Stability of Cahn–Hilliard Fronts in Three Dimensions

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Academic dissertation

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Preface

The topic of this work was suggested to me in 1997 by Professor Kupiainen, who had been working on the one-dimensional case and was then finishing [BKT99] with the other authors. Thus began a joint project where I worked on detailed analysis of the linearised problem, while he and Jari Taskinen were looking at the nonlinear analysis. Kupiainen and Taskinen were trying to formulate assumptions on the linear semigroup that would be sufficient for carrying out the nonlinear analysis, I was trying to find out what I could actually prove about the linear semigroup, and many meetings were held trying to fit the two pieces together.

The bulk of the present work consists of the linear analysis, which borrows heavily from [BKT99] but is otherwise my work. The nonlinear analysis of Chapter 6 and parts of the introduction were written jointly with Kupiainen and Taskinen.

I wish to thank Professors Kupiainen and Taskinen for their help in creating this work and Alain Schenkel for suggesting many improvements. I would also like to thank Mikko Stenlund, Ville Hakulinen, all my other friends and colleagues in our research group and my parents.

Some parts of this work, Chapter 3 in particular, involve lengthy routine computations. Mathematica and Maxima were used for these. I would like to extend my thanks to the people who have contributed to the development of Maxima and other free software, which provides for most of my computing needs.

Contents

Chapter 1

Introduction

1.1 The equation

In this work we consider the Cahn–Hilliard equation, which we write

$$\frac{\partial \psi}{\partial t} = \triangle(-\triangle \psi - \tfrac{1}{2}\psi + \tfrac{1}{2}\psi^3). \tag{1.1}$$

Denote the spatial coordinates by

$$\underline{x} = (x, \vec{x}) \in \mathbb{R}^d, \quad x \in \mathbb{R}, \quad \vec{x} \in \mathbb{R}^{d-1}.$$

(1.1) has the time-independent solution

$$\psi_0(\underline{x}) := \tanh(\frac{x}{2}), \tag{1.2}$$

which is also independent of $\vec{x}$. Our aim is to establish the stability of this solution. This has been done in one dimension in [BKT99], where the following asymptotic behaviour was established:

$$\psi(x,t) = \psi_0(x-a) + \frac{A}{\sqrt{t}}\frac{\mathrm{d}}{\mathrm{d}x}\big(\psi_0(x)e^{-\frac{x^2}{4t}}\big) - \frac{B}{\sqrt{t}}\frac{\mathrm{d}}{\mathrm{d}x}e^{-\frac{x^2}{4t}} + o(\frac{1}{t}).$$

Thus at large times there is a translated front and two perturbation terms near the origin: one of magnitude proportional to $1/\sqrt{t}$ and constant width and another of magnitude proportional to $1/t$ and width to $\sqrt{t}$.

The proof in [BKT99] is not directly applicable to several dimensions although we shall use rather similar techniques. One would also expect the equation to behave somewhat differently: in several dimensions an initial perturbation of finite mass should not be able to translate the whole front. We shall prove the following:

Theorem 1.1. *Let $d \geq 3$, $r > d+1$ and assume*

$$|\eta_0(\underline{x})| \leq \frac{\delta}{(1+|\underline{x}|)^r}$$

for a small enough δ. Then (1.1) *with the initial condition $\psi(0,\underline{x}) = \psi_0(x) + \eta_0(\underline{x})$ has a unique classical solution, which for $t \geq 1$ satisfies*

$$\psi(t,\underline{x}) = \psi_0(x) + \frac{A}{2}\partial_x\psi_0(x)\phi(t,\vec{x}) + \tilde{\psi}(t,\underline{x}) \tag{1.3}$$

where

$$A = \int_{\mathbb{R}^d} \eta_0 \,\mathrm{d}\underline{x}, \qquad \hat{\phi}(t,\vec{k}) = e^{-\frac{1}{3}|\vec{k}|^3 t}, \qquad |\tilde{\psi}(t,\underline{x})| \leq \frac{C\delta}{t^{\frac{d-1}{3}+\frac{1}{12}}}.$$

(1.3) describes by linear approximation a local translation of the front by $\frac{A}{2}\phi(t,\vec{x})$ at each $\vec{x}$. Because $\phi(t,\vec{x}) = t^{-\frac{d+1}{3}}\phi(1,\vec{x}/t^{1/3})$, as t increases the perturbation grows wider but reduces in magnitude appropriately to preserve mass. The decay rate is different from normal diffusion where one would have $\sqrt{t}$ instead of $t^{1/3}$. Such scaling was to be expected in light of the work on the linear problem in [SO93] and [BR96].

Unfortunately technical problems prevent us from establishing this result in two dimensions. We do not know whether it holds there. We will try to point out the problems by generally assuming $d > 1$ and indicating explicitly when $d > 2$ is needed.

In bounded domains quite a bit more is known about (1.1) because with appropriate boundary conditions the free energy

$$\int (\tfrac{1}{2}|\nabla\psi|^2 + \tfrac{1}{8}(\psi^2-1)^2)\,\mathrm{d}x$$

is a Lyapunov function. It can also be useful in $\mathbb{R}$ as demonstrated in [CCO01] because in one dimension ψ_0 has finite free energy. That, however, is not true in higher dimensions.

Writing $\psi = \psi_0 + \eta$ we get an equation for η:

$$\frac{\partial\eta}{\partial t} = \triangle(H\eta + \tfrac{3}{2}\psi_0\eta^2 + \tfrac{1}{2}\eta^3), \tag{1.4}$$

where $H := -\triangle + 1 + V$ and

$$V(\underline{x}) := -\tfrac{3}{2}\cosh(\tfrac{x}{2})^{-2}.$$

This is equivalent to the integral equation

$$\eta(t) = e^{t\triangle H}\eta(0) + \int_0^t e^{(t-s)\triangle H}\,\triangle(\tfrac{3}{2}\psi_0\eta^2 + \tfrac{1}{2}\eta^3)\,\mathrm{d}s. \tag{1.5}$$

The idea is to solve this using the contraction mapping principle. That requires estimates for $e^{t \triangle H}$ and $e^{t \triangle H} \triangle$. Hence we need to study the semigroup which solves the linearised equation

$$\partial_t u = \triangle H u. \tag{1.6}$$

The independence of V of the $d-1$ spatial directions can be exploited by applying the Fourier transform in these directions. We denote, somewhat unconventionally

$$\hat{f}(x, \vec{k}) := \int_{\mathbb{R}^{d-1}} e^{-\mathrm{i}\vec{k}\cdot\vec{x}} f(x, \vec{x}) \, \mathrm{d}\vec{x} \tag{1.7}$$

and hope that this does not cause too much confusion.[1] Then

$$\widehat{\triangle H u} = -D_k H_k \hat{u} \tag{1.8}$$

where $k = |\vec{k}|$,

$$\begin{aligned} D_k &= -\partial_x^2 + k^2, \\ H_k &= D_k + 1 + V. \end{aligned}$$

$-\triangle H$ is quasi-accretive[2] in $L^2(\mathbb{R}^d)$:

$$\begin{aligned} \operatorname{Re}\langle u, -\triangle H u\rangle &= \|\triangle u\|^2 - \operatorname{Re}\langle \triangle u, (1+V)u\rangle \\ &= \|\triangle u - \tfrac{1}{2}(1+V)u\|^2 - \tfrac{1}{4}\|(1+V)u\|^2 \geq -\tfrac{1}{4}\|u\|^2. \end{aligned}$$

In fact using the second Neumann series

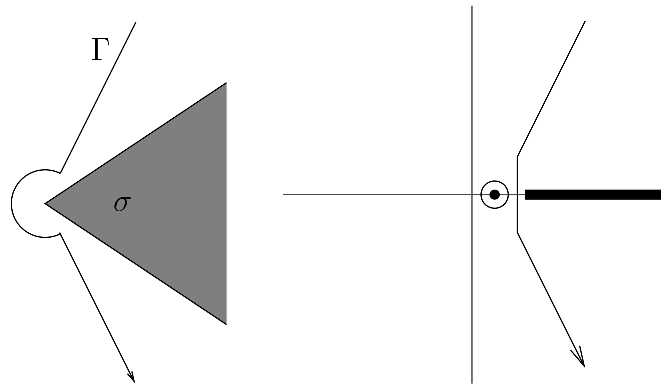
$$(\zeta + \triangle H)^{-1} = \sum_{k=0}^{\infty} ((\zeta - \triangle^2 + \triangle)^{-1} \triangle V)^k (\zeta - \triangle^2 + \triangle)^{-1},$$

which converges for ζ far enough from the positive real axis, we see that $-\triangle H$ is sectorial and generates a quasi-bounded analytic semigroup. That means we can write the integral kernel of the semigroup as a Dunford–Cauchy integral, using the integral kernel of the resolvent:

$$\begin{aligned} e^{t \triangle H}(\underline{x}, \underline{\xi}) &= \int_{-\Gamma} e^{\zeta t} (\zeta - \triangle H)^{-1}(\underline{x}, \underline{\xi}) \, \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} \\ &= \int_{-\Gamma} e^{\zeta t} \int_{\mathbb{R}^{d-1}} e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{\xi})} (\zeta + D_k H_k)^{-1}(x, \xi) \, \mathrm{d}\frac{\vec{k}}{2\pi} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} \\ &= \int_{\mathbb{R}^{d-1}} e^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{\xi})} \int_{\Gamma} e^{-\zeta t} (\zeta - D_k H_k)^{-1}(x, \xi) \, \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} \, \mathrm{d}\frac{\vec{k}}{2\pi} \end{aligned} \tag{1.9}$$

[1]Unfortunately we shall sometimes need Fourier transform in the remaining x variable, which will also be denoted by $\hat{\ }$. Sorry about that.

[2]See [Kat66] for an introduction to semigroup theory.



(a) Generic (b) k-dependent

Figure 1.1: Integration paths for the analytic semigroup. Due to analyticity the precise shape of the path does not matter.

where Γ is a suitable curve around the spectrum of $-\triangle H$ as in Figure 1.1(a). However, after reaching the last line above we can replace Γ with a k-dependent path tailored to fit the spectrum of $D_k H_k$. This will of course require some knowledge about the spectrum. For most interesting values of k the path will look like Figure 1.1(b), where the spectrum is represented by the thick parts.

1.2 On the spectrum

The problem with integrating around the spectrum of $D_k H_k$ is that one would like to stay in the right half plane in order to keep $e^{-\zeta t}$ bounded (rather than just quasi-bounded). Although $D_k H_k$ is positive there is not much room when k is small. Here we present some coarse estimates which serve to illustrate the structure of the spectrum. These are based on [SO93].

The spectrum of H_0 is known [LL81, page 79]: there are two isolated eigenvalues at 0 and 3/4 and a continuous spectrum $[1, \infty)$. The eigenfunctions are V and $x \mapsto \sinh(x/2)/\cosh(x/2)^2$.

For $k > 0$ D_k is positive and self-adjoint. Its inverse can be represented with the convolution kernel

$$G_k(x) = \frac{1}{2k} e^{-k|x|}.$$

Alternatively, on the Fourier side we have

$$\hat{G}_k(p) = \frac{1}{k^2 + p^2},$$

which is bounded by $1/k^2$.

