + V_{\min}$.[7] Typical orbits one can think of are elliptical. However, for a general elliptical orbit the calculation cannot be done in a closed form. For a circular orbit the result is trivial since then

[6] In [18] the expansion of the universe was neglected, which just corresponds to setting $R = 1$. Here we have included the expansion in the matter dominated case, which is valid during inflaton oscillations. We find that there is a slight difference to [18], where $\delta\varphi_{\mathbf{k}}/\varphi \sim \exp(t)$ whereas here for $t \gg t_\phi$ one obtains $\delta_{\mathbf{k}\phi} \to t^{-1/3}\exp(t^{1/3})$. However, the perturbations still grow exponentially, which was the main conclusion in [18].

[7] The maximum potential energy corresponds to minimum kinetic energy and vice versa. Since the difference between the maximum and minimum of the kinetic and potential energy are equal, the minimum and maximum also have to be equal if the minimum potential energy is zero.

$V_{\max} = V_{\min}$, so $w = 0$. This means that the scalar field behaves like matter in a circular orbit regardless of the potential. Another simple situation is the pure oscillation for which $V_{\min} = 0$ and using the potential $V(\phi) \sim |\phi|^n$ one gets [20]

$$w = \frac{n-2}{n+2}. \tag{98}$$

As one can see the pressure is negative if the potential grows slower than ϕ^2. This is essentially the situation for the mass terms Eqs. (85, 87) or Eq. (88). For the potential (85) the calculation was done in [53], where it was pointed out that the mass term behaves as $|\phi|^{2+2K}$ and so $w \approx K/2$.

The negative pressure has another effect besides causing the exponential growth of perturbations. According to the continuity equation in the expanding universe,

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad \Leftrightarrow \qquad \frac{\partial(\rho R^3)}{\partial t} = -p\frac{\partial R^3}{\partial t}, \tag{99}$$

where R is the scale factor of the universe. If $p < 0$ the energy density in the co-moving volume increases. One can think of this as stretching a rubber band, so that its tension increases. This also increases its energy in the form of increased tension. Of course at some point the band has been stretched too much and it breaks into pieces. The pieces collapse together. In the same way the AD condensate fragments if its pressure is negative. The fragments of the AD condensate collapse to a smaller volume than the size of the original fragment producing Q-balls (see Section 5).

4.4 Fragmentation of the AD condensate

In this section we present a more thorough analysis of the growth of perturbations of the AD condensate. The spectrum of perturbations is [18]

$$\delta\phi(\lambda) \approx \frac{1}{2\pi m_\phi H_I^{1/2} \lambda^{5/2}}, \tag{100}$$

where λ is the perturbation length scale at $H \sim m_\phi$. Using $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$ and assuming $U(1)$-violating terms are negligible the equations of motion in the expanding universe, taking the inhomogeneity into account, are [23]:

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{R^2}\nabla^2\varphi - \dot{\theta}^2\varphi + \frac{1}{R^2}|\nabla\theta|^2\varphi + \frac{\partial V}{\partial\varphi} = 0, \tag{101}$$

$$\ddot{\theta} + \left(3H + \frac{2\dot{\varphi}}{\varphi}\right)\dot{\theta} - \frac{1}{R^2}\nabla^2\theta - \frac{2}{R^2\varphi}(\nabla\varphi)\cdot(\nabla\theta) = 0, \tag{102}$$

where R is the scale factor of the universe. The perturbations can be studied by linearizing Eq. (101) with $\theta \to \theta + \delta\theta$, $\varphi \to \varphi + \delta\varphi$ assuming a homogeneous backround *i.e.* $\nabla\varphi = \nabla\theta = 0$:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{1}{R^2}\nabla^2\delta\varphi - 2\varphi\dot{\theta}\delta\dot{\theta} - \dot{\theta}^2\delta\varphi + \frac{\partial^2 V}{\partial\varphi^2}\delta\varphi = 0, \tag{103}$$

$$\delta\ddot{\theta} + \left(3H + \frac{2\dot{\varphi}}{\varphi}\right)\delta\dot{\theta} + \frac{2\dot{\theta}}{\varphi}\delta\dot{\varphi} - \frac{2\dot{\theta}\dot{\varphi}}{\varphi^2}\delta\varphi - \frac{1}{R^2}\nabla^2\delta\theta = 0. \tag{104}$$

We consider perturbations of φ and θ, where both $\delta\varphi$ and $\delta\theta$ are proportional to $\exp(S(t) - i\mathbf{k}\cdot\mathbf{x})$, which we use as an *ansatz* for the solution of perturbation equations. An exponentially growing mode is present if $\mathrm{Re}\,\dot{S} > 0$. We substitute the *ansatz* into Eq. (103), collect terms proportional to $\delta\varphi$ and $\delta\theta$, and require that the solution is non-trivial, which just means that the determinant of the matrix multiplying the perturbations has to vanish, and we obtain

$$\left[\alpha^2 + 3H\alpha + \left(\frac{k}{R}\right)^2 - \dot{\theta}^2 + V''(\varphi)\right]\left[\alpha^2 + \left(3H + \frac{2\dot{\varphi}}{\varphi}\right)\alpha + \left(\frac{k}{R}\right)^2\right] + 4\dot{\theta}^2\left(\alpha - \frac{\dot{\varphi}}{\varphi}\right)\alpha = 0, \tag{105}$$

where an adiabatic approximation $\dot{\alpha} \ll \alpha$ has been used with $\alpha = \dot{S}$.[8] Terms proportional to H can be neglected since $H \ll m_\phi$. In order to have solutions of α that have non-zero real components one must have $V''(\varphi) - \dot{\theta}^2 < 0$. The band of growing modes (modes having $\mathrm{Re}\,\alpha > 0$) can then be solved from Eq. (105):

$$0 < \left(\frac{k}{R}\right)^2 < \left(\frac{k_{\max}}{R}\right)^2 = \dot{\theta}^2 - V''(\varphi). \tag{106}$$

If $k_{\max}$ is constant or grows with time, the instabilities can develop indefinitely. At some stage the perturbations grow non-linear. If, on the other hand, $k_{\max}$ decreases

[8] Adiabatic approximation is actually valid only if the orbit is circular. If the orbit is not circular then $\dot{\theta}$ and φ change rapidly, so that also $\dot{\alpha}$ should change rapidly, too.

with time, it is possible that the perturbations are not able to grow significantly before they are red-shifted away from the resonance.

For a gravity mediated scenario, Eq. (85), one obtains the following band of growing modes [26, 27]

$$0 < \left(\frac{k}{R}\right)^2 < 3|K|m_S^2, \tag{107}$$

where $K \approx -0.1 \ldots -0.01$ and $m_S \sim 100$ GeV. Since the right-hand side is a constant, the instabilities can grow indefinitely. The perturbations grow as [79]

$$\delta\varphi = \left(\frac{R_0}{R}\right)^{\frac{3}{2}} \delta\varphi_0 e^{i\mathbf{k}\cdot\mathbf{x}} \exp\left(\int dt \sqrt{\frac{|K|m^2k^2}{2\dot{\theta}^2R^2}}\right), \tag{108}$$

$$\delta\theta = \delta\theta_0 e^{i\mathbf{k}\cdot\mathbf{x}} \exp\left(\int dt \sqrt{\frac{|K|m^2k^2}{2\dot{\theta}^2R^2}}\right), \tag{109}$$

where subscript 0 marks the initial values at $t = t_0$ and it has been assumed that $k^2/R^2 \lesssim 2|K|m_S^2$, $H < m_S$ and $|K| \ll 1$. For the band of growing modes in the gauge mediated scenario, Eqs. (87, 88), one obtains [25, 27]

$$0 < \frac{k}{R} < \frac{2m_\phi^2}{\varphi_{\min}}, \tag{110}$$

where $\varphi_{\min}$ is calculated at $H \sim m_\phi$. So in this case instabilities grow indefinitely, too.

The instability of the AD condensate has been numerically studied in [25–27]. As a result the band of growing modes for a circular orbit is approximately given by the analytical expressions of Eqs. (107, 108), although the bands are not so sharply peaked. This just indicates that in the non-linear regime the coupling between different modes becomes important. However, for an oscillating orbit there appeared several bands of growing modes [27]. This indicates that all sizes of lumps are formed from the fragmenting condensate. This situation is further analyzed in Section 5 when discussing Q-balls.

5 Q-balls

5.1 Solitons

In field theories with a certain amount of non-linearity stable bound states can exist on classical and quantum mechanical level. These bound states are called solitons. Unlike wave packets, which are superpositions of plane waves, the soliton solutions are non-dispersive. They have a finite stable shape in space and they can travel with a constant velocity. Such solutions were first discovered in hydrodynamics by Russell in 1845 [62]. These were later explained in terms of the soliton solutions of a non-linear hydrodynamical equation called Korteweg-de Vries equation [63]. Since then many other soliton-type solutions have been found. Most of them exist only in one space dimension due to a theorem by Derrick [64]. The theorem implies that one has to include gauge fields of non-zero spin or consider time-dependent but non-dispersive solutions in order to have soliton solutions in two or more spatial dimensions. If one is restricted to relativistic field theories, then all soliton solutions can be classified into two general types.

(1) *Topological solitons.* For these it is necessary that the boundary condition of the soliton state at infinity is topologically different from that of the physical vacuum state. This requires degenerate vacuum states. Examples of topological solitons are the magnetic monopole solution of t'Hooft and Polyakov [65] and the sphaleron [66].