To overcome the lack of self-adjointness of $D_k H_k$ we study instead the positive semidefinite and self-adjoint operator $A_k := D_k^{1/2} H_k D_k^{1/2}$.

Lemma 1.2. *When $k > 0$ A_k and $D_k H_k$ have the same eigenvalues with same multiplicities.*

Proof. Let $D_k H_k u = \zeta u$. Set $v = G_k^{1/2} u$. Now $D_k H_k D_k^{1/2} v = \zeta D_k^{1/2} v$ and v is an eigenvector of A_k.

Any eigenvector v of A_k obviously has to be in the domain of $D_k^{1/2}$, thus we can set $u = D_k^{1/2} v$ and reverse the calculation above. $\square$

The bottom of the spectrum of A_k can be obtained using the Rayleigh quotient. We get

$$\zeta_0 = \inf_u \frac{\langle u, H_k u\rangle}{\langle u, G_k u\rangle} \tag{1.10}$$

(by substituting $G_k^{1/2} u$ for the variable v in the usual Rayleigh quotient for A_k, which is permissible since any v in the domain of A_k is of such form). Plugging the normalised eigenfunction u_0 of H_0 (corresponding to the zero eigenvalue) into (1.10) we immediately get an upper bound:

$$\zeta_0 \leq \frac{k^2}{\langle u_0, G_k u_0\rangle} = \frac{2k^3}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x) e^{-k|x-y|} u_0(y)\, \mathrm{d}x\, \mathrm{d}y}.$$

The denominator can be bounded from below by a constant C^{-1} for, say, $k \leq 1$, and we get $\zeta_0 \leq Ck^3$ for small k. We also get a lower bound:

$$\zeta_0 \geq \inf_u \frac{k^4 \|u\|^2}{\langle u, k^2 G_k u\rangle} \geq k^4.$$

To see that ζ_0 is a discrete eigenvalue of multiplicity one we proceed further with the minimax principle:

$$\begin{aligned}
\zeta_1 &= \sup_v \inf_{u \in v^\perp} \frac{\langle u, A_k u\rangle}{\langle u, u\rangle} = \sup_v \inf_{\substack{u \\ \langle G_k^{1/2} u, v\rangle = 0}} \frac{\langle u, H_k u\rangle}{\langle u, G_k u\rangle} \\
&= \sup_v \inf_{u \in v^\perp} \frac{\langle u, H_k u\rangle}{\langle u, G_k u\rangle} \geq \sup_v \inf_{u \in v^\perp} \frac{\langle u, H_0 u\rangle}{\|u\|^2} \frac{k^2 \|u\|^2}{\langle u, k^2 G_k u\rangle} \geq \tfrac{3}{4} k^2,
\end{aligned}$$

which for small k is larger than ζ_0.

$D_k H_k$ has asymptotically constant coefficients, thus its essential spectrum is the same as that of $D_k^2 + D_k$ [CE90, Proposition 26.2], which is real and bounded from below by $k^2 + k^4$.

We have gathered these facts:

Lemma 1.3. *For any k the spectrum of $D_k H_k$ is real and bounded from below by k^4.*

For small k the bottom of the spectrum is an isolated eigenvalue ζ_0 of multiplicity one with $\zeta_0 \leq Ck^3$. The remaining part of the spectrum is bounded from below by $\frac{3}{4}k^2$.

1.3 Notation

C will be used as generic notation for constants. The actual values of C may vary (typically increasing) from expression to expression. For clarity, one might label every C with a distinct subscript. For brevity, we usually don't. When we do equip C with a subscript it denotes a specific constant.

R will be used to denote the resolvent $(\zeta - D_k H_k)^{-1}$. K will be used to denote the semigroup $e^{-tD_k H_k}$. Operators and their integral kernels are identified, e.g., we may write $(RD_k)(x, \xi)$.

The $O(\cdot)$ notation will be used in the usual sense, except that we sometimes don't bother writing absolute value signs: $O(f(\lambda))$ denotes an expression which is bounded by $|f(\lambda)|$ when λ is in some region, which hopefully will be clear in each context. Our laxness with absolute values extends to writing $O(\lambda^n)$ when λ is a finite-dimensional vector, in which case we naturally mean $O(|\lambda|^n)$.

Chapter 2

The Pego–Weinstein formalism

2.1 Introduction

The linear analysis is based on an approach used in [PW92]. Let us study a system of ordinary linear differential equations in $\mathbb{C}^n$:

$$y'(x) = A(\lambda, x)y(x) + b(x), \tag{2.1}$$

where the coefficient matrix depends analytically on one or more complex parameters λ.[1] Assume $A_\infty(\lambda) := \lim_{x\to\pm\infty} A(\lambda, x)$, i.e., both limits exist and are the same and that $R(\lambda, x) := A(\lambda, x) - A_\infty(\lambda)$ can be bounded by $Ce^{-\rho|x|}$ uniformly in λ. We will usually not bother to write the λ dependence explicitly. Here is a summary of relevant notation:

Definition 2.1 (notation). μ_i, $i \in \{1, \ldots, n\}$, are the eigenvalues of A_∞. v_i and w_i are the corresponding right and left eigenvectors, i.e., $A_\infty v_i = \mu_i v_i$, $w_i A_\infty = \mu_i w_i$.

Some of the μ_i and their eigenvectors may be equal to each other for some values of λ.

It proves useful to study the homogeneous equation corresponding to (2.1) along with the so-called transposed equation:

$$y'(x) = A(x)y(x) \tag{2.2}$$
$$z'(x) = -z(x)A(x). \tag{2.3}$$

The motivation for (2.3) is that if y and z are solutions of (2.2) and (2.3) the product $z(x)y(x)$ is independent of x.

[1] Thus λ is a complex vector of unspecified dimension and in some unspecified region.

As the coefficient matrix $A(x)$ tends to the constant A_∞ when $x \to \pm\infty$ one would also expect the solutions of (2.2) and (2.3) to tend to the solutions of the corresponding constant coefficient equations. Indeed, Pego and Weinstein show in [PW92] that if there is a simple eigenvalue μ_1 with a smaller real part than any other eigenvalue (2.2) admits a unique solution y_1^+ with the property that $e^{-\mu_1 x} y_1^+(x) \to v_1$ as $x \to \infty$. Similarly (2.3) has an unique solution z_1^- with $e^{\mu_1 x} z_1^-(x) \to w_1$ as $x \to -\infty$.

2.2 More solutions of the homogeneous equation

For our purposes it is useful to extract solutions of (2.2) and (2.3) corresponding to each eigenvalue μ_j, which is possible under suitable assumptions. We prove things for y_j^+ but the z_j^- case is similar.

Choose some λ-dependent μ and substitute $y(x) =: e^{\mu x} v(x)$ in (2.2) to get

$$v'(x) = (B + R(x))v(x), \tag{2.4}$$

where $B := A_\infty - \mu$ and $R(x) := A(x) - A_\infty$. We want to divide the eigenvalues of A_∞ into two sets: those with smaller real part than μ and the rest. However, we need some margins in order to keep everything uniform in λ because the eigenvalues move around.

Let $E_i := \bigcup_{n=1}^\infty \ker(\mu_i - A_\infty)^n$. Assume that there are constants α and β and a set of indices I such that $\mathrm{Re}(\mu_i - \mu) \leq \alpha < 0$ when $i \in I$ and $\mathrm{Re}(\mu_i - \mu) \geq \beta > -\rho$ whenever $i \notin I$. α, β and I are assumed independent of λ. Also, if some eigenvalues of A_∞ coincide for some λ, such a group of eigenvalues shall not be split between I and its complement: they must all be in one or the other. Let P be the projection onto $\bigoplus_{i \in I} E_i$ which commutes with A_∞ and let $Q := 1 - P$. Fix some positive x_0 and define

$$\begin{aligned}
(\mathcal{F}_P v)(x) &:= \int_{x_0}^{x} e^{(x-\xi)B} P R(\xi) v(\xi)\,\mathrm{d}\xi, \\
(\mathcal{F}_Q v)(x) &:= -\int_{x}^{\infty} e^{(x-\xi)B} Q R(\xi) v(\xi)\,\mathrm{d}\xi, \\
\mathcal{F} &:= \mathcal{F}_P + \mathcal{F}_Q
\end{aligned} \tag{2.5}$$

for bounded continuous functions v on $[x_0, \infty]$.

Lemma 2.2. *When x_0 is sufficiently large $\mathcal{F}$ is a contraction in the norm $\|v\| = \sup_{x \in [x_0, \infty)} |v(x)|$.*

Proof. Some eigenvalues of B may be defective and solutions corresponding to them may have additional polynomial growth rather than just the usual exponential behaviour. Fix a small ϵ to absorb that. Choose $\epsilon \in (0, -\alpha - \rho)$ if possible, otherwise just $\epsilon \in (0, -\alpha)$. Letting C denote a generic constant independent of x_0 (but dependent on ϵ) we have

$$\begin{aligned}
|\mathcal{F}_P v(x)| &= \left| \int_{x_0}^{x} e^{(x-\xi)B} PR(\xi) v(\xi) \, \mathrm{d}\xi \right| \\
&\leq C \int_{x_0}^{x} e^{(\alpha+\epsilon)(x-\xi)} e^{-\rho\xi} \|v\| \, \mathrm{d}\xi \\
&\leq C e^{(\alpha+\epsilon)x} \|v\| \int_{x_0}^{x} e^{-(\rho+\alpha+\epsilon)\xi} \, \mathrm{d}\xi \\
&\leq \begin{cases} C e^{-\rho x} \|v\| & \text{if } \alpha < -\rho, \\ C e^{-(\rho+\alpha+\epsilon)x_0} e^{(\alpha+\epsilon)x} \|v\| & \text{otherwise} \end{cases}
\end{aligned}$$

and, taking $\epsilon < \rho + \beta$,

$$\begin{aligned}
|\mathcal{F}_Q v(x)| &= \left| \int_{x}^{\infty} e^{(x-\xi)B} QR(\xi) v(\xi) \, \mathrm{d}\xi \right| \\
&\leq C \int_{x}^{\infty} e^{(\beta-\epsilon)(x-\xi)} e^{-\rho\xi} \|v\| \, \mathrm{d}\xi \\
&= C e^{(\beta-\epsilon)x} \|v\| \int_{x}^{\infty} e^{-(\beta+\rho-\epsilon)\xi} \, \mathrm{d}\xi \\
&\leq C e^{-\rho x} \|v\|,
\end{aligned}$$

Thus we can make the operator norm of $\mathcal{F}$ arbitrarily small by taking a large enough x_0. ☐

Corollary 2.3. $(\mathcal{F}v)(x) = O(e^{\max(-\rho, \alpha+\epsilon)x})$ *for large* x *and any* $\epsilon > 0$.

Hence $v = \tilde{v} + \mathcal{F}v$ can be solved for v given any $\tilde{v}$. For such a solution

$$\begin{aligned}
(v - \tilde{v})' &= PRv + B\mathcal{F}_P v + QRv + B\mathcal{F}_Q v \\
&= B(v - \tilde{v}) + Rv.
\end{aligned}$$

In particular if $\tilde{v}$ is a bounded solution of $\tilde{v}' = B\tilde{v}$ (a constant coefficient equation) for large x then v will be a solution of (2.4) with the same asymptotic behaviour as $\tilde{v}$ in the sense that $v - \tilde{v} = \mathcal{F}v$ tends to zero exponentially fast as $x \to \infty$.