(2) *Non-topological solitons.* The boundary condition at infinity for a non-topological soliton is the same as that for the vacuum state. Thus, there is no need for degenerate vacuum states. The necessary condition for the existence of non-topological solitons is that there is an additive conservation law. There are several examples of non-topological solitons in the literature, most of which can be found in the review by Lee and Pang [67]. In this work we concentrate on a specific example of non-topological solitons called Q-balls [24].

Q-balls are minimum energy configurations in scalar field theories with a non-zero conserved charge. Simplest type of Q-ball is related to global $U(1)$-symmetries [24]. There are also other types of Q-balls studied in the literature: non-abelian Q-balls

[68, 69] and gauged Q-balls [70].

5.2 Q-balls in any space dimension D

The existence of Q-balls requires a conserved charge, which is usually taken to be related to a global $U(1)$-symmetry. The Lagrangian density of a scalar field is given by

$$\mathcal{L} = |\partial_\mu \phi|^2 - U(|\phi|), \tag{111}$$

where $\mu = 0, 1, \ldots, D$, $x^0 = t$. The $U(1)$-symmetry transformation is

$$\phi \leftarrow e^{iq\alpha}\phi, \quad \phi^* \leftarrow e^{-iq\alpha}\phi^*, \tag{112}$$

where q is the charge of the scalar field. With a field redefinition one can choose $q = 1$ (in the following we will assume this to be done). The symmetry trasformation, Eq. (112), induces a conserved current and charge

$$j_\mu = i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) = \varphi^2\partial_\mu\theta, \quad \partial_\mu j^\mu = 0, \quad Q = \int d^D\mathbf{x}\, j^0, \tag{113}$$

where a parameterization $\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}$ is used.

In order to find Q-ball solutions [24, 67] one has to minimize the energy with a non-zero conserved charge (if the charge is zero one gets the usual vacuum state as the minimum of energy). This is best achieved with the method of Lagrange multipliers

$$E = \int d^D\mathbf{x} \left[|\dot{\phi}|^2 + |\nabla\phi|^2 + U(|\phi|)\right] + \omega\left[Q - i\int d^D\mathbf{x}(\phi\dot{\phi}^* - \phi^*\dot{\phi})\right], \tag{114}$$

where ω is a Lagrange multiplier and Q is the charge of the field configuration (one can add the unit of charge by redefinitions $\omega = q\omega'$, $Q = Q'/q$ and one can assume Q, $\omega > 0$ since the energy is symmetric in $Q \to -Q$, $\omega \to -\omega$). With a reordering of terms in Eq. (114) one can see that the time dependence of the field can be solved completely for the minimum energy configuration

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2}}\varphi(\mathbf{x})e^{i\omega t}, \tag{115}$$

where φ can be chosen to be real and positive by a gauge transformation. The quantity ω can be interpreted as the angular velocity of ϕ in the field space. With the field configuration Eq. (115) one obtains from Eq. (114) the energy

$$E = \int d^D\mathbf{x} \left[\frac{1}{2}|\nabla\varphi|^2 + U(\varphi) - \frac{1}{2}\omega^2\varphi^2\right] + \omega Q \tag{116}$$

and charge

$$Q = \omega \int d^D\mathbf{x}\, \varphi(\mathbf{x})^2. \tag{117}$$

Thus the problem has now become that of finding a solution to the euclidean equation of motion in D dimensions with an effective potential $U_\omega(\varphi) = U(\varphi) - \frac{1}{2}\omega^2\varphi^2$ and the constraint (117). This is a well known problem, and it has been solved in [71]. For this purpose we define the spherical rearrangement of function φ, where φ is positive and vanishes at infinity. The spherical rearrangement of a function is the spherically symmetric monotonically decreasing function φ_R, for which the condition

$$\mu\{x|\varphi_R(|x|) \geq M\} = \mu\{x|\varphi(x) \geq M\}, \qquad \text{for all}\, M \geq 0 \tag{118}$$

holds [49], where μ is the Lebesque measure. The necessary condition for the field configuration to have a finite energy is that the field vanishes at infinity. Thus we can form the spherical rearrangement of the field φ in Eq. (116). The energy related to the effective potential does not change with $\varphi \to \varphi_R$. On the other hand the gradient energy is minimized using the spherical rearrangement theorem [72]

$$\int d^D\mathbf{x}|\nabla\varphi|^2 \geq \int d^D\mathbf{x}|\nabla\varphi_R(|x|)|^2. \tag{119}$$

So the energy is minimized with a spherical field configuration. The equation of motion is found either directly from the Lagrangian given in Eq. (111) or from the energy (116):

$$\varphi''(r) + \frac{D-1}{r}\varphi'(r) - U'_\omega(\varphi) = 0. \tag{120}$$

Equation Eq. (120) is analogous to the equation related to the decay of the false vacuum [73, 74]. Therefore solutions to Eq. (120) can be found in the same manner. It is helpful to think of Eq. (120) as a Newtonian equation of motion in the inverted

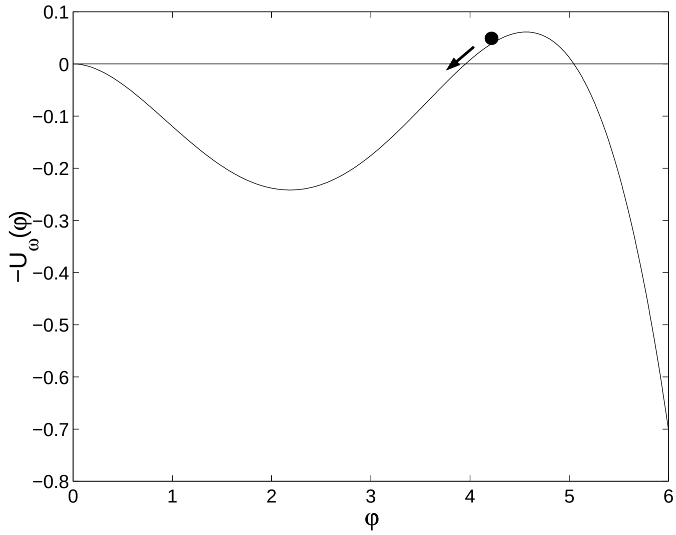

Figure 5: Example of an inverted potential with $U_\omega(\varphi) = \frac{1}{2}(m^2 - \omega^2)\varphi^2 - A\varphi^3 + \lambda\varphi^4$ and parameters $m = 1$, $A = 0.09$, $\lambda = 0.01$ and $\omega = 1.005\omega_c$. The position of the particle at $r = 0$ is shown.

potential $-U_\omega(\varphi)$ with friction. φ and r are interpreted as position and time respectively. Naturally one cannot expect to find solutions for arbitrary potentials. We show in Fig. 5 a typical example of the type of inverted potential, for which solution to Eq. (120) exists. We require that the potential fulfills the following conditions: it is $U(1)$-symmetric, $U(0) = 0$ can be chosen by convention, its global minimum is at $\varphi = 0$ so that symmetry is not spontaneously broken, $U'(0) = 0$ and $U''(0) = m^2 > 0$. The last two conditions imply that there exist massive scalar mesons. We also require that

$$\omega_c^2 = \min_\varphi \left(\frac{2U(\varphi)}{\varphi^2}\right) = \frac{2U(\varphi_c)}{\varphi_c^2} < m^2, \quad \varphi_c > 0. \tag{121}$$

One also has to specify the relevant boundary conditions for the solution. Since the energy has to be finite, in the limit $r \to \infty$ the potential and kinetic energy (gradient energy) has to vanish, which is accomplished by setting $\varphi(\infty) = \varphi'(\infty) = 0$. The solution has to be regular at $r = 0$, so $\varphi'(0) = 0$. $\varphi(0) = \varphi_0$ is to be determined from the solution. The specified boundary conditions just mean that the particle starts at rest at φ_0, reaches the origin $\varphi = 0$ at infinite time and stops at rest there. For this to be possible the origin has to be a maximum of the inverted potential (see Fig. 5) and the potential at φ_0 has to be larger than at $\varphi = 0$. This results in $\omega_c < \omega < m$. The behaviour of the potential is plotted in Fig. 6 for different values of ω. Fig. 6 also

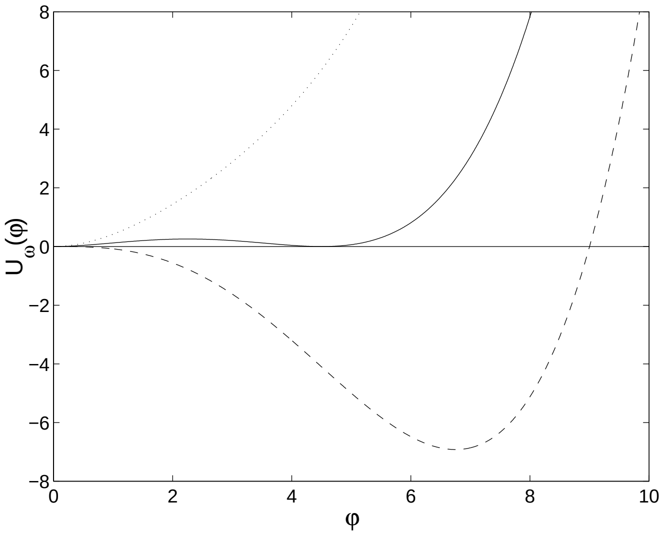

Figure 6: The behaviour of the effective potential $U_\omega(\varphi) = \frac{1}{2}(m^2 - \omega^2)\varphi^2 - A\varphi^3 + \lambda\varphi^4$ with parameters $m = 1$, $A = 0.09$, $\lambda = 0.01$ and $\omega = 0$ (dotted), $\omega = \omega_c$ (solid) and $\omega = m$ (dashed).