Corollary 2.4. *For each eigenvalue μ_i there is a solution y_i of* (2.2) *which behaves asymptotically (as $x \to \infty$) like $e^{\mu_i x} v_i$:*

$$|e^{-\mu_i x} y_i^+(x) - v_i| < C e^{-\frac{\rho}{2} x}$$

for some C and all $x > 0$.

Proof. Pick $\mu = \mu_i$, $\alpha = -\frac{5}{8}\rho$, $\beta = -\frac{7}{8}\rho$, $\epsilon = \frac{1}{16}\rho$ and $\tilde{v}(x) = v_i$. It is clear that a suitable I can be found and $\mathcal{F}$ can be used to get a solution for $x > x_0$, which can then be extended to the whole real line. □

In general the solutions of Corollary 2.4 are not unique even after fixing normalisation of v_i because one can add similar solutions corresponding to μ_j with smaller real parts than μ_i.

Corollary 2.5. *If the μ_i and v_i are analytic functions of λ in some domain and we can fix the I in the previous proof uniformly for all λ then the solutions in Corollary 2.4 are analytic in this domain when evaluated at some fixed x. If y_i is such a solution then $|e^{-\mu_i(\lambda)x} y_i(\lambda; x) - v_i(\lambda)| < C e^{-\frac{\rho}{2} x}$ for $x > 0$ and λ in compact subsets of the domain (C depends on the subset).*

2.3 Application to the resolvent of $D_k H_k$

Write $(\zeta - D_k H_k)u = f$ in the form $y' = Ay + b$ by setting

$$y = (u, u', u'', u''')^T, \qquad b = (0, 0, 0, -f)^T \tag{2.6}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \zeta - k^2 - k^4 - k^2 V + V'' & 2V' & 1 + 2k^2 + V & 0 \end{pmatrix}. \tag{2.7}$$

The eigenvalues of A_∞ are the zeros of the polynomial

$$\zeta - (-\mu^2 + k^2)^2 - (-\mu^2 + k^2)$$

which are

$$\mu_j = \pm\sqrt{\tfrac{1}{2} + k^2 \pm \tfrac{1}{2}\sqrt{1 + 4\zeta}}. \tag{2.8}$$

Unfortunately they have somewhat poor analyticity for k and ζ near zero. This problem can be overcome by writing

$$\zeta = k^2 + k^4 + (1 + 2k^2)^2 \tau^2 \tag{2.9}$$

and using $\lambda = (k, \tau)$ as the parameters. Then (2.7) and (2.8) become

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (1+2k^2)^2\tau^2 - k^2V + V'' & 2V' & 1+2k^2+V & 0 \end{pmatrix}$$

and

$$\mu_j = \pm\sqrt{1+2k^2}\sqrt{\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4\tau^2}}. \tag{2.10}$$

To fix the branches when the second sign is negative set

$$\pm\sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+4\tau^2}} = \frac{\pm\mathrm{i}\tau}{\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4\tau^2}}} \tag{2.11}$$

and choose μ_2 to have the positive sign with the principal branch in the denominator. Now the μ_i are analytic functions of k and τ when $k \in \mathbb{C} \setminus \pm\mathrm{i}[1/\sqrt{2}, \infty)$ and $\tau \in \mathbb{C} \setminus \pm\mathrm{i}[\frac{1}{2}, \infty)$. We are mostly interested in real k, considering that k used to be the length of something. Then the cuts in the τ plane correspond to $\zeta \in (-\infty, -\frac{1}{4}]$. $\tau \in \pm\mathrm{i}[0, \frac{1}{2}k] \cup \mathbb{R}$ is mapped onto $[\frac{3}{4}k^2 - k^6, \infty)$ which by Lemma 1.3 contains all of the spectrum of $D_k H_k$ except the lowest eigenvalue, which should be somewhere around $\tau \approx \mathrm{i}k$ when k is small. For larger k it is more useful to use the k^4 bound: $\tau \in \pm\mathrm{i}[0, k/(1+2k^2)] \cup \mathbb{R}$ is mapped onto $[k^4, \infty)$ which contains the entire spectrum.

τ is mapped two-to-one to ζ and will usually be kept in the upper half plane. There $\operatorname{Re}\mu_2 < 0$ when k is real. However, we need some results to be valid also in a small complex neighbourhood of $k = \tau = 0$ because we will develop a power series there later.

Set $\mu_1 := -\sqrt{1+2k^2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1+4\tau^2}}$, $\mu_4 := -\mu_1$ and $\mu_3 := -\mu_2$. These equations are sometimes useful:

$$\begin{aligned} \mu_1^2 + \mu_2^2 &= 1 + 2k^2, \quad \mu_1\mu_2 = -\mathrm{i}\tau(1+2k^2), \\ \mu_1^2 - \mu_2^2 &= (1+2k^2)\sqrt{1+4\tau^2} = \sqrt{1+4\zeta}. \end{aligned} \tag{2.12}$$

The eigenvectors v_j and w_j are easy to express in terms of μ_j and are thus also analytic. Normalise the eigenvectors so that $(v_j)_1 = (w_j)_4 = 1$. This results in

$$\begin{aligned} v_j &= \begin{pmatrix} 1, & \mu_j, & \mu_j^2, & \mu_j^3 \end{pmatrix}^T, \\ w_j &= \begin{pmatrix} \mu_j(\mu_j^2 - 1 - 2k^2), & \mu_j^2 - 1 - 2k^2, & \mu_j, & 1 \end{pmatrix}. \end{aligned}$$

According to Corollary 2.4 for $j \in \{1, 2, 4\}$ there is a y_j^+ which solves the homogeneous equation $\partial_x y_j^+ = A y_j^+$ and behaves like $e^{\mu_j x} v_j$ as $x \to \infty$. There is also a z_j^- which solves the transposed equation $\partial_x z_j^- = -z_j^- A$ and behaves like $e^{-\mu_j x} w_j$ as $x \to -\infty$. It would also be possible to define y_3^+ and z_3^- in this way but that would not be very useful because A_∞ is defective at $\tau = 0$: the eigenvectors v_2 and v_3 collide and we would not have a set of four linearly independent solutions. Thus with some abuse of notation we require instead

$$y_3^+(x) \sim \frac{e^{\mu_3 x} v_3 - e^{\mu_2 x} v_2}{\mu_3 - \mu_2} \quad \text{and} \quad z_3^-(x) \sim \frac{e^{-\mu_3 x} w_3 - e^{-\mu_2 x} w_2}{\mu_3 - \mu_2}$$

as $x \to \infty$ and $x \to -\infty$, respectively. As $\mu_3 - \mu_2 \to 0$ these converge (pointwise in x) to solutions with linear asymptotes.

Our expression for the integral kernel of $(\zeta - D_k H_k)^{-1}$ will contain only y_j^+ and z_j^- for $j \in \{1, 2\}$. However, the other two values of j are needed for understanding the behaviour of the kernel.

Theorem 2.6. *Assume*

$$\mathrm{Re}(\mu_1 \pm \mu_2) < -\tfrac{5}{8} \quad \textit{and} \quad \mathrm{Re}(\mu_2) < \tfrac{1}{32}. \tag{2.13}$$

Then (2.2) *with* (2.7) *has solutions* y_i^+ *such that* $|y_i^+(x) - e^{\mu_i x} v_i| < C|e^{(\mu_i - \frac{1}{2})x}|$ *for* $i \in \{1, 2, 4\}$ *and*

$$|y_3^+(x) - \frac{e^{\mu_3 x} v_3 - e^{\mu_2 x} v_2}{2\mu_3}| < C|e^{(\mu_3 - \frac{1}{2})x}|$$

when x is bounded from below. Furthermore,

$$\begin{aligned} |\partial_x^n y_i^+(x) - \mu_i^n e^{\mu_i x} v_i| &< C|e^{(\mu_i - \frac{1}{2})x}|, \qquad n \in \{1, 2\}, \quad i \neq 3 \\ |\partial_x y_3^+(x) - \tfrac{1}{2}(e^{\mu_3 x} v_3 + e^{\mu_2 x} v_2)| &< C|e^{(\mu_3 - \frac{1}{2})x}|, \\ |\partial_x^2 y_3^+(x) - \tfrac{1}{2}(\mu_3 e^{\mu_3 x} v_3 + \mu_2 e^{\mu_2 x} v_2)| &< C|e^{(\mu_3 - \frac{1}{2})x}|. \end{aligned}$$

Similarly, (2.3) *has solutions* z_i^- *such that* $|z_i^-(x) - e^{-\mu_i x} w_i| < Ce^{(\frac{1}{2} - \mu_i)x}$ *for* $i \in \{1, 2, 4\}$ *and*

$$|z_3^-(x) - \frac{e^{-\mu_3 x} w_3 - e^{-\mu_2 x} w_2}{2\mu_3}| < C|e^{(\frac{1}{2} - \mu_3)x}|$$

when x is bounded from above. The corresponding estimates for the derivatives hold.

For $i \in \{1, 2, 4\}$ $y_i^+(x)$ and $z_i^-(x)$ are analytic in λ wherever the assumptions hold. $y_3^+(x)$ and $z_3^-(x)$ are continuous and when

$$\mathrm{Re}(\mu_2) > -\tfrac{1}{4} \tag{2.14}$$

also analytic. C above depends on λ but can be fixed in any compact subset.

Before the proof we would like to comment on the assumptions of the theorem. Specifying them in terms of μ_1 and μ_2 was perhaps a bit opaque. However, there are two cases which interest us. The first is $\lambda \approx 0$. In a small enough neighbourhood of the origin both (2.13) and (2.14) clearly hold. The second case is when k is real and τ in the upper half plane. Then μ_2 is in the left half plane. For the first inequality of (2.13) use (2.12) to get $(\mu_1 \pm \mu_2)^2 = (1+2k^2)(1 \mp 2\mathrm{i}\tau)$. From this we see that the inequality is equivalent to τ lying beneath a parabola whose apogee is at $\mathrm{i}(128k^2+39)/(256k^2+128)$. This is always above the values corresponding to the spectrum.

Proof. We prove this for the y_i^+, the z_i^- case is similar enough. $\rho = 1$. Apply Corollary 2.5 with

$$I = \begin{cases} \emptyset, & \text{when } i = 1 \\ \{1\}, & \text{when } i = 2 \\ \{1,2,3\}, & \text{when } i = 4. \end{cases}$$

When $i = 3$ choose $\mu = \mu_3 + \frac{1}{16}$ and

$$\tilde{v}(x) = e^{-\frac{1}{8}x} \frac{v_3 - e^{2\mu_2 x} v_2}{2\mu_3}$$

which is bounded even when $\mu_3 \to 0$. Unfortunately we have to split this up as follows: when $\mathrm{Re}(\mu_2) > -\frac{3}{8}$ choose $I = \{1\}$ and when $\mathrm{Re}(\mu_2) < -\frac{11}{32}$ choose $I = \{1,2\}$. $\alpha = -\frac{5}{8}$, $\beta = -\frac{7}{8}$ will do for both regions just as in Corollary 2.4. Solve $v = \tilde{v} + \mathcal{F}v$ for v in each region separately, glue together with a partition of unity and define $y_3^+(x) = e^{\mu x} v(x)$. As $\alpha + \frac{1}{16} < -\frac{1}{2}$ we still get the claimed estimate.

That gave us the solutions and the basic estimates. For the derivatives we need to use (2.2). When $i \neq 3$ $y_i^+(x) = e^{\mu_i x} v_i + r_i(x)$ and $|r_i(x)| \leq C|e^{(\mu_i - \frac{1}{2})x}|$. Thus

$$\begin{aligned} \partial_x y_i^+(x) &= A(x) y_i^+(x) = (A_\infty + R(x))(e^{\mu_i x} v_i + r_i(x)) \\ &= \mu_i e^{\mu_i x} v_i + e^{\mu_i x} R(x) v_i + A(x) r_i(x) \\ &= \mu_i e^{\mu_i x} v_i + O(e^{(\mu_i - \frac{1}{2})x}), \\ \partial_x^2 y_i^+(x) &= A(x) \partial_x y_i^+(x) + \partial_x R(x) y_i^+(x) \\ &= \mu_i^2 e^{\mu_i x} v_i + O(e^{(\mu_i - \frac{1}{2})x}). \end{aligned}$$

The $i = 3$ is slightly but not substantially different:

$$\begin{aligned}
y_3^+(x) &= \frac{e^{\mu_3 x} v_3 - e^{\mu_2 x} v_2}{2\mu_3} + r_3(x) \qquad \text{where } |r_3(x)| \leq C|e^{(\mu_3 - \frac{1}{2})x}|, \\
\partial_x y_3^+(x) &= \tfrac{1}{2}(e^{\mu_3 x} v_3 + e^{\mu_2 x} v_2) + R(x)\frac{e^{\mu_3 x} v_3 - e^{\mu_2 x} v_2}{2\mu_3} + A(x) r_3(x) \\
&= \tfrac{1}{2}(e^{\mu_3 x} v_3 + e^{\mu_2 x} v_2) + O(e^{(\mu_3 - \frac{1}{2})x}), \\
\partial_x^2 y_3^+(x) &= A(x)\partial_x y_3^+(x) + \partial_x R(x) y_3^+(x) \\
&= \tfrac{1}{2}(\mu_3 e^{\mu_3 x} v_3 + \mu_2 e^{\mu_2 x} v_2) + O(e^{(\mu_3 - \frac{1}{2})x}).
\end{aligned}$$

□

Lemma 2.7. *Under the assumptions* (2.13) *there are also solutions* y_j^- *and* z_j^+ *of the homogeneous* (2.2) *and transposed equation* (2.3), *respectively, with prescribed behaviour for* $x \to -\infty$ *and* $x \to \infty$: *(e.g.* $y_1^-(x) \sim e^{-\mu_1 x} v_4$ *as* $x \to -\infty$*—note that* $\mu_1 = -\mu_4$*). These can be expanded as linear combinations of* y_j^+ *and* z_j^-. *The coefficients of this expansion are continuous and when* (2.14) *holds also analytic.*

Proof. This is an immediate consequence of the symmetry of (1.6) under the reflection $x \mapsto -x$: we have $y_j^-(x) = J y_j^+(-x)$ and $z_j^+(x) = -z_j^-(-x)J$, where $J = \mathrm{diag}(1, -1, 1, -1)$. The coefficients of the expansion can be determined at some fixed point, say $x = 0$; the $y_j^\pm(0)$ are continuous in the parameters and linearly independent. □

The solutions $y_j^\pm$ can of course naturally be written in terms of a scalar valued function $Y_j^\pm$. Similarly the solutions of the transposed equation can be written in terms of a single function. A straightforward computation yields:

Lemma 2.8.

$$\begin{aligned}
y_j^\pm &= \left(Y_j^\pm, \ \ \partial_x Y_j^\pm, \ \ \partial_x^2 Y_j^\pm, \ \ \partial_x^3 Y_j^\pm\right)^T, \\
z_j^\mp &= \left((H_k \partial_x + k^2 \partial_x - V')Z_j^\mp, \ \ -(H_k + k^2) Z_j^\mp, \ \ -\partial_x Z_j^\mp, \ \ Z_j^\mp\right)
\end{aligned}$$

with $(\zeta - D_k H_k) Y_j^\pm = 0$ *and* $(\zeta - H_k D_k) Z_j^\mp = 0$. *For* $j \in \{1, 2, 4\}$ *we have*

$$\lim_{x \to \infty} e^{-\mu_j x} Y_j^+(x) = \lim_{x \to -\infty} e^{\mu_j x} Z_j^-(x) = 1$$

while

$$\lim_{x \to \infty} e^{-\mu_3 x} Y_3^+(x) - \frac{e^{2\mu_2 x} - 1}{2\mu_2} = \lim_{x \to -\infty} e^{\mu_3 x} Z_3^-(x) - \frac{e^{-2\mu_2 x} - 1}{2\mu_2} = 0$$

(when $\mu_2 = 0$*, replace the fractions by their limits, i.e.,* x *and* $-x$*).*

We want to write a solution of $y' = Ay + b$ in the form

$$y(x) = \int_{-\infty}^{x} y^+(x)(\Omega^+)^{-1} z^-(\xi) b(\xi)\,\mathrm{d}\xi + \int_{\infty}^{x} y^-(x)(\Omega^-)^{-1} z^+(\xi) b(\xi)\,\mathrm{d}\xi, \tag{2.15}$$

where $y^\pm = (y_1^\pm, y_2^\pm)$, $z^\mp = \begin{pmatrix} z_1^\mp \\ z_2^\mp \end{pmatrix}$ and $\Omega^\pm = z^\mp y^\pm$. These products are independent of x and it is also easy to see that $\Omega^- = -\Omega^+$. For simplicity we shall from now on denote Ω^+ just by Ω. For this formula to make sense Ω of course needs to be invertible.

Theorem 2.9. *Assume* $\operatorname{Re}\mu_1 < \operatorname{Re}\mu_2 < 0$ *and* $\zeta \in \rho(D_k H_k)$ *(the resolvent set). Then* $\Omega = z^- y^+$ *is invertible.*

Proof. Assume the contrary: let $\Omega(\alpha_1, \alpha_2)^T = 0$, i.e., $z_i^- y = 0$ for $i \in \{1, 2\}$, $y := \alpha_1 y_1^+ + \alpha_2 y_2^+ \neq 0$. y can also be written as $\sum_{j=1}^4 \beta_j y_j^-$. We have

$$e^{\mu_i x} z_i^-(x) \to w_i, \qquad e^{-\mu_i x} y_{5-i}^-(x) \to \begin{cases} \frac{1}{2\mu_3} v_2 & \text{when } i = 2, \\ v_i & \text{otherwise} \end{cases}$$

as $x \to -\infty$. Thus $z_1^-(x) y_j^-(x) \to \delta_{4j} w_1 v_1$ but the left side is actually independent of x. Since μ_1 is a simple eigenvalue we must have $w_1 v_1 \neq 0$, consequently $z_1^- y = 0$ implies $\beta_4 = 0$. In a similar vein $z_2^- y = 0$ implies $\beta_3 = 0$. Hence $y = \beta_1 y_1^- + \beta_2 y_2^-$ decreases exponentially at $\pm\infty$, giving a nontrivial L^2 solution to $(\zeta - D_k H_k)u = 0$. Such a solution was assumed not to exist. $\square$

Lemma 2.10. *Under the assumptions of the previous theorem* (2.15) *solves* $y' = Ay + b$.

Proof. Differentiating (2.15) and recalling that $y^\pm$ solve the homogeneous equation we get

$$y'(x) = A(x)y(x) + \big(y^+(x)\Omega^{-1} z^-(x) - y^-(x)\Omega^{-1} z^+(x)\big) b(x).$$

Thus we need to show that the matrix inside the parentheses is the identity.

$y^+(x)\Omega^{-1} z^-(x)$ is a projection onto the space spanned by $\{y_1^+(x), y_2^+(x)\}$. These two vectors are linearly independent. Similarly $-y^-(x)\Omega^{-1} z^+(x)$ is a projection onto the two-dimensional space spanned by $\{y_1^-(x), y_2^-(x)\}$. The two projections are orthogonal to each other as $z^\pm(x) y^\pm(x) = 0$: the product would have exponential decay as $x \to \pm\infty$ but is actually independent of x. Thus their sum is the identity. $\square$

Recalling (2.6) we get for the original equation:

Theorem 2.11. *Under the assumptions of Theorem 2.9 the integral kernel of the resolvent is given by*

$$R(x,\xi) := (\zeta - D_k H_k)^{-1}(x,\xi) = \begin{cases} -Y^+(x)\Omega^{-1}Z^-(\xi) & \text{for } \xi < x, \\ -Y^+(-x)\Omega^{-1}Z^-(-\xi) & \text{for } \xi > x, \end{cases} \tag{2.16}$$

where $Y^+ = (Y_1^+, Y_2^+)$ *and* $Z^- = (Z_1^-, Z_2^-)^T$.

Note that from $y^+(x)\Omega^{-1}z^-(x) - y^-(x)\Omega^{-1}z^+(x) = 1$ it follows in particular that the (i,j) component of the left hand side is 0 for $i < j$. From this we see that our resolvent kernel has continuous derivatives with respect to x or ξ up to total order two.

Chapter 3

Leading terms of the resolvent

3.1 Outline

Here we estimate the resolvent for very small k and $|\zeta|$. We use (2.9) and assume $k < \epsilon$, $|\tau| < \epsilon$ for a small enough ϵ.[1] Then the assumptions of Theorem 2.6 are satisfied, including the analyticity condition (2.14). We shall expand Ω of (2.16) into a power series to get some explicit leading terms for the resolvent. Thus this chapter is mostly about computing derivatives of things at $k = \tau = 0$.

3.2 Solutions of the homogeneous equation

We'll denote the solutions at $k = \tau = 0$ with a ring. $\mathring{\mu}_1 = -1$, $\mathring{\mu}_2 = 0$ and

$$\begin{aligned} \mathring{Y}_1^+(x) &= \frac{1}{4\cosh(\frac{x}{2})^2} \\ \mathring{Y}_2^+(x) &= \frac{-1 - 6e^x + 5e^{2x} + 2e^{3x} + 6e^{2x}x}{2e^x(1+e^x)^2} \\ \mathring{Z}_1^-(x) &= \log(e^x + 1) \\ \mathring{Z}_2^-(x) &= 1. \end{aligned} \tag{3.1}$$

By a mechanical computation we get

$$\mathring{\Omega} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[1]We start with some ϵ but Cauchy's estimates and such nibble at it, and we shall end up with a somewhat smaller ϵ than we started with.