gives an idea that the minimum of effective potential is achieved for $\omega > \omega_c$. Otherwise the minimum is not a true vacuum in the sector of non-zero conserved charge. The solution is stable with respect to decay to free scalar quanta if the energy of the Q-ball, given in Eq. (116), is less than the mass of the corresponding amount of free quanta:

$$E_Q < mQ. \tag{122}$$

The solution to Eq. (120) is found by an argument of continuity. If the particle in Fig. 5 starts at a point that has a smaller potential energy than the origin, it never reaches the origin. If the particle starts close to the top and stays there for a long time (to large values of r), so that the friction term becomes negligible, it has enough energy to go past the origin. By continuity there has to be a solution for which the particle just reaches the origin. This corresponds to the Q-ball solution [24]. In general finding solutions to the equation of motion Eq. (120) is a numerical problem. In some limiting cases analytical approximations can be used. Examples are thin-wall and thick-wall approximations.

We have restricted ourselves here to potentials which do not produce spontaneous symmetry breaking, *i.e.*, $\omega_c^2 \geq 0$. In [75] also the case of spontaneous symmetry breaking, *i.e.*, $\omega_c^2 < 0$ has been investigated. In the latter case there is a maximum charge and energy that Q-balls can have, see [75].

5.3 Thin-wall approximation

In the thin-wall approximation one considers a Q-ball configuration that has a constant value up to a radius R, after which the field vanishes, *i.e.*, the field configuration is a step function $\varphi(r) = \varphi_0\,\theta(R-r)$ (surface contributions are omitted in this configuration). The energy and charge of such a configuration can be obtained from Eqs. (116, 117):

$$\begin{aligned} E &= V_D \left(U(\varphi_0) + \frac{1}{2}\omega^2\varphi_0^2 \right), \\ Q &= \omega\varphi_0^2 V_D, \end{aligned} \tag{123}$$

where V_D is the volume of a D dimensional sphere with radius R. Now one can solve ω from the second equation and solve the energy of the Q-ball in terms of volume and charge:

$$E = V_D U(\varphi_0) + \frac{Q^2}{2\varphi_0^2 V_D}. \tag{124}$$

For a fixed charge Q, E acquires its minimum at $V_D = |Q|/(\varphi_0\sqrt{2U(\varphi_0)})$, thus one obtains

$$E = Q\sqrt{\frac{2U(\varphi_0)}{\varphi_0^2}}. \tag{125}$$

Minimizing this with respect to φ_0 one gets $E = \omega_c Q$ from Eq. (121). Hence the thin-wall limit corresponds to $\omega \to \omega_c$ and $\varphi_0 \to \varphi_c$.

The surface energy can be taken into account by dropping the dissipation term in Eq. (120) and then inserting the solved φ into Eq. (116) (details can be found in [24]):

$$E \approx \omega_c Q + S_D^{\frac{1}{D}} \left(\frac{DQ}{2\omega_c\varphi_c^2} \right)^{\frac{D-1}{D}} \int\limits_0^{\varphi_c} d\varphi\, \sqrt{2U_{\omega_c}(\varphi)}. \tag{126}$$

One should note that here we have only shown that a Q-ball is a stationary point of energy E with a fixed charge Q. Coleman [24] has shown that for large Q the Q-ball solution is the minimum energy configuration, and that Q-balls exist and are stable with respect to the decay to ϕ quanta, if

$$\inf E_Q < mQ. \tag{127}$$

He also showed that if stable thin-walled Q-balls exist for some $Q > 0$, then Q-balls in the thin-wall limit exist and are stable for any $Q' > Q$ [24]. However, Coleman's arguments are purely classical. In [76] Graham has shown that Q-balls are quantum mechanically stable for $Q \gtrsim 7$.

5.4 Thick-wall approximation

Q-balls can also exist for small charges. Kusenko has showed that in the thick-wall limit [77] small stable Q-balls exist in the case of the potential $U(\varphi) = \frac{1}{2}m^2\varphi^2 - A\varphi^3 + \lambda\varphi^4$. Multamäki and Vilja [78] extended this for more general potentials and calculated the thick-wall properties for arbitrary spatial dimensions D.

The idea of the thick-wall approximation is the following: Let us consider a Q-ball configuration with $\omega > \omega_c$. When ω is increased, U_ω dips deeper and φ_0 moves closer to the origin, so that overshoot of origin is avoided, see Figs. (5, 6). When $\omega \to m$ the particle starts so far away from the top that only the lowest order terms have to be taken into account. Whether this approximation is valid depends on the potential. Under these conditions Eq. (121) is not sufficient to determine the existence of thick-wall Q-balls (however, Eq. (121) is a necessary condition for Q-ball stability).

We give here the results of [78]. Consider a potential

$$U(\varphi) = \frac{1}{2}m^2\varphi^2 - A\varphi^B + \lambda\varphi^C + \mathcal{O}(\varphi^{C+1}), \tag{128}$$

where A, $\lambda > 0$ and $C > B > 2$. With redefinitions

$$\xi_i = x_i\sqrt{m^2 - \omega^2}, \qquad \psi = \varphi\left[\frac{A}{m^2 - \omega^2}\right]^{\frac{1}{B-2}} \tag{129}$$

the energy Eq. (116) becomes

$$E \approx A^{-\frac{2}{B-2}}(m^2 - \omega^2)^{1-\frac{D}{2}+\frac{2}{B-2}} S_\psi^B + \omega Q, \tag{130}$$

where

$$S_\psi^B = \int d^D\xi \, \left[\frac{1}{2} |\nabla_\xi \psi|^2 + \frac{1}{2}\psi^2 - \psi^B \right]. \tag{131}$$

S_ψ^B is just a D dimensional action of a bounce in the potential $\frac{1}{2}\psi^2 - \psi^B$. Clearly the action, S_ψ^B can be minimized without knowing ω. Minimizing with respect to ω one obtains after some calculation that Q-balls obey the stability bound Eq. (121) if

$$Q < (1+\zeta)(2\zeta)^\zeta (1+2\zeta)^{-\zeta-\frac{1}{2}} \, 2A^{-\frac{2}{B-2}} S_\psi^B m^{1+2\zeta}, \tag{132}$$

where $\zeta = -D/2 + 2/(B-2)$.

5.5 Q-balls in the MSSM

The flat directions support the existence of Q-balls, since in low energy scale the flat direction grows slower than ϕ^2 Eqs. (85, 88), if $K < 0$.

In the gravity mediated case the potential reads, omitting the A-terms and the Hubble induced corrections,

$$U(|\phi|) = m_{3/2}^2 \left[1 + K \log\left(\frac{|\phi|^2}{M^2} \right) \right] |\phi|^2 + \frac{|\lambda|^2}{M_p^{2(d-3)}} |\phi|^{2(d-1)}. \tag{133}$$

In order to have Q-balls at all we must have $K < 0$. The non-renormalizable term is important in the thin-wall case. The thin-wall behaviour can be solved straightforwardly. From Eq. (121) one obtains

$$\omega_c^2 = m_{3/2}^2 \left[1 + \frac{|K|}{d-2} - |K| \log\left(\frac{\varphi_c^2}{2M^2} \right) \right],$$

$$\varphi_c = \sqrt{2} \left[\frac{M_p^{d-3} m_{3/2}}{|\lambda|} \sqrt{\frac{|K|}{d-2}} \right]^{\frac{1}{d-2}}. \tag{134}$$

One notes that $\omega_c > m_{3/2}$ unless $\varphi > M$, but the correct mass at $\varphi = 0$ is $U''(0) = \infty$, so that this is not a problem. In principle one has to compare ω_c with the renormalized mass, which is $m_R = m_{3/2}(1+\alpha|K|)$ where $\alpha \sim 1$ [18]. The same phenomenon happens in the case of the thick-wall solution.

Thick-wall behaviour is determined by the mass term (85) alone. Solving the equation of motion of Eq. (120) with a gaussian ansatz one obtains [79]

$$\varphi(r) = \varphi_0 \, e^{-\frac{r^2}{R^2}}, \tag{135}$$

where

$$
\begin{aligned}
R^2 &= \frac{2}{|K|m_{3/2}^2}, \\
\omega^2 &= m_{3/2}^2\left[1+(D-1)|K|-|K|\log\left(\frac{\varphi_0^2}{2M^2}\right)\right],
\end{aligned}
\tag{136}
$$

leading to the energy and charge given by

$$
\begin{aligned}
E &= m_{3/2}Q\left(1+\frac{D}{2}|K|\right), \\
Q &= \left(\frac{\pi}{2}\right)^{\frac{D}{2}}\omega\varphi_0^2 R^D.
\end{aligned}
\tag{137}
$$

The profile has been verified numerically and it has been observed that a better fit is achieved by $R \approx |K|^{-1/2}m_{3/2}^{-1}$ [18, 83].