Although they do not appear in Ω, we also list for later use the solutions which are asymptotically equal to $x \mapsto |x|$, i.e., $\mathring{Y}_3^+(x)/x \to 1$ as $x \to \infty$ and $\mathring{Z}_3^-(x)/x \to -1$ as $x \to -\infty$. After arduous but routine calculations[2] we find these:

$$\begin{aligned}\mathring{Y}_3^+(x) &= \frac{1}{2(1+e^x)^2}\Big(1 + 4e^x - e^{2x} + 16e^x \log 2 - 6x - xe^{-x} + 7xe^x + 2xe^{2x} \\ &\qquad + 3x^2e^x + (-8 - e^{-x} + 8e^{2x} + e^{3x}) \log(1+e^{-x}) + 12e^x \operatorname{Li}_2(-e^{-x})\Big) \\ \mathring{Z}_3^-(x) &= -x\end{aligned}$$

where Li_2 is the dilogarithm:

$$\operatorname{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_z^0 \frac{\log(1-s)}{s}\,\mathrm{d}s.$$

These products will be needed later:

$$\begin{pmatrix} \mathring{z}_1^- \\ \mathring{z}_2^- \\ \mathring{z}_3^- \end{pmatrix} \mathring{y}_3^+ = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathring{z}_3^- \begin{pmatrix} \mathring{y}_1^+ & \mathring{y}_2^+ & \mathring{y}_3^+ \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}. \tag{3.2}$$

We shall also also need the rapidly growing solution

$$\mathring{Y}_4^+(x) = 4\cosh(\frac{x}{2})^2.$$

3.3 First order

Define $v_j^+(x) = e^{-\mu_j x} y_j^+(x)$ and $w_j^-(x) = e^{\mu_j x} z_j^-(x)$ for $j \in \{1,2\}$. By Corollary 2.5 they are analytic functions of λ for each fixed x and satisfy $|v_j^+(x) - v_j| < Ce^{-x/2}$ for $x > 0$ and $|w_j^-(x) - w_j| < Ce^{x/2}$ for $x < 0$ where C can be fixed independently of k and τ. They will also satisfy the obvious differential equations, which we now differentiate, then set $k = \tau = 0$:

$$\begin{aligned}\partial_x \partial_\lambda v_j^+ &= (A - \mu_j)\partial_\lambda v_j^+ - \partial_\lambda \mu_j v_j^+ \\ \partial_x \partial_\lambda w_j^- &= -\partial_\lambda w_j^- (A - \mu_j) + \partial_\lambda \mu_j w_j^-\end{aligned}$$

(note that $\partial_\lambda A = 0$). These are just inhomogeneous versions of the equations satisfied by v_j^+ and w_j^-. The "initial conditions" are again asymptotic:

[2] Admittedly somewhat less arduous for $\mathring{Z}_3^-$ than for $\mathring{Y}_3^+$.

$|\partial_\lambda v_j^+(x) - \partial_\lambda v_j| < Ce^{-x/2}$ and $|\partial_\lambda w_j^-(x) - \partial_\lambda w_j| < Ce^{x/2}$ by Cauchy's estimates. The first component of $\partial_\lambda v_j$ and the last component of $\partial_\lambda w_j$ are zero because of the normalisation used.

The $j = 1$ case is trivial: we have $\partial_\lambda \mu_1 = 0$, $\partial_\lambda v_1 = 0$ and the equations above are actually homogeneous. They have no nontrivial solutions with the required asymptotic behaviour, and we conclude $\partial_\lambda v_1^+ = 0$, $\partial_\lambda w_1^- = 0$.

For $j = 2$ we easily find special solutions: just take $-\partial_\lambda \mu_2 x v_2^+$ and $\partial_\lambda \mu_2 x w_2^-$. These of course grow linearly, which can be countered by adding some $\mathring{y}_3^+$ and $\mathring{z}_3^-$. When we do that we end up with solutions whose first and last components, respectively, decrease exponentially. We are happy with that:

$$\begin{aligned}
\partial_\lambda v_2^+ &= \partial_\lambda \mu_2 (\mathring{y}_3^+ - x\mathring{y}_2^+) \\
\partial_\lambda w_2^- &= \partial_\lambda \mu_2 (\mathring{z}_3^- + x\mathring{z}_2^-).
\end{aligned}$$

Note that y_2^+ and z_2^- are only unique up to multiples of y_1^+ and z_1^-. We fixed the values at $\lambda = (0, 0)$ earlier and now we fix their λ-derivatives so that no $\mathring{y}_1^+$ or $\mathring{z}_1^-$ appears in the expressions above.

It is also worth noting that $\mathring{Z}_3^- + x\mathring{Z}_2^- = 0$. Not all components of the vector $\partial_\lambda w_2^-$ vanish, though.

Thus we end up with these reasonably simple expressions:

$$\partial_\lambda y_j^+ = \partial_\lambda \mu_j \mathring{y}_3^+, \qquad\qquad \partial_\lambda z_j^- = \partial_\lambda \mu_j \mathring{z}_3^- \tag{3.3}$$

and, using (3.2),

$$\partial_\lambda \Omega = \partial_\lambda z^- y^+ + z^- \partial_\lambda y^+ = \begin{pmatrix} 0 & 0 \\ 0 & -2\partial_\lambda \mu_2 \end{pmatrix},$$

with $\partial_\lambda \mu_1 = \partial_k \mu_2 = 0$ and $\partial_\tau \mu_2 = \mathrm{i}$.

3.4 Second order

So now we have come up with some non-zero leading terms for the right column of Ω. Let us also try to get something to the left column in order to eventually compute some non-zero terms for $\det \Omega$. Here we shall deal with

$$\partial_\lambda^2 (z_i^- y_1^+) = \partial_\lambda^2 z_i^- y_1^+ + z_i^- \partial_\lambda^2 y_1^+$$

—remember that $\partial_\lambda y_1^+ = 0$ at $\lambda = 0$. It is convenient to estimate the terms on the right hand side at separate values of x, which is possible with the following trick:

$$\partial_x (\partial_\lambda^2 z_i^- y_1^+) = -\partial_\lambda^2 (z_i^- A) y_1^+ + \partial_\lambda^2 z_i^- A y_1^+ = -z_i^- \partial_\lambda^2 A y_1^+$$

(recalling $\partial_\lambda A = 0$). Now we can write

$$\partial_\lambda^2(z_i^- y_1^+) = (\partial_\lambda^2 z_i^- y_1^+)(x_0) + (z_i^- \partial_\lambda^2 y_1^+)(x_1) - \int_{x_0}^{x_1} z_i^- \partial_\lambda^2 A y_1^+ \, \mathrm{d}x$$

and take the limits $x_0 \to -\infty$ and $x_1 \to \infty$. This works splendidly because $\mathring{y}_1^+$ decreases very fast in both the positive and the negative direction—this is quite different from $\mathring{y}_2^+$ which makes similar examination of $\partial_\lambda^2(z_i^- y_2^+)$ more difficult. The $\mathring{z}_i^-$ behave nicely in the negative direction and do not give too much trouble in the positive direction either: there the worse case is the linearly growing z_1^-, easily taken care of by y_1^+. These facts and a couple of Cauchy's estimates get rid of everything but the integral at the limit.

There are two nontrivial cases for the integral:

$$\partial_k^2 A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2V & 0 & 4 & 0 \end{pmatrix} \quad \text{and} \quad \partial_\tau^2 A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

The integrations can be carried out to yield

$$\partial_k^2(z^- y_1^+) = \begin{pmatrix} -\frac{7}{3} \\ -2 \end{pmatrix}, \qquad \partial_k \partial_\tau(z^- y_1^+) = 0, \qquad \partial_\tau^2(z^- y_1^+) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

3.5 Third order

One more term to go: we want to expand $z_2^- y_1^+$ to the third order as a counterpart to the zeroth order term $z_1^- y_2^+$ for computing $\det \Omega$ to the third order. Thus we need to compute

$$\partial_\lambda^3(z_2^- y_1^+) = \partial_\lambda^3 z_2^- y_1^+ + \{\partial_\lambda z_2^- \partial_\lambda^2 y_1^+\} + z_2^- \partial_\lambda^3 y_1^+,$$

where $\{\cdot\}$ has been used to denote the sum over cyclic permutations.[3] Using the same trick as before and recalling that $\partial_\lambda^3 A = 0$:

$$\partial_x(\partial_\lambda^3 z_2^- y_1^+) = -\{\partial_\lambda z_2^- \partial_\lambda^2 A y_1^+\}$$

we have

$$\partial_\lambda^3(z_2^- y_1^+) = (\partial_\lambda^3 z_2^- y_1^+)(x_0) + \{\partial_\lambda z_2^- \partial_\lambda^2 y_1^+\}(x_1) + (z_2^- \partial_\lambda^3 y_1^+)(x_1) \\ - \int_{x_0}^{x_1} \{\partial_\lambda z_2^- \partial_\lambda^2 A y_1^+\} \, \mathrm{d}x.$$

[3]If you don't want to wallow in the linear algebra just think of it like this: each of the three ∂_λ gets to hit z_2^- in turn, producing three terms. Depending whether each ∂_λ is really a ∂_k or a ∂_τ, some or all of these terms are identical.

Again we take the limits $x_0 \to -\infty$ and $x_1 \to \infty$, and again everything but the integral vanishes (recall (3.3)). The integral also vanishes as the integrand is odd (see (3.4)). Thus

$$\partial_\lambda^3(z_2^- y_1^+) = 0.$$

3.6 x and ξ dependence

Collecting everything from the previous sections we get

Lemma 3.1.

$$\Omega = \begin{pmatrix} -\frac{7}{6}k^2 - \tau^2 & 1 \\ -k^2 - \tau^2 & -2\mathrm{i}\tau \end{pmatrix} + \begin{pmatrix} O(\lambda^3) & O(\lambda^2) \\ O(\lambda^4) & O(\lambda^2) \end{pmatrix},$$
$$\det\Omega = k^2 + \tau^2 + 2\mathrm{i}\tau^3 + \tfrac{7}{3}\mathrm{i}\tau k^2 + O(k^4) + O(\tau^4). \tag{3.5}$$

According to (2.16) $R(x,\xi) = R(-x,-\xi)$. Thus it suffices to consider the case $\xi < x$. Then

$$R(x,\xi) = \frac{-1}{\det\Omega} Y^+(x) \mho Z^-(\xi).$$

where

$$\mho = \begin{pmatrix} -2\mathrm{i}\tau & -1 \\ k^2 + \tau^2 & -\frac{7}{6}k^2 - \tau^2 \end{pmatrix} + \begin{pmatrix} O(\lambda^2) & O(\lambda^2) \\ O(\lambda^4) & O(\lambda^3) \end{pmatrix}.$$

In our small neighbourhood $(k,\tau) \approx 0$ the only possible singularity is that $\det\Omega$ may become zero, producing a pole in the resolvent. To examine the x and ξ dependence of the resolvent it is easier to work with $F(\lambda; x, \xi) := -Y^+(x)\mho Z^-(\xi)$ which has no such singularity.

$$F(\lambda;x,\xi) = Y_1^+(\lambda;x)(Z_2^-(\lambda;\xi) + 2\mathrm{i}\tau Z_1^-(\lambda;\xi)) + \sum_{i=1}^{2}\sum_{j=1}^{2} \gamma_{ij}(\lambda) Y_i^+(\lambda;x) Z_j^-(\lambda;\xi),$$

where the γ_{ij} are analytic functions of $\lambda = (k,\tau)$ and $O(\lambda^2)$. For $\xi < 0 < x$ this is easy to bound as

$$|e^{-\mu_i x} Y_i^+(x) - 1| < C e^{-\frac{1}{2}x}, \qquad |e^{\mu_i \xi} Z_i^-(\xi) - 1| < C e^{\frac{1}{2}\xi} \tag{3.6}$$

(see Theorem 2.6). It turns out that we need to look at the $Y_2^+ Z_1^-$ term a bit more carefully so let us write it more explicitly:

$$F = Y_1^+ Z_2^- + 2\mathrm{i}\tau Y_1^+ Z_1^- - (k^2 + \tau^2 + O(\lambda^4)) Y_2^+ Z_1^- + \sum_{j=1}^{2} \gamma_{1j} Y_1^+ Z_j^- + \gamma_{22} Y_2^+ Z_2^-. \tag{3.7}$$

As these expressions for F get longer we drop the arguments to reduce clutter. Y is always evaluated at x, Z at ξ and everything depends on λ. Also, $Y_i^-(x) = Y_i^+(-x)$ and similarly for Z.