In the gauge mediated case the mass term is

$$
U(\phi) = m_\phi^4 \log\left(1+\frac{|\phi|^2}{m_\phi^2}\right). \tag{138}
$$

The equation of motion (120) is no longer analytically solvable. Even the thin-wall values are not analytically calculable, unless we ignore the non-renormalizable terms, in which case the thin-wall values are $\varphi_c = \infty$ and $\omega_c = 0$. The thin-wall regime is never reached with potentials that grow slower than ϕ^2. Even if an exact solution is not possible, one can approximate the potential (138) with a tree level mass term, $U(\phi) = m_\phi^2|\phi|^2$, for low energy scales and a flat potential, $U(\phi) = U_0 = const.$, for high energy scale (see Eq. (87)). Previously the profile and energy has been solved for $D = 3$ with a flat potential $U(\phi) = const.$ for all ϕ [32]. Here we have solved the problem by approximating the potential with

$$
U(\varphi) = \begin{cases} \frac{1}{2}m_\phi^2\varphi^2, & \varphi \le \varphi_R, \\ U_0 = const., & \varphi > \varphi_R. \end{cases} \tag{139}
$$

In Appendix we give the complete solution to the Q-ball equation, Eq. (120), with this potential. We also make a comparison with the numerical solution obtained in [84] and find that the resulting profile is a very good approximation to the numerical solution, as long as one trusts eye-ball fitting. Another key point to note is that here we do not choose $U_0 = m_\phi^{D+1}$ and $D = 3$, as is usually done. The analytical approximation to the profile is (see Appendix)

$$\varphi(r) \approx \begin{cases} \varphi_0 \, \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega r}\right)^{\frac{D}{2}-1} J_{D/2-1}(\omega r), & r \leq R \\ \varphi_0 \, \Gamma\left(\frac{D}{2}\right)\frac{J_{D/2-1}(\omega R)}{K_{D/2-1}(kR)}\left(\frac{2}{\omega r}\right)^{\frac{D}{2}-1} K_{D/2-1}(kr), & r > R, \end{cases} \tag{140}$$

where R is chosen in such a way that $\varphi(R) = \varphi_R$ and $k = \sqrt{m_\phi^2 - \omega^2}$. In the limit $Q \to \infty$ all the parameters can be given:

$$\begin{aligned} \varphi_0 &\approx \frac{1}{\sqrt{2}J_{D/2}(x_D)}\left(\frac{x_D}{2}\right)^{\frac{D}{2}-1}\left(\frac{2}{\sqrt{\pi}\Gamma(D/2)x_D}\right)^{\frac{D}{D+1}} U_0^{\frac{D-1}{2(D+1)}} Q^{\frac{1}{D+1}}, \\ \omega &\approx \left(\frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\right)^{\frac{1}{D+1}} U_0^{\frac{1}{D+1}} Q^{-\frac{1}{D+1}}, \\ R &\approx \left(\frac{\Gamma(D/2)x_D}{2\pi^{D/2}}\right)^{\frac{1}{D+1}} U_0^{-\frac{1}{D+1}} Q^{\frac{1}{D+1}}, \\ E &\approx \frac{D+1}{D}\left(\frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\right)^{\frac{1}{D+1}} U_0^{\frac{1}{D+1}} Q^{\frac{D}{D+1}}, \end{aligned} \tag{141}$$

where x_D is the first positive zero of the Bessel function $J_{D/2-1}$. These results reproduce the previous results for $U_0 = m^4$ and $D = 3$.

Multamäki and Vilja have solved numerically the Q-ball profiles in both the gravity mediated [83] and gauge mediated case [84]. The results in the gravity mediated case were found to correspond to the Gaussian profile quite well. The gauge mediated case was found to be better described by a kink solution. The analytic solution given here is found to fit quite well the numerical solution of the logarithmic potential in Eq. (138), given in Fig. 1(b) at [84].

5.6 Decay and scattering of Q-balls

The Q-balls formed through gauge mediation mechanism are absolutely stable for large enough Q because the mass increases as $Q^{3/4}$. Therefore for large enough Q, its mass is less than the mass of corresponding amount of quarks, *i.e.*, $m_Q < m_N Q$, where $m_N \approx 1$ GeV is the lightest nucleon mass. For the gravity mediated case this does not occur since the Q-ball mass increases linearly with respect to charge. Therefore the

Q-balls are formed by gravity mediation decay when the temperature falls below m_N [18].

Depending on the interactions a Q-ball can decay into bosons or fermions (decay into its own scalar quanta is forbidden by the stability of Q-ball). If the decay mode is to fermions, then the decay happens by evaporation from the surface [31]. Cohen *et al.* [31] showed that a field theory described by the Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - U(|\phi|) + \psi^\dagger i \sigma^\mu \partial_\mu \psi - ig\phi\psi^\dagger \sigma_2 \psi^* + ig\phi^* \psi^T \sigma_2 \psi, \tag{142}$$

where ψ is a two component Weyl spinor, ϕ is the Q-ball scalar field and σ are the Pauli matrices, leads to an evaporation rate of the Q-ball which is bounded from above:

$$\frac{dN}{dt\, dA} \leq \frac{\omega^3}{192\pi^2}. \tag{143}$$

Here ω is the rotation velocity of the Q-ball. The bound was calculated in the thin-wall limit. The numerical simulations show that the rate almost saturates the bound [31]. The step function is not an applicable profile in all cases. With a generic Q-ball the calculation of the evaporation rate is a numerical problem and has been solved in [78]. In the limit of small coupling the evaporation rate can be calculated analytically. One finds [31]

$$\frac{DN}{dt\, dA} = \frac{g\varphi_0 \omega^2}{64\pi}, \tag{144}$$

where g is the coupling between the Q-ball and fermions.

Q-balls can also decay through scattering against thermal particles [18, 23]. There are two kinds of processes, dissolution and dissociation. In dissociation thermal particles collide with the Q-ball "hard core", which corresponds to a region where thermal particles cannot penetrate. Thermal particles penetrate to a radius at which $g\varphi(r) \approx 3T$. They transfer energy to the Q-ball when they stop. If the time over which the particle stops is short compared to the time over which Q-ball can absorb energy, the particle gets ejected, leaving some of its energy into the hard part. If sufficient energy is delivered to overcome the binding energy of the Q-ball, then the Q-ball dissociates. If less energy is delivered, the Q-ball is able to radiate the extra energy away. In dissolution charge is removed from the soft edge of the Q-ball. At large temperatures there is an equilibrium between the soft edge and the ambient thermal bath, and if the width of the edge is larger than the mean free path of the thermal backround particles, charge diffuses out of the Q-ball. Thermal equilibrium between a Q-ball and thermal particles has also been considered in [80, 81]. When a Q-ball collides with its own quanta in a thermal path, it becomes excited. Q-ball excitation and its relaxation has been numerically studied in [82] in the MSSM with gravity mediated SUSY breaking. It was found that Q-balls can absorb a large amount of extra energy without decaying and that they just radiate the extra energy out.

The scattering properties of the Q-balls have been solved numerically in [78, 83, 84]. The cross-section in the Q-ball collisions was found to be approximately given by the geometrical cross section for all kinds of processes: elastic scattering, fusion and charge-exchange. The gauge mediated Q-balls tend to become larger in collisions. This a process called solitosynthesis and has been studied in [23, 85].

5.7 Thermally distributed Q-balls

Q-balls are formed from the fragmented lumps of the AD condensate as mentioned at the end of Section 4. Since the negative pressure causes the energy of the AD condensate to increase and the spatial perturbations to grow, it is natural that the condensate deforms into a configuration that corresponds to the minimum of energy. This yields a distribution of Q-balls. The form of the Q-ball distribution depends on the energy-to-charge ratio of the AD condensate

$$x = \frac{\rho(\phi)}{mq(\phi)}, \tag{145}$$

where $\rho(\phi)$ is the energy density and $q(\phi)$ the charge density of the AD condensate before fragmentation. According to numerical simulations [25–29] the charge of the AD condensate is almost completely stored in the Q-balls. Then one can write down the equations for the conservation of energy and charge [28]

$$E_{tot} = \int d^D\mathbf{p} \int dQ \, [N_+(\mathbf{p}, Q) + N_-(\mathbf{p}, Q)] \, E(\mathbf{p}, Q), \tag{146}$$

$$Q_{tot} = \int d^D\mathbf{p} \int dQ \, [N_+(\mathbf{p}, Q) - N_-(\mathbf{p}, Q)] \, Q, \tag{147}$$

where $E(\mathbf{p}, Q) = \sqrt{|\mathbf{p}|^2 + m_Q^2}$ is the energy of a Q-ball with charge Q and momentum $\mathbf{p}$, $N_\pm(\mathbf{p}, Q)$ are the number distributions of Q-balls (+) and anti-Q-balls (-), E_{tot} and Q_{tot} are the total energy and charge stored in Q-balls, which are the same as the ones the AD condensate originally had. Since the mass of the Q-ball in the gravity mediated case is $m_Q \approx m|Q|$ (see Eq. (137)), it follows that $x = E_{tot}/mQ_{tot} \geq 1$ independent of the distribution. If $x = 1$ only Q-balls but no anti-Q-balls appear. This is also seen in numerical simulations [26, 28, 29] both in 2-D and 3-D. The case $x = 1$ is most often used in the simulations. At the other extreme $x \gg 1$ a large amount of Q-balls and anti-Q-balls appear [28, 29]. The total charges in the Q-balls and anti-Q-balls are much larger than the total charge Q_{tot} itself. This can simply be understood from energy conservation: the extra energy has to go somewhere, in this case to anti-Q-balls (the other choice, kinetic energy alone, would make Q-balls ultrarelativistic, but one still expects Q-balls to be relativistic [22, 28]). Numerical solutions in [22] indicate that $x \geq 1.1$.

In Section 4 it was noted that the pressure-to-energy density ratio w of the AD condensate acquires its largest negative value for an oscillating orbit and vanishes for a circular orbit. Hence w is roughly proportional to $\epsilon - 1$, where $\epsilon = B/A$ is the ellipticity of the orbit and A (B) is the semi-major (semi-minor) axis of the ellipse, see Section 4.3. The energy-to-charge ratio x behaves roughly as $x \approx 1/\epsilon$.

After the AD condensate fragments, the Q-balls are expected to interact vigorously. Therefore the Q-ball distributions are expected to be obtained by maximizing the entropy. The necessary condition for this is that the interaction rate is greater than the expansion rate, $\Gamma = n_{tot}\sigma v > H$, where n_{tot} is the total number density of Q-balls and anti-Q-balls, $\sigma \approx \pi R_Q^2$ is the geometric cross-section of a Q-ball-Q-ball collision and v is the average velocity of a Q-ball. Considerations supporting this has been given in [22, 28]. Therefore one can describe the Q-ball distribution by an equilibrium distribution.

The grand canonical partition function is given by

$$Z_G = \sum_{N=0}^{\infty} \eta^N Z_N, \tag{148}$$

where $\eta = \exp(\beta\mu_N)$ is the fugacity and μ_N is the chemical potential related to the number of Q-balls, Z_N is the canonical partition function of N Q-balls and β is the inverse temperature of Q-balls. If the Universe has expanded enough, so that Q-balls can be distuingished, one can treat Q-balls as free particles with no interaction. Then the N-particle partition function is given simply by $Z_N = (1/N!)Z_1^N$, since Q-balls are bosons. The grand canonical partition function becomes $Z_G = \exp(\eta Z_1)$. So one just has to calculate the one-particle partition function of Q-balls, which can be done by assuming that the Q-balls obey Boltzmann statistics:

$$\begin{aligned} Z_1 &= g\int \frac{d^D\mathbf{x}d^D\mathbf{p}}{(2\pi)^D}\int_{-\infty}^{\infty} dQ\, e^{-\beta E(p,Q)+\mu Q} \\ &= 2gV_D\beta\int_{-\infty}^{\infty} dQ\left(\frac{m_Q}{2\pi\beta}\right)^{\frac{D+1}{2}} e^{\mu Q}\, K_{\frac{D+1}{2}}(\beta m_Q), \end{aligned} \tag{149}$$

where g is the number of internal degrees of freedom, which for scalars is unity, μ is the chemical potential related to the charge, $E(p,Q) = \sqrt{p^2+m_Q^2}$ is the energy of the Q-ball, m_Q is the mass of the Q-ball with charge Q and $K_\nu(x)$ is the modified Bessel function. For the gravity mediated case we have $m_Q \approx m|Q|$ and the partition function is completely solvable and it has been solved in [28]. For the gauge mediated it is not possible to calculate the partition function for general μ. However, for the special case $\mu = 0$, which corresponds to $Q_{tot} = 0$, it can be calculated and has been done in [34]. Numerical simulations [28, 29] show that the equilibrium distribution is a very good approximation in the gravity mediated case. For the gauge mediated case the energy-to-charge ratio can be lower [22]. However, the mass of the Q-ball in this case is $m_Q \approx m|Q|^{D/(D+1)}$. Then thermalization should be possible even with lower x. In addition, the negative pressure is larger in this case, so that the energy grows faster and there exist more instabilities for the oscillating case [27].

6 Q-balls in cosmology

The copious production of Q-balls from a fragmentation of Affleck-Dine condensate may leave the Universe filled with Q-balls. Their existence in the Universe before reheating may have significant consequences for the evolution of the Universe and could leave a detectable trace. A number of different cosmological scenarios involving Q-balls have been suggested. For a more complete list of cosmological applications see the review by Enqvist and Mazumdar [37].

6.1 Baryogenesis

Depending on the flat direction the Q-ball can carry baryon and/or lepton number. The lifetime of the Q-ball in the gravity mediated case is [18]

$$\tau \approx \frac{48\pi}{R^2\omega^3}\, Q, \tag{150}$$

where R is the radius of the thick-walled Q-ball and the upper bound of the evaporation rate has been assumed to be saturated. The temperature of the Universe at which the Q-balls decay is [79]

$$T_d \approx 0.06 \left(\frac{f_s}{|K|}\right)^{\frac{1}{2}} \left(\frac{m}{100\ \mathrm{GeV}}\right)^{\frac{1}{2}} \left(\frac{10^{20}}{Q}\right)^{\frac{1}{2}}\ \mathrm{GeV}, \tag{151}$$

where f_s is a possible enhancement factor of the decay rate due to decay into pairs of light scalars. $f_s \approx 1$ for purely baryonic directions and for directions containing leptons $f_s \approx 170(1 + 2.1g^4)|K|^{-1/2}$, where g is the coupling between the condensate scalars and the decay products [79].

The numerical simulations show that at least for $x \approx 1$ the largest Q-balls are in the range $10^{16} \ldots 10^{23}$ [26]. Then it is possible that Q-balls decay after the electroweak phase transition. The sphaleron transitions wash out the baryon number produced by the conventional AD mechanism, if the temperature of the Universe is above the electroweak scale. Baryon number in the Q-balls is protected from the electroweak wash-out if the Q-balls decay after the electroweak phase transition [18, 79].

6.2 Dark matter

The experimental observations of large scale structure [86], supernovas [87, 88] and cosmic microwave backround together indicate [89] that the Universe is composed of baryonic matter $\Omega_b h^2 = 0.023 \pm 0.003$ or $\Omega_b = 0.05 \pm 0.02$, cold dark matter (CDM) $\Omega_{CDM} h^2 = 0.14^{+0.03}_{-0.02}$ and total amount of matter (baryons and dark matter) $\Omega_m = 0.34 \pm 0.07$, cosmological constant or dark energy $\Omega_\Lambda = 0.65^{+0.05}_{-0.06}$ with $h = 0.67 \pm 0.09$ and total density $\Omega_{tot} = 0.99^{+0.03}_{-0.04}$. $\Omega_i = \rho_i/\rho_c$ is the mass fraction of the component i where ρ_i is the energy density of the component i and ρ_c is the critical energy density. $h = H_0/(100 kms^{-1}Mpc^{-1})$ where H_0 is the Hubble parameter today.

Q-balls are natural candidates for dark matter. In the gauge mediation scenario Q-balls are absolutely stable with respect to decay to SM baryons due to kinematical reasons, if their charge is large enough:

$$Q \gtrsim \left(\frac{m_\phi}{m_b}\right)^4 \sim 10^{12} \tag{152}$$

for $m_\phi \sim 1$ TeV and $m_b \sim 1$ GeV. Therefore Q-balls with charges larger than 10^{12} can act as dark matter. Such Q-balls could be detected in experiments. The possible experimental signatures have been discussed in [90]. The constraints are mainly dependent on whether the Q-ball is a supersymmetric electrically charged (SECS) or neutral (SENS) soliton. The MACRO search [91] produces constraint to the flux of SECS

$$F_{SECS} \lesssim 10^{-14} cm^{-2} s^{-1} sr^{-1}. \tag{153}$$

The Baikal deep underwater array, "Gyrlyanda", and Kamiokande Cherenkov detector constrain the flux of SENS as [92]

$$F_{SENS} \lesssim \begin{cases} 10^{-16} cm^{-2} s^{-1} sr^{-1}, & \text{for } \sigma > 1.9 \cdot 10^{-22} cm^2, \\ 10^{-15} cm^{-2} s^{-1} sr^{-1}, & \text{for } \sigma = 10^{-26} cm^2. \end{cases} \tag{154}$$

These constraints translate into lower limits of baryon number of the dark matter Q-balls: for SECS $Q \gtrsim 10^{21}$ and for SENS $Q \gtrsim 10^{22}$ assuming $m_\phi = 1$ TeV, $\rho_{DM} \approx 0.3$ GeV$/cm^3$ [90] and all dark matter is in Q-balls. Q-ball dark matter in the gauge mediated scenario is also constrained by the difficulty of having Q-balls to produce both the observed baryon number and dark matter without having a large reheating temperature [27].