In the other cases where $\xi < x$ we need to make use of Lemma 2.7. When $0 < \xi < x$ we write

$$Z_i^-(\lambda;\xi) = \sum_{j=1}^{4} b_{ij}(\lambda) Z_j^+(\lambda;\xi).$$

Explicit computation yields

$$\begin{aligned} b_{1j} &= \delta_{1j} + \delta_{3j} + O(\lambda) \\ b_{2j} &= \delta_{2j} - 2\mathrm{i}\tau\delta_{3j} + O(\lambda^2). \end{aligned}$$

Thus

$$F = Y_1^+ Z_2^+ + 2\mathrm{i}\tau Y_1^+ Z_1^+ + \sum_{i=1}^{2}\sum_{j=1}^{4} \beta_{ij} Y_i^+ Z_j^+$$

for some coefficients β_{ij}, which are analytic functions of λ and $O(\lambda^2)$. Many values of λ in the neighbourhood of origin correspond to the resolvent set.[4] There F must be a bounded function which means that $\beta_{24}(\lambda)$ must vanish for these and consequently for all λ. The $j = 3$ terms, however, are a bit tricky: we have

$$|e^{-\mu_3\xi} Z_3^+(\xi) - \frac{e^{2\mu_2\xi} - 1}{2\mu_2}| < Ce^{-\frac{1}{2}\xi}$$

and $\mu_2 \approx \mathrm{i}\tau$. $\beta_{23} = -k^2 - \tau^2 + O(\lambda^3)$, which is more important than β_{13} because of the rapid decay of Y_1^+. Let us write F a bit more explicitly and also bring in the $Y_2^+ Z_1^+$ term:

$$\begin{aligned} F = Y_1^+ Z_2^+ + 2\mathrm{i}\tau Y_1^+ Z_1^+ - (k^2 + \tau^2 + O(\lambda^3))(Y_2^+ Z_1^+ + Y_2^+ Z_3^+) \\ + \sum_{j=1}^{4} \beta_{1j} Y_1^+ Z_j^+ + \beta_{22} Y_2^+ Z_2^+. \quad (3.8) \end{aligned}$$

When $\xi < x < 0$ we do the same thing but with Y:

$$\begin{aligned} Y_i^+ &= \sum_{j=1}^{4} a_{ij} Y_j^-, \\ a_{1j} &= \delta_{1j} + O(\lambda^2), \\ a_{2j} &= 8\delta_{1j} - \delta_{2j} - \tfrac{1}{2}\delta_{4j} + O(\lambda). \end{aligned}$$

[4] E.g., whenever $k \in \mathbb{R}$ and $\tau \notin \mathbb{R} \cup \mathrm{i}\mathbb{R}$, because then $\zeta \notin \mathbb{R}$.

Plugging this in we get

$$F = Y_1^- Z_2^- + 2\mathrm{i}\tau Y_1^- Z_1^- + \sum_{i=1}^{4}\sum_{j=1}^{2} \alpha_{ij} Y_i^- Z_j^-$$

with some coefficients $\alpha_{ij}(\lambda) = O(\lambda^2)$. Again α_{42} must vanish to keep the kernel bounded. To compute the $Y_3^- Z_2^-$ term coefficient to second order it seems that we would need a_{13} to second order, which seems difficult to compute. However, we can get around this difficulty by using twice continuous differentiability of F when $\xi \approx x \ll 0$. The limits

$$\begin{aligned}
\lim_{\xi\to x-} F(x,\xi) &= \alpha_{32} Y_3^-(x) Z_2^-(x) + \alpha_{41} Y_4^-(x) Z_1^-(x) + O(e^{2\mu_2|x|}) \\
\lim_{\xi\to x+} F(x,\xi) &= \lim_{\xi\to x+} F(-x,-\xi) \\
&= \beta_{23} Y_2^+(-x) Z_3^+(-x) + \beta_{14} Y_1^+(-x) Z_4^+(-x) + O(e^{2\mu_2|x|}).
\end{aligned}$$

must be equal. Working in the region where $\operatorname{Re}\mu_2 < 0$ and taking the limit $x \to -\infty$ yields

$$\frac{\alpha_{32}}{2\mu_3} + \alpha_{41} = \frac{\beta_{23}}{2\mu_3} + \beta_{41}.$$

Repeat the same for $\partial_x^2 F$:

$$\begin{aligned}
\lim_{\xi\to x-} \partial_x^2 F(x,\xi) &= \alpha_{32}\partial_x^2 Y_3^-(x) Z_2^-(x) + \alpha_{41}\partial_x^2 Y_4^-(x) Z_1^-(x) + O(e^{2\mu_2|x|}) \\
\lim_{\xi\to x+} \partial_x^2 F(x,\xi) &= \lim_{\xi\to x+} \partial_x^2 F(-x,-\xi) \\
&= \beta_{23}\partial_x^2 Y_2^+(-x) Z_3^+(-x) + \beta_{14}\partial_x^2 Y_1^+(-x) Z_4^+(-x) + O(e^{2\mu_2|x|}),
\end{aligned}$$

hence

$$\frac{\mu_3}{2}\alpha_{32} + \mu_4^2\alpha_{41} = \frac{\mu_3}{2}\beta_{23} + \mu_1^2\beta_{14}.$$

The two equations for α_{32} and α_{41} are linearly independent (recall (2.12)) and their obvious solution is $\alpha_{ij} = \beta_{ji}$. Thus we get

$$\begin{aligned}
F = Y_1^- Z_2^- + 2\mathrm{i}\tau Y_1^- Z_1^- - (k^2 + \tau^2 + O(\lambda^3))(Y_3^- Z_2^- - Y_2^- Z_1^-) \\
+ \sum_{i\in\{1,3,4\}} \alpha_{i1} Y_i^- Z_1^- + \sum_{i=1}^{2} \alpha_{i2} Y_i^- Z_2^-. \quad (3.9)
\end{aligned}$$

Something more explicit is still needed because we only know Y_i^+ and Z_i^+ at $\lambda = 0$ for $i \in \{1,2,3\}$. For $i \in \{1,2\}$ we have (3.6) uniformly in λ and the things inside the absolute value signs are analytic functions of λ. Thus

$$Y_i^+(x) = e^{(\mu_i - \mathring{\mu}_i)x}\mathring{Y}_i^+(x) + O(\lambda e^{(\mu_i - \frac{1}{2})x})$$

and similarly for the Z_i^+. In some cases the first λ-derivatives of $e^{-\mu_i x}Y_i^+(x)$ and $e^{-\mu_i \xi}Z_i^+(\xi)$ also vanish at $\lambda = 0$ (recall Section 3.3 and $\mathring{Z}_2^+(x) = 1$):

$$
\begin{aligned}
Y_1^+(x) &= e^{(\mu_1+1)x}\mathring{Y}_1^+(x) + O(\lambda^2 e^{(\mu_1-\frac{1}{2})x}) \\
Z_1^+(\xi) &= e^{(\mu_1+1)\xi}\mathring{Z}_1^+(\xi) + O(\lambda^2 e^{(\mu_1-\frac{1}{2})\xi}) \\
Z_2^+(\xi) &= e^{\mu_2\xi} + O(\lambda^2 e^{(\mu_2-\frac{1}{2})\xi}).
\end{aligned}
$$

By Theorem 2.6

$$
\left| Y_3^+(x) - \frac{e^{\mu_2 x} - e^{-\mu_2 x}}{2\mu_2} \right| < Ce^{(\mu_3-\frac{1}{2})x},
$$

similarly for Z_3^+. We also need a couple of derivatives with respect to ξ:

Lemma 3.2. *For $n \in \{1, 2\}$*

$$
\begin{aligned}
|\partial_\xi^n Z_1^+ - \partial_\xi^n \mathring{Z}_1^+| &< C|\lambda^2 e^{-\frac{1}{2}\xi}|, \\
|\partial_\xi^n Z_2^+ - \mu_2^n e^{\mu_2\xi}| &< C|\lambda^2 e^{(\mu_2-\frac{1}{2})x}|, \\
|\partial_\xi^n Z_3^+ - \tfrac{1}{2}(\mu_3^{n-1}e^{\mu_3\xi} + \mu_2^{n-1}e^{\mu_2\xi})| &< C|\lambda e^{(\mu_3-\frac{1}{2})x}|.
\end{aligned}
$$

Proof. $e^{-\mu_2\xi}\partial_\xi^n Z_2^+(\xi) - \mu_2^n$ is known to be $O(e^{-x/2})$ by Theorem 2.6 and vanishes at $\lambda = 0$. Its λ-derivative at $\lambda = 0$ can be computed (recall $\partial_\lambda Z_2^+$ from Section 3.3) and also vanishes. Z_3^+ is similar except without the λ-derivative. For Z_1^+ use

$$
|e^{-\mu_1\xi}\partial_\xi^n Z_1^+ - e^\xi \partial_\xi^n \mathring{Z}_1^+ - (\mu_1^n - (-1)^n)| < C|\lambda^2 e^{-\frac{1}{2}\xi}|.
$$

□

To simplify further we use $\mu_2 = -\mu_3 = \mathrm{i}\tau(1 + O(\lambda^2))$. Let c be an upper bound for the $O(\lambda^2)$ term, assumed to be conveniently small. If τ is in the sector $\{\mathrm{Im}(\tau) > 2c|\tau|\}$ and $\xi > 0$ we have, denoting $z := \mathrm{i}\tau\xi$,

$$
\begin{aligned}
|e^{\mu_2\xi}| < |e^{z+c|z|}| &< |e^{\frac{1}{2}\mathrm{i}\tau\xi}|, \\
|e^{\mu_2\xi} - e^{\mathrm{i}\tau\xi}| = |e^z(e^{O(\lambda^2)z} - 1)| &< C|\lambda^2 e^{z+c|z|}| < C|\lambda^2 e^{\frac{1}{2}\mathrm{i}\tau\xi}|.
\end{aligned}
$$

If in addition $x - \xi > 0$

$$
\left| \frac{e^{\mu_2(x+\xi)} - e^{\mu_2(x-\xi)}}{2\mu_2} - \frac{e^{\mathrm{i}\tau(x+\xi)} - e^{-\mathrm{i}\tau(x-\xi)}}{2\mathrm{i}\tau} \right| < C\left| \frac{\lambda^2}{\tau} e^{\frac{1}{2}\mathrm{i}\tau(x-\xi)} \right|.
$$

Putting all this together leads into a terrible mess. The case $x\xi < 0$ is reasonably simple, having terms with four kinds of different behaviour in the two variables but when $x\xi > 0$ there are seven kinds of different behaviour. In the case $\xi < 0 < x$ we have from (3.7) the following terms and estimates for $n \in \{0, 1, 2\}$:

$$
\begin{aligned}
Y_1\partial_\xi^n Z_2 : \quad & \mathring{Y}_1^+(x)\partial_\xi^n e^{\mathrm{i}\tau|\xi|} + O(\lambda^2\tau^n e^{-\frac{1}{2}|x|+\frac{1}{2}\mathrm{i}\tau|\xi|}) + O(\lambda^2 e^{-\frac{1}{2}(|x|+|\xi|)}),\\
Y_1\partial_\xi^n Z_1 : \quad & 2\mathrm{i}\tau\mathring{Y}_1^+(x)\partial_\xi^n \mathring{Z}_1^+(|\xi|) + O(\lambda^2 e^{-\frac{1}{2}(|x|+|\xi|)}),\\
Y_2\partial_\xi^n Z_1 : \quad & -(k^2+\tau^2+O(\lambda^3))\big(e^{\mathrm{i}\tau|x|}\partial_\xi^n \mathring{Z}_1^+(|\xi|)\\
& \qquad + O(e^{-\frac{1}{2}(|x|+|\xi|)}) + O(\lambda^2 e^{\frac{1}{2}\mathrm{i}\tau|x|-\frac{1}{2}|\xi|})\big),\\
Y_2\partial_\xi^n Z_2 : \quad & O(\lambda^2\tau^n e^{\frac{1}{2}\mathrm{i}\tau(|x|+|\xi|)}).
\end{aligned}
$$

When $0 < \xi < x$ equation (3.8) produces the above terms plus the following:

$$
\begin{aligned}
Y_1\partial_\xi^n Z_3 : \quad & O(\lambda^2 e^{-\frac{1}{2}|x-\xi|}),\\
Y_2\partial_\xi^n Z_3 : \quad & -(k^2+\tau^2+O(\lambda^3))\bigg(\partial_\xi^n \frac{e^{\mathrm{i}\tau(|x|+|\xi|)} - e^{\mathrm{i}\tau|x-\xi|}}{2\mathrm{i}\tau}\\
& \qquad + O(\lambda^2\tau^{n-1} e^{\frac{1}{2}\mathrm{i}\tau|x-\xi|}) + O(e^{-\frac{1}{2}|x-\xi|})\\
& \qquad + O(\lambda e^{\frac{1}{2}\mathrm{i}\tau|x-\xi|-\frac{1}{2}|\xi|})\bigg),\\
Y_1\partial_\xi^n Z_4 : \quad & O(\lambda^2 e^{-\frac{1}{2}|x-\xi|}).
\end{aligned}
$$

When $\xi < x < 0$ equation (3.9) gives the first four terms except for a sign change in Y_2Z_1 and

$$
\begin{aligned}
Y_3\partial_\xi^n Z_1 : \quad & O((|k^2+\tau^2|+|\lambda|^3)e^{-\frac{1}{2}|x-\xi|}),\\
Y_3\partial_\xi^n Z_2 : \quad & -(k^2+\tau^2+O(\lambda^3))\bigg(\partial_\xi^n \frac{e^{\mathrm{i}\tau(|x|+|\xi|)} - e^{\mathrm{i}\tau|x-\xi|}}{2\mathrm{i}\tau}\\
& \qquad + O(\lambda^2\tau^{n-1} e^{\frac{1}{2}\mathrm{i}\tau|x-\xi|}) + O(\lambda^2 e^{-\frac{1}{2}|x-\xi|})\\
& \qquad + O(\tau^n e^{\frac{1}{2}\mathrm{i}\tau|x-\xi|-\frac{1}{2}|x|})\bigg),\\
Y_4\partial_\xi^n Z_1 : \quad & O((|k^2+\tau^2|+|\lambda|^3)e^{-\frac{1}{2}|x-\xi|}).
\end{aligned}
$$

The case $x < \xi$ reduces to the preceding cases by $F(x,\xi) = F(-x,-\xi)$. We treat the leading terms separately and summarise:

Theorem 3.3. *There exists $\epsilon > 0$ such that for $|\lambda| < \epsilon$ $F := (\det\Omega)R =$*

$F_0 + F_1$ *and* $F_1 = F_{10} + F_{11} + F_{12}$ *where*

$$\begin{aligned}
F_0(x,\xi) &= \mathring{Y}_1^+(x)(e^{\mathrm{i}\tau|\xi|} + 2\mathrm{i}\tau\mathring{Z}_1^+(|\xi|)),\\
F_{10}(x,\xi) &= \begin{cases} -\frac{k^2+\tau^2}{2\mathrm{i}\tau}(e^{\mathrm{i}\tau|x+\xi|} - e^{\mathrm{i}\tau|x-\xi|}) & \textit{when } x\xi > 0,\\ 0 & \textit{when } x\xi < 0,\end{cases}\\
F_{11}(x,\xi) &= -\operatorname{sgn}(x(x-\xi))(k^2+\tau^2)e^{\mathrm{i}\tau|x|}\mathring{Z}_1^+(|\xi|)\\
|\partial_\xi^n F_{12}(x,\xi)| &< C\big(|\lambda^3\tau^{n-1}e^{\frac{1}{2}\mathrm{i}\tau|x-\xi|}| + |\lambda^2 e^{-\frac{1}{2}|x-\xi|}| + |\lambda^3 e^{\frac{1}{2}\mathrm{i}\tau|x|-\frac{1}{4}|\xi|}|\big)
\end{aligned}$$

for $n \in \{0,1,2\}$.

F_0 has two continuous derivatives with respect to ξ. Although we have tried to write these expressions so that they would be valid everywhere the other pieces are a bit rough around the edges. Hence the estimate for the derivatives is only valid for $x \neq \xi \neq 0 \neq x$. F itself has two continuous derivatives, see the comment after Theorem 2.11.

Chapter 4

Perturbation theory

4.1 The resolvent

Here we estimate $R := (\zeta - D_k H_k)^{-1}$ when $|\zeta|$ is large. We use the second Neumann series

$$\begin{aligned}(\zeta - D_k H_k)^{-1} &= (\zeta - D_k^2 - D_k - D_k V)^{-1}\\ &= \sum_{j=0}^{\infty} ((\zeta - D_k^2 - D_k)^{-1} D_k V)^j (\zeta - D_k^2 - D_k)^{-1},\end{aligned} \tag{4.1}$$

where V is to be interpreted as the operator of multiplying a function pointwise by $V(x)$, thus $D_k V$ denotes a composition rather than D_k applied to V. Let $R_\infty := (\zeta - D_k^2 - D_k)^{-1}$. On the Fourier side we have a multiplier representation for the constant coefficient part in the Neumann series

$$\widehat{R_\infty D_k}(p) = \frac{p^2 + k^2}{\zeta - (p^2 + k^2)^2 - p^2 - k^2}. \tag{4.2}$$

$|V(x)| \le 6e^{-|x|}$ and we want exponential decay also for $R_\infty D_k$, which can be accomplished by the usual imaginary shift of the Fourier variable p. This means we need an L^1 estimate for $\widehat{R_\infty D_k}$ along horizontal lines near the real axis. Hence we need the poles of $\hat{R}_\infty$, as a function of p, to lie outside a strip around the real axis. These poles are i times the eigenvalues μ_j of the A_∞ of Chapter 2. To facilitate combination with other estimates we replace ζ with τ according to (2.9), leading into (2.10) and (2.12). Expanding in partial fractions we get

$$\begin{aligned}\hat{R}_\infty(p) &= \frac{1}{\mu_1^2 - \mu_2^2}\left(\frac{1}{p^2 + \mu_1^2} - \frac{1}{p^2 + \mu_2^2}\right)\\ \widehat{R_\infty D_k}(p) &= \frac{1}{\mu_1^2 - \mu_2^2}\left(\frac{k^2 - \mu_1^2}{p^2 + \mu_1^2} - \frac{k^2 - \mu_2^2}{p^2 + \mu_2^2}\right).\end{aligned}$$

When $|\tau|$ is large $\mu_i = O(\sqrt{\tau})$ and the coefficients above are $O(1/\tau)$ and $O(1)$ respectively (recall (2.12)). Assume $\operatorname{Re}\mu_j < -2a$ for some $a > 0$ and $j \in \{1,2\}$. Shift the integration path by a:

$$\int_{\mathbb{R}+\mathrm{i}a} \frac{1}{|p^2+\mu_j^2|}\,\mathrm{d}\frac{p}{2\pi} < \int_{\mathbb{R}+\mathrm{i}a} \frac{1}{|p+\mathrm{i}\mu_j|^2}\,\mathrm{d}\frac{p}{2\pi} < \int_{-\infty}^{\infty} \frac{1}{a^2+p^2}\,\mathrm{d}\frac{p}{2\pi} < \frac{\pi}{a}.$$

Thus $|R_\infty D_k(x,\xi)| < \frac{C}{a}e^{-a|x-\xi|}$. $\mu_i = O(\sqrt{\tau})$ for large $|\tau|$. If τ is an appropriate sector away from the real axis we can take $a = O(\sqrt{\tau})$ and get convergence for the Neumann series:

Theorem 4.1. *For any $\alpha \in (0, \frac{\pi}{4})$ there exist $C, c, r > 0$ such that for all τ with $|\tau| > r$, $\arg\tau \in (\alpha, \pi-\alpha)$ and $n \in \{0,1\}$*

$$|(RD_k^n)(x,\xi)| < \frac{C}{|\tau|^{\frac{3}{2}-n}} e^{-c\sqrt{|\tau|}|x-\xi|}.$$

Proof. We have established that $|(R_\infty D_k^n)(x,\xi)| \le C_n e^{-a|x-\xi|}$ where $a = c\sqrt{\tau}$ and $C_n = O(\tau^{n-\frac{3}{2}})$. Also, $|V(x)| < 6e^{-|x|}$. We can now estimate the series (4.1) as follows:

$$\begin{aligned}
&|\sum_{j=0}^{\infty}((R_\infty D_k V)^j R_\infty D_k^n)(x,\xi)| \\
&\le \sum_j \Big| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (R_\infty D_k)(x-x_j)V(x_j)\cdots \\
&\qquad\qquad \cdots (R_\infty D_k)(x_2-x_1)V(x_1)(R_\infty D_k^n)(x_1-\xi)\,\mathrm{d}x_1\cdots\mathrm{d}x_j \Big| \\
&\le \sum_j C_n(6C_1)^j \int\cdots\int \exp(-a|x-x_j| - |x_j| - \dots \\
&\qquad\qquad \dots - a|x_2-x_1| - |x_1| - a|x_1-\xi|)\,\mathrm{d}x_1\cdots\mathrm{d}x_j \\
&\le \sum_j C_n(6C_1)^j e^{-a|x-\xi|}\left(\int_{-\infty}^{\infty} e^{-|\tilde{x}|}\,\mathrm{d}\tilde{x}\right)^j \\
&= \frac{C_n}{1-12C_1}e^{-a|x-\xi|},
\end{aligned}$$

provided $C_1 < \frac{1}{12}$, which is the case when $|\tau|$ is large enough. ◻

4.2 The semigroup

In this section we estimate $e^{-tD_kH_k}$ when k is large or t is small:

Theorem 4.2. *There exist $C, c > 0$ such that for any $a > 0$*

$$|e^{-tD_kH_k}(x,\xi)| < \frac{C}{t^{1/4}} e^{(\alpha(k,a)+c)t-a|x-\xi|}$$
$$|(e^{-tD_kH_k}D_k)(x,\xi)| < \frac{C}{t^{3/4}} e^{(\alpha(k,a)+c)t-a|x-\xi|}$$

where

$$\alpha(k,a) := -\tfrac{7}{8}(k^4+k^2) + c(a^4+a^2).$$

There are a couple of interesting options for choosing a so that the t dependency is held in check: for bounded t we can choose $a = t^{-1/4}$ and get rapid decrease in $|x-\xi|$ or for large k we can choose $a = \epsilon k$ for an ϵ such that $\alpha(k,a) + c < -\frac{3}{4}k^4$.