Q-ball dark matter in the gravity mediated case has also been considered [79]. Since in this case the Q-balls decay, DM has to be formed of the decay products. If R-parity is conserved the Q-balls produce both baryons and lightest supersymmetric particles (LSP), typically the neutralino. Based on this Enqvist and McDonald [79] showed that baryon to dark matter ratio can be accounted for if

$$3.7 \text{ GeV} \lesssim \left(\frac{N_\chi}{3}\right) f_B m_\chi \lesssim 67 \text{ GeV}, \tag{155}$$

where N_χ is the number of LSPs produced per baryon number, m_χ is the mass of the neutralino and f_B is the fraction of baryons in Q-balls. Numerical simulations indicate that after the AD condensate fragmentation nearly all of the baryon number finds its way to Q-balls [26, 28]. If the AD condensate is the only source of baryons $f_B \approx 1$. This applies to the gauge mediated scenario, too [25]. This would mean that all the baryon number is in the Q-balls, which is unacceptable for the gauge mediated scenario. Thermal effects transport charge from Q-balls to the surrounding plasma thus decreasing f_B. In [80] it was estimated that the correct baryon-to-photon ratio for $m_\phi \sim 1$ TeV if the initial charge of the Q-balls is in the range $Q \sim 10^{23} \ldots 10^{28}$. However, in [86] it was argued that the rate at which the charge diffuses away is suppressed resulting into a bound $Q \gtrsim 10^{21} \ldots 10^{23}$.

6.3 Problems of Cold Dark Matter

The cosmological model that is in good agreement with the CMB observations and large scale structure, on scales $\gg 1$ Mpc that corresponds to scales of galaxy clusters, is based on colliosionless cold dark matter (CDM). The profile of CDM halos is well approximated by the Navarro-Frenk-White profile [93]. The CDM consists of weakly interacting massive particles (WIMP) (or axions), which means that both the interactions with ordinary matter and other DM is weak. On galactic and sub-galactic scales, $\leq$ a few Mpc, there are a number of discrepancies with observations compared to analytical calculations and numerical simulations.

The generic prediction for weakly self-interacting DM is that CDM forms triaxial halos with dense cores and significant dense substructures. Howeve, lensing observations of clusters [94] reveal central regions with nearly spherical low density cores on galactic scale. Dwarf irregular galaxies appear to have low density cores [95–98] with much shallower profiles than predicted by numerical simulations [99, 100]. Observations of the Local Group reveal less than 100 galaxies [101], while numerical simulations [102, 103] and analytical theory [104, 105] predict that there should be roughly 1000 discrete dark matter halos within the Local Group. However, it seems that the observational status is quite unclear [106].

6.4 Self-Interacting Dark Matter

The success of the CDM model on large scales suggests that a modification of the CDM properties may be the best approach for resolving the problems on small scales. Allowing the DM to be warm alleviates some of the problems, but this seems to be ruled out [107]. Another approach suggested by Spergel and Steinhardt is to allow DM to be self-interacting [108]. The interactions of DM with ordinary matter have stringent constraints but as long as the DM annihilation cross section is much smaller than the scattering cross section, the self-interaction is only mildly constrained.

The mean free path of the DM particle should be on the range $\lambda = 1\text{kpc} \dots 1\text{Mpc}$, so that it does not change the structure on large scales ($\gg 1$ Mpc) [108]. The mean free path is defined by

$$\lambda = \frac{1}{\sigma_{DD} n_{DM}}, \tag{156}$$

where σ_{DD} is the DM-DM scattering cross section and $n_{DM} \approx \rho_{DM}/m_{DM}$ is the number density of DM, where $\rho_{DM} \approx 0.4$ GeV$/cm^3$ is the mean density of DM at the solar radius and m_{DM} is the typical mass of the DM particle. From Eq. (156) it follows that

$$\frac{\sigma_{DD}}{m_{DM}} \approx 8 \cdot 10^{-25} \dots 10^{-22} \, \frac{\text{cm}^2}{\text{GeV}}. \tag{157}$$

Actually the upper limit drops down to $10^{-23}\text{cm}^2\ \text{GeV}^{-1}$ according to numerical simulations on cosmological structure formation [109]. Eq. (157) implies a cross section that is typical for a hadron with a mass $m_{DM} \sim 1$ GeV, so SIDM is strongly interacting.

When the structures form via gravitational instability, the central density increase. When the density has increased enough the collisions of DM particles become more frequent, so that DM particles start to scatter out of the center as fast as they are accreted. Then the growth of central density stops and DM forms a core (a region where the density is essentially constant). The collisionless CDM does not scatter, so the central density increases more to form a cusp in the center.

6.5 Q-balls as Self-Interacting Dark Matter

Q-balls in certain extensions of MSSM, for instance in the hidden sector, have been proposed as candidates for SIDM [33]. Since Q-balls are extended objects, they have naturally large cross sections comparable to their size (the Q-ball cross sections are given by their geometrical cross sections $\sigma_{QQ} = \pi R^2$ with R given by Eqs. (135, 140)). Q-balls can fuse in scatterings, but total annihilation seems to be difficult.

In paper IV we used the constraint Eq. (157) to find phenomenological constraints for the properties of the Q-ball dark matter. We acquired the following constraints for thin- and thick-walled Q-balls:

$$
\begin{aligned}
3 \cdot 10^{13} \quad &< \quad \frac{\omega_c^5 \phi_0^4}{\mathrm{MeV}^9} Q \quad < \quad 5 \cdot 10^{16}, \quad (\mathrm{thin-wall}) && (158)\\
3 \cdot 10^{18} \quad &< \quad \left(\frac{m_\phi}{\mathrm{MeV}}\right)^{12} Q \quad < \quad 7 \cdot 10^{22}, \quad (\mathrm{thick-wall, gauge}) && (159)\\
2 \cdot 10^{6} \quad &< \quad |K| \left(\frac{m_{3/2}}{\mathrm{MeV}}\right)^{3} Q \quad < \quad 2 \cdot 10^{7}, \quad (\mathrm{thick-wall, grav.}). && (160)
\end{aligned}
$$

The commonly considered Q-balls carrying baryon number [23, 79] are clearly unacceptable candidates for SIDM, since the free scalar has a mass of the order of SUSY breaking mass, which is $m_\phi \sim 1$ TeV. However, if there are Q-balls in models where the $U(1)$-symmetry is not related to the baryon or lepton number, it might be possible to satisfy the bounds (158)-(160).

Another choide is to have thermally distributed dark matter, where the one-particle partition function is given by Eq. (149). This leads to bounds that can be seen in Fig. 2 of Paper III [34].

As discussed in Section 5.6 the Q-ball evaporation rate depends on its couplings. However, the evaporation rate has an upper bound (Eq. (143)). If one assumes that Q-ball dark matter has to be present at the time of galaxy formation and that the bound Eq. (143) is saturated, this leads to constraint $Q \gtrsim 10^{29} m_\phi/$ GeV for the gauge mediated Q-ball.[9] For the gravity mediated Q-ball the same bound is $10^{36} m_{3/2}/$ GeV. These bounds are much higher than the typical sizes of the largest Q-balls seen in simulations [25, 26].

[9] However, the decay of large Q-balls can be forbidden because of kinematical reasons, see Eq. (152).

7 Conclusions

All supersymmetric theories have in the globally supersymmetric limit many flat directions, where the scalar potential vanishes. These flat directions stay flat even under radiative corrections due to non-renormalization theorems. The only corrections to their potential come from soft SUSY breaking and non-renormalizable operators. Couplings through the Kähler potential generate terms that are more important in the Early Universe than the standard soft breaking terms. These set the value of the field along the flat direction to a high energy scale during inflation. After inflation the fields evolve to low energy scales and usually end up rotating around the origin and hence producing a charged AD condensate. If the AD condensate is formed of fields in the MSSM, which carry $B - L$, it is possible to produce the baryon asymmetry required by observations.

If the mass term along the flat direction grows slower than ϕ^2, the condensate has a negative pressure and is is unstable with respect to spatial perturbations. The magnitude of the pressure depends on the ellipticity of the orbit. The largest absolute value for the pressure is obtained for pure oscillation, whereas the pressure vanishes for a circular orbit. The energy-to-charge ratio is also linked to the orbit. The energy of the AD condensate is approximately independent of the orbit. However, the charge depends strongly on the orbit: it vanishes for the oscillation and is maximal for a circular orbit. This leads to an energy-to-charge ratio, which is unlimited for oscillation and acquires its minimum for a circular orbit.

The negative pressure makes perturbations grow and become non-linear fragmenting the condensate. The fragmented charged lumps form non-topoligical solitons, Q-balls. If the energy-to-charge ratio of the AD condensate is large, large numbers of both Q-balls and anti-Q-balls appear. Analytical considerations show that the thermalization of the Q-ball distribution is likely such that the distribution can be described by an equilibrium distribution. This has been verified in numerical simulations for both 2 and 3 spatial dimensions in the gravity mediated scenario for $x \gg 1$. If $x = 1$, only Q-balls appear, as is expected by analytical considerations. In this case the orbit is close to circular and only a small instability band appears both in the gauge and the gravity mediated case. For pure oscillation in the gauge mediated case it has been numerically shown that there appear more unstable bands, which can produce more Q-balls.

Besides baryogenesis the Q-balls can provide dark matter. For MSSM with gauge mediated SUSY breaking large Q-balls are absolutely stable against decay and could have survived until today and act as dark matter. Outside MSSM there are other choices. Lately there has been much consideration regarding Self-Interacting Dark Matter (SIDM), where the standard cold dark matter is allowed to have a strong self-interaction. The required self-interaction strength is comparable to hadronic interactions. Q-balls are extended objects and have a large geometrical cross section. Therefore they are natural candidates for SIDM. This in general requires that the Q-balls are formed from flat directions outside MSSM.