Write $A = D_k^2 + D_k$, $B = D_kV$ (interpreted as the operator composition of multiplication by V followed by D_k rather than D_k applied to V). Now $D_kH_k = A + B$ and formally we can write the following series, which will turn out to converge:

$$e^{-tD_kH_k} = \sum_{n=0}^{\infty} T_n(t)$$
$$T_n(t) = (-1)^n \int \cdots \int_{\{0\le t_1\le\cdots\le t_n\le t\}} e^{-(t-t_n)A} B e^{-(t_n-t_{n-1})A} B \cdots e^{-t_1A}\, \mathrm{d}t_1\cdots \mathrm{d}t_n.$$

Lemma 4.3. *There exist $C, c, M > 0$ such that for any $a > 0$ and $j \in \{0,1\}$*

$$|(e^{-tA}D_k^j)(x,\xi)| < \frac{C}{t^{\frac{1}{4}+\frac{j}{2}}} e^{\alpha(k,a)t-a|x-\xi|}$$
$$|(e^{-tA}B)(x,\xi)| < \frac{M}{t^{3/4}} e^{\alpha(k,a)t-a|x-\xi|-|\xi|}.$$

Proof. We'll use the usual imaginary translation trick on

$$\widehat{e^{-tA}D_k^j}(p) = (p^2+k^2)^j e^{-t((p^2+k^2)^2+p^2+k^2)},$$

for $j \in \{0,1\}$, i.e.,

$$\begin{aligned}
|(e^{-tA}D_k^j)(x,\xi)| &= \left|\int_{\mathbb{R}+\mathrm{i}a} (p^2+k^2)^j e^{-((p^2+k^2)^2+(p^2+k^2))t+\mathrm{i}p(x-\xi)}\, \mathrm{d}\frac{p}{2\pi}\right| \\
&\le \frac{C}{t^{j/2}} \int_{-\infty}^{\infty} e^{(-\frac{7}{8}p^4-\frac{7}{8}(k^4+k^2)+c(a^4+a^2))t-a(x-\xi)}\, \mathrm{d}p \\
&\le \frac{C}{t^{\frac{1}{4}+\frac{j}{2}}} e^{(-\frac{7}{8}(k^4+k^2)+c(a^4+a^2))t-a(x-\xi)}
\end{aligned}$$

for some C and c, which do not depend on a. This gives the desired result if $x > \xi$, otherwise replace a with $-a$. Multiplication by V gives another $e^{-|\xi|}$. □

Lemma 4.4.

$$\int_0^t (t-s)^{q-1} s^{p-1}\,\mathrm{d}s = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} t^{q+p-1}$$

for any positive p and q.

Proof. Laplace transform. □

Lemma 4.5.

$$|T_n(t;x,\xi)| \le \frac{C(2M\Gamma(\frac{1}{4}))^n}{\Gamma(\frac{n+3}{4})} t^{\frac{n-1}{4}} e^{\alpha(k,a)t - a|x-\xi|},$$

$$|(T_n D_k)(t;x,\xi)| \le \frac{C(2M\Gamma(\frac{1}{4}))^n}{\Gamma(\frac{n+1}{4})} t^{\frac{n-3}{4}} e^{\alpha(k,a)t - a|x-\xi|}.$$

Proof. By induction. T_0 is taken care of by Lemma 4.3.

$$\begin{aligned}
|T_{n+1}(t;x,\xi)| &= \left| \int_0^t \int_{-\infty}^{\infty} (e^{-sA}B)(x,w) T_n(t-s;w,\xi)\,\mathrm{d}w\,\mathrm{d}s \right| \\
&\le \int_0^t \int_{-\infty}^{\infty} \left| \frac{M}{s^{3/4}} e^{\alpha(k,a)s - a|x-w| - |w|} T_n(t-s;w,\xi) \right| \mathrm{d}w\,\mathrm{d}s.
\end{aligned}$$

The $T_n D_k$ case is similar. □

The Γ in the denominator guarantees convergence and we can estimate

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\frac{n+3}{4})} &= \frac{1}{\Gamma(\frac{3}{4})} + \sum_{m=0}^{\infty} \sum_{n=0}^{3} \frac{s^{4m+n+1}}{\Gamma(m+1+\frac{n}{4})} \\
&\le \frac{1}{\Gamma(\frac{3}{4})} + \sum_{m=0}^{\infty} \sum_{n=0}^{3} \frac{s^{4m+n+1}}{m!\Gamma(1+\frac{n}{4})} \\
&< 1 + C(s+s^2+s^3+s^4)e^{s^4} < C_\epsilon e^{(1+\epsilon)s^4} \\
\sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\frac{n+1}{4})} &= \frac{1}{\Gamma(\frac{1}{4})} + \frac{s}{\Gamma(\frac{1}{2})} + \frac{s^2}{\Gamma(\frac{3}{4})} + \sum_{m=0}^{\infty} \sum_{n=0}^{3} \frac{s^{4m+n+3}}{\Gamma(m+1+\frac{n}{4})} \\
&< C_\epsilon e^{(1+\epsilon)s^4}
\end{aligned}$$

leading to Theorem 4.2.

Chapter 5

The semigroup

5.1 Outline

We shall use various techniques to estimate the semigroup of D_kH_k for different k: when k is large we use semigroup perturbation theory (Theorem 4.2), elsewhere we use the Dunford–Cauchy integral (1.9) and estimate the resolvent. There we have more cases: for small k and ζ we use the results of Chapter 3, for large ζ we use perturbation theory for the resolvent (Theorem 4.1) and for other k and ζ just Theorem 2.11.

For small but positive k there is a spectral gap between the lowest eigenvalue at $O(k^3)$ and the rest of the spectrum from $O(k^2)$ onward.

When t is small we can use semigroup perturbation theory regardless of k. In the following sections we assume that t is bounded away from zero.

5.2 The pole

Here we estimate the residue K^{pole} of (1.9) at the lowest eigenvalue when k is small. Changing the integration variable to the τ of (2.9) and using the notation of Chapter 3 the integral becomes

$$\int e^{-\zeta(k,\tau)t}\frac{2\tau(1+2k^2)^2}{\det\Omega}F(k,\tau;x,\xi)\,\frac{\mathrm{d}\tau}{2\pi\mathrm{i}}. \tag{5.1}$$

The integration path can be taken to run in the upper half plane. Near the origin there is a pole, corresponding to the smallest eigenvalue of D_kH_k. To circle the spectrum the integration path would need to pass above the pole. However, we now estimate the residue of (5.1) at the pole and shall later combine this with an estimate for the integral where the path passes below the pole.

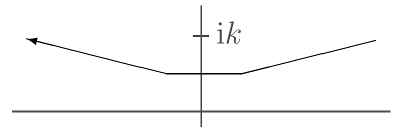


Figure 5.1: Integration path of Section 5.3.

We see from (3.5) that the zero of $\det \Omega$ is at

$$p(k) = \mathrm{i}k - \frac{\mathrm{i}k^2}{6} + O(k^3).$$

We have $\det \Omega = (\tau - p(k))(2\mathrm{i}k + O(\tau - p(k)) + O(k^2))$ and

$$\lim_{\tau \to p(k)} \frac{2\tau(\tau - p(k))}{\det \Omega} = \frac{p(k)}{\mathrm{i}k + O(k^2)} = 1 + O(k). \tag{5.2}$$

$p(k)$ corresponds to $\zeta_0(k) = \frac{1}{3}k^3 + O(k^4)$. The terms of Theorem 3.3 produce $K^{\text{pole}} = K_0^{\text{pole}} + K_1^{\text{pole}}$ with

$$\begin{aligned} K_0^{\text{pole}}(t; x, \xi) &= (e^{-\frac{1}{3}k^3 t} + O(ke^{-\frac{7}{24}k^3 t}))\mathring{Y}_1^+(x)(e^{-(k+O(k^2))|\xi|} - 2k\mathring{Z}_1^+(|\xi|)), \\ |\partial_\xi^n K_1^{\text{pole}}(t; x, \xi)| &< Ce^{-\frac{1}{4}k^3 t}\big(|k^{2+n}e^{-\frac{1}{4}k|x-\xi|}| \\ &\qquad + |k^2 e^{-\frac{1}{2}|x-\xi|}| + |k^3 e^{-\frac{1}{4}k|x| - \frac{1}{4}|\xi|}|\big). \end{aligned}$$

where the $O(\cdot)$ term comes from (5.2) and replacing $\zeta_0(k)$ by $\frac{1}{3}k^3$ (sorry about the silly fraction but we needed something between $\frac{1}{4}$ and $\frac{1}{3}$).

5.3 The rest

Here we integrate around the rest of the spectrum. In terms of τ of (2.9) this means going above the real axis but below the pole at $\approx \mathrm{i}k$ as in Figure 5.1. k is still small. We stay far enough from the pole so that $\frac{|\tau|^2 + k^2}{k^2 + \tau^2}$ remains bounded. This allows us to write for small τ

$$\frac{2\tau}{\det \Omega} = \frac{2\tau}{k^2 + \tau^2} + O(1)$$

c.f. (3.5). Denote $\tilde{R} := \frac{\partial\zeta}{\partial\tau}(\zeta(k,\tau) - D_k H_k)^{-1}$. Combining with Theorem 3.3 we split $\tilde{R}$ as follows:

$$
\begin{aligned}
\tilde{R} &= \tilde{R}_{00} + \tilde{R}_{01} + \tilde{R}_{10} + \tilde{R}_{11} + \tilde{R}_{12} \\
\tilde{R}_{00}(x,\xi) &= \frac{2\tau}{k^2+\tau^2}\mathring{Y}_1^+(x)(e^{\mathrm{i}\tau|\xi|} + 2\mathrm{i}\tau\mathring{Z}_1^+(|\xi|)), \\
\tilde{R}_{01}(x,\xi) &= \omega(k,\tau)\mathring{Y}_1^+(x)(e^{\mathrm{i}\tau|\xi|} + 2\mathrm{i}\tau\mathring{Z}_1^+(|\xi|)), \\
\tilde{R}_{10}(x,\xi) &= \begin{cases} \mathrm{i}(e^{\mathrm{i}\tau|x+\xi|} - e^{\mathrm{i}\tau|x-\xi|}) & \text{when } x\xi > 0, \\ 0 & \text{when } x\xi < 0, \end{cases} \\
\tilde{R}_{11}(x,\xi) &= -2\tau\,\mathrm{sgn}(x(x-\xi))e^{\mathrm{i}\tau