In summary, the Affleck-Dine mechanism provides an interesting possibility for baryogenesis. In the MSSM there are several flat directions along which the AD mechanism can be realized. They also support the existence of Q-balls, into which the AD condensates decay in some cases. Q-balls can have important cosmological consequences, and they can be of relevance to the evolution of the Early Universe. They may also be a candidate for dark matter, in some cases also self-interacting. Most importantly one should note that if the Higgs mass $m_H > 120$ GeV, then there is no electroweak baryogenesis in the MSSM [9], and the alternative within MSSM is the Affleck-Dine mechanism along flat directions. But even if it were found in future experiments that $m_H < 120$ GeV, the flat directions are still there. Along them baryon asymmetry can still be produced and one would obtain constraints on the possible form of the non-renormalizable operators lifting the flat directions. One should also point out that the electroweak baryogenesis requires that the reheating temperature has to be larger than electroweak scale, whereas the Affleck-Dine mechanism does not suffer from such a restriction. So whatever the result concerning Higgs mass, the flat directions will remain important for the cosmological evolution of the Early Universe.

8 Appendix

Q-ball solutions in a flat potential

We argued in Section 5.5 that Eq. (140) gives a very good approximation to the numerical solution with potential given in Eq. (138). Here we solve the Q-ball equation (120) with the potential

$$U(\varphi) = \begin{cases} \frac{1}{2}m^2\varphi^2, & \varphi \leq \varphi_R \\ U_0 = const., & \varphi > \varphi_R. \end{cases} \tag{161}$$

We leave U_0 arbitrary, but require that the potential is continuous. The Q-ball equation Eq. (120) in the two regimes are

$$\varphi''(r) + \frac{D-1}{r}\varphi'(r) + \omega^2\varphi(r) = 0, \quad 0 \leq r \leq R, \tag{162}$$

$$\varphi''(r) + \frac{D-1}{r}\varphi'(r) - (m^2 - \omega^2)\varphi(r) = 0, \qquad r > R, \tag{163}$$

where R is chosen such that $\varphi(R) = \varphi_R$. Eq. (162) can be transformed into the Bessel equation and the modified Bessel equation. Therefore the complete solution is given by

$$\varphi(r) = \begin{cases} r^{-(\frac{D}{2}-1)}\left[A_1 J_{D/2-1}(\omega r) + A_2 Y_{D/2-1}(\omega r)\right], & r \leq R, \\ r^{-(\frac{D}{2}-1)}\left[B_1 K_{D/2-1}(kr) + B_2 I_{D/2-1}(kr)\right], & r \geq R, \end{cases} \tag{164}$$

where J_ν, Y_ν are the Bessel and Neumann functions, I_ν, K_ν are the modified Bessel and Neumann functions, $k = \sqrt{m^2 - \omega^2}$, and A_i, B_i are constants to be chosen from the boundary conditions. The behaviour of the Bessel functions is known [110], so one obtains from the boundary conditions $\varphi'(0) = \varphi(\infty) = 0$ and $\varphi(0) = \varphi_0 < \infty$ that

$$\varphi(r) = \begin{cases} \varphi_0\Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega r}\right)^{\frac{D}{2}-1} J_{D/2-1}(\omega r), & r \leq R, \\ \varphi_0\Gamma\left(\frac{D}{2}\right)\frac{J_{D/2-1}(\omega R)}{K_{D/2-1}(kR)}\left(\frac{2}{\omega r}\right)^{\frac{D}{2}-1} K_{D/2-1}(kr), & r > R, \end{cases} \tag{165}$$

where the solution has been required to be continuous at $r = R$. We also require the continuity of the derivative at $r = R$ and the continuity of the potential which give rise to the extra conditions

$$\omega\,\frac{J_{D/2}(\omega R)}{J_{D/2-1}(\omega R)} = k\,\frac{K_{D/2}(kR)}{K_{D/2-1}(kR)}, \tag{166}$$

$$U_0 = \frac{1}{2} m^2 \varphi_0^2 \, \Gamma\left(\frac{D}{2}\right)^2 \left(\frac{2}{\omega R}\right)^{D-2} J^2_{D/2-1}(\omega R). \tag{167}$$

With the help of the recursion relations of Bessel functions [110] one obtains two other conditions from Eq. (166)

$$-\,\omega\, \frac{J_{D/2-2}(\omega R)}{J_{D/2-1}(\omega R)} \;=\; k\, \frac{K_{D/2-2}(kR)}{K_{D/2-1}(kR)}, \tag{168}$$

$$\omega^2 \left[\frac{J_{D/2+1}(\omega R)}{J_{D/2-1}(\omega R)} + 1\right] \;=\; k^2 \left[\frac{K_{D/2+1}(kR)}{K_{D/2-1}(kR)} - 1\right]. \tag{169}$$

We want to find the energy and charge of the Q-ball solution

$$E \;=\; \int d^D\mathbf{x} \left[\frac{1}{2}|\nabla|^2 + U(\varphi) - \frac{1}{2}\omega^2\varphi^2\right] + \omega Q, \tag{170}$$

$$Q \;=\; \omega \int d^D\mathbf{x} \varphi^2. \tag{171}$$

In order to calculate the energy and charge we need the following integral, which is valid for all Bessel functions [110]:

$$\int dx\, x\, Z^2_\nu(\alpha x) = \frac{1}{2} x^2 \left[Z^2_\nu(\alpha x) - Z_{\nu-1}(\alpha x) Z_{\nu+1}(\alpha x)\right]. \tag{172}$$

We tabulate the results of all integrals needed for the energy, Eq. (170), and charge, Eq. (171), of the Q-ball solution, Eq. (165):

$$\int\limits_{r\le R} d^D\mathbf{x}\, \varphi(r)^2 \;=\; \pi^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega}\right)^{D-2} R^2 \varphi_0^2 \times$$
$$\left[J^2_{D/2-1}(\omega R) - J_{D/2-2}(\omega R) J_{D/2}(\omega R)\right], \tag{173}$$

$$\int\limits_{r> R} d^D\mathbf{x}\, \varphi(r)^2 \;=\; \pi^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega}\right)^{D-2} R^2 \varphi_0^2 J^2_{D/2-1}(\omega R) \times$$
$$\left[\frac{K_{D/2-2}(kR) K_{D/2}(kR)}{K^2_{D/2-1}(kR)} - 1\right], \tag{174}$$

$$\int\limits_{r\le R} d^D\mathbf{x}\, |\nabla\varphi|^2 \;=\; \pi^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega}\right)^{D-2} \omega^2 R^2 \varphi_0^2 \times$$
$$\left[J^2_{D/2}(\omega R) - J_{D/2-1}(\omega R) J_{D/2+1}(\omega R)\right], \tag{175}$$

$$\int\limits_{r> R} d^D\mathbf{x}\, |\nabla\varphi|^2 \;=\; \pi^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)\left(\frac{2}{\omega}\right)^{D-2} k^2 R^2 \varphi_0^2 \frac{J^2_{D/2-1}(\omega R)}{K^2_{D/2-1}(kR)} \times$$

$$
\left[K_{D/2-1}(kR)K_{D/2+1}(kR) - K^2_{D/2}(kR)\right], \tag{176}
$$

$$
\int\limits_{r\leq R} d^D\mathbf{x} \;=\; \frac{\pi^{\frac{D}{2}}R^D}{\Gamma(D/2+1)}. \tag{177}
$$

Using the constraints, Eqs. (166)-(169), the energy and charge, Eqs. (170, 171), of the Q-ball are

$$
E \;=\; \omega Q + \frac{\pi^{D/2}R^D U_0}{\Gamma(D/2+1)}, \tag{178}
$$

$$
Q \;=\; -\frac{2\pi^{D/2}}{\Gamma(D/2)}\,\frac{\omega R^D U_0}{k^2}\,\frac{J_{D/2-2}(\omega R)\,J_{D/2}(\omega R)}{J^2_{D/2-1}(\omega R)}. \tag{179}
$$

One still has to solve $R = R(\omega)$ from the constraint (166). Then in principle everything is known and one can solve $\omega = \omega(Q)$ from Eq. (179) and finally obtain from Eq. (178) $E = E(Q)$. Note that there is no minimization with respect to any parameter left. Since we originally chose to solve ω, $Q > 0$ and the Q-ball solution has to be positive decreasing function, we must have $J_{D/2-2}(\omega R) < 0$ and one gets $\omega R >$ 1st positive zero of $J_{D/2-2}$. On the other hand charge has to be finite, so $\omega R <$ 1st positive zero of $J_{D/2-1}$. Zeros of Bessel functions can be found in tables [110].

Let us turn to the solution of R as a function of ω. Eq. (166) is not solvable for generic D but for $D = 3$ Bessel functions can be given in terms of elementary functions and then one obtains

$$
R = \frac{\pi - \mathrm{Arccot}\left(\sqrt{\frac{\mathrm{k}}{\omega}}\right)}{\omega}, \tag{180}
$$

where $0 < \omega < m$ and $k = \sqrt{m^2 - \omega^2}$. With the help of this we have plotted charge vs. ω in Fig. 7. We have also plotted energy vs. charge in Fig. 8.

One can see from Fig. 7 that when $\omega \to 0$ the charge $Q \to \infty$. However, also the limit $\omega \to m$ produces $Q \to \infty$. The correct Q-ball solution can be found from Fig. 8. The limit $\omega \to 0$ corresponds to the curve below the stability line $E = mQ$. From the detail one can see that when $\omega \to m$ the mass of the Q-ball first crosses the stability line and then turns over to approach the stability line from above. The turning point corresponds to the minimum at Fig. 7. This in itself contains no physics. Rather it signals the breakdown of the approximation used, since the piece of the flat potential becomes smaller in in this limit and at some point the main contribution

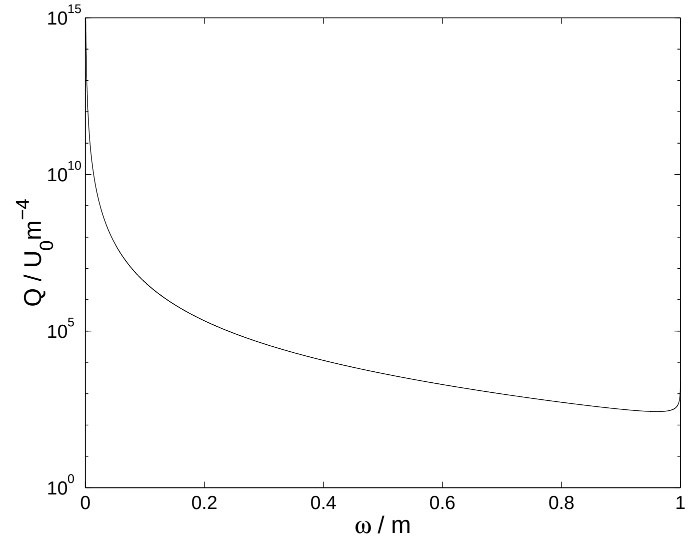

Figure 7: Charge of the Q-ball vs. ω in $D = 3$.

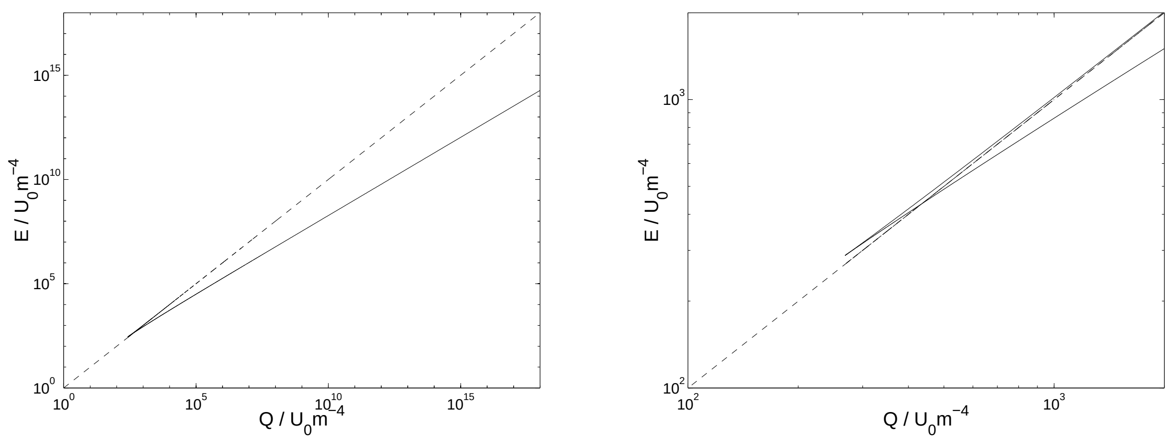

Figure 8: Energy vs. Charge of the Q-ball $D = 3$. The right figure is a detail of the left figure. Q-ball (solid) and stability line (dashed)

to the potential is received from the φ^2 part. The approximation is valid only for $Q \gtrsim 10^3$.

Since the Bessel functions behave qualitatively in the same way, Figs. (7, 8) give the correct qualitative behaviour for all D. With this as quidance we calculate the $\omega \to 0$, $Q \to \infty$, limit for general D. Eq. (180) implies that $R \to \infty$ and $\omega R \to x_D$ in this limit, where $J_{D/2-1}(x_D) = 0$. Then Eq. (166) gives the first order correction to $\omega R = x_D - \Delta(\omega R)$

$$\Delta(\omega R) = \frac{\omega}{k} = \frac{\omega}{m} + \mathcal{O}(\omega). \tag{181}$$

Next one expands Eq. (167) and Eqs. (181, 179) around $\omega R = x_D$ using Eq. (181) to obtain

$$U_0 = \frac{1}{2}\omega^2\varphi_0^2\Gamma\left(\frac{D}{2}\right)^2\left(\frac{2}{x_D}\right)^{D-2} J_{D/2}^2(x_D), \tag{182}$$

$$Q = \frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\frac{U_0}{\omega^{D+1}}, \tag{183}$$

$$E = \frac{D+1}{D}\frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\frac{U_0}{\omega^D}. \tag{184}$$

Then one can solve $\omega = \omega(Q)$ from Eq. (183) to obtain

$$\omega = \left(\frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\right)^{\frac{1}{D+1}} U_0^{\frac{1}{D+1}} Q^{-\frac{1}{D+1}}. \tag{185}$$

With this result one can solve everything as a function of Q-ball charge Q. We get

$$R \approx \frac{x_D}{\omega} \approx \left(\frac{\Gamma(D/2)x_D}{2\pi^{D/2}}\right)^{\frac{1}{D+1}} U_0^{-\frac{1}{D+1}} Q^{\frac{1}{D+1}}, \tag{186}$$

$$\varphi_0 \approx \frac{1}{\sqrt{2}J_{D/2}(x_D)}\left(\frac{x_D}{2}\right)^{\frac{D}{2}-1}\left(\frac{2}{\sqrt{\pi}\Gamma(D/2)x_D}\right)^{\frac{D}{D+1}} U_0^{\frac{D-1}{2(D+1)}} Q^{\frac{1}{D+1}}, \tag{187}$$

$$E \approx \frac{D+1}{D}\left(\frac{2\pi^{D/2}x_D^D}{\Gamma(D/2)}\right)^{\frac{1}{D+1}} U_0^{\frac{1}{D+1}} Q^{\frac{D}{D+1}}. \tag{188}$$

The Q-ball profile is given by Eq. (165). One can check that all the quantities given in Eqs. (185)-(188) reduce in $D = 3$ to the previous results [32, 80], where $x_3 = \pi$ is used.

The value of U_0 has been left undetermined here. Usually one has considered $U_0 = m^4$ in $D = 3$. However, this is not a good approximation to the logarithmic potential given in Eq. (88). For example if $Q = 10^{12}$ one gets $\varphi_0 = 10^3 m$ and this results to a potential value $U(\varphi_0) \approx 13.8m^4$. Besides if one plots the profile in Eq. (165) with $U_0 = m^4$, one finds that the result has the same profile as the numerical solution, Fig. 2 in [84], but the scale of the solution is of order one half of the numerical solution. A better way is to fit the values of φ_0 and ω to the numerical solution. If one takes exactly the same values as acquired in the numerical solution one gets already a very good fit, Fig. 9. One can do better, if one takes different ω for the numerical solution and the analytic approximation given here. Then one obtains almost perfect fit, if one trusts eye-ball fitting, see Fig. 9. The comparison in $D = 3$ is done with the accurate analytical formulas. We have not used the approximation $Q \to \infty$ in producing Fig. 9.

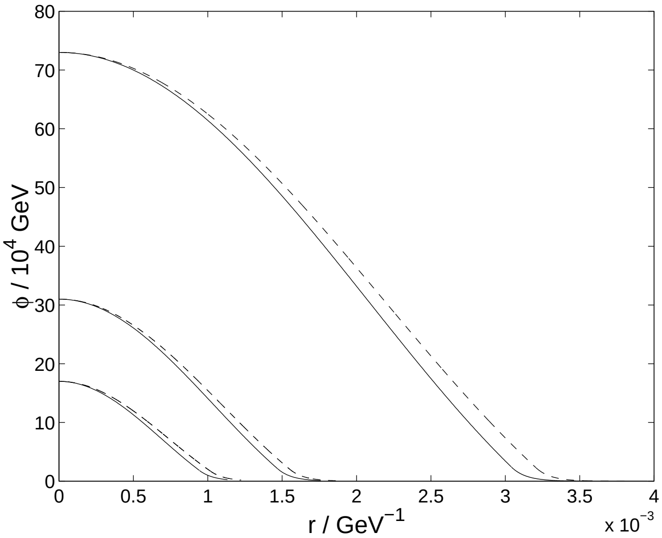

Figure 9: The Q-ball profile plotted in the same units as in [84]. The starting values have been approximated as $\varphi_0/m^4 = 73,\ 31,\ 17$ with $\omega/m = 0.1,\ 0.2,\ 0.3$ from top to bottom in $D = 3$ (solid lines). Comparison with [84] Fig. 2 gives that profile is almost the same except at the region $r \sim R$ the approximation is slightly below the numerical one. This has been fixed by using here $\omega/m = 0.095,\ 0.19,\ 0.28$ (dashed lines).

The comparison with the numerical profile indicates that the approximation of the logarithmic with flat potential works very well. One should clearly do a full numerical fitting in order to achieve better results. Besides the $D = 2$ one cannot solve Eq. (166) analytically, but it has to be done numerically.

In summary, we have presented an analytic approximation to the Q-ball profile solved numerically in [84] and find that the approximation fits very well. The approximation also reproduces the previous approximations for a totally flat potential in the $Q \to \infty$ limit.

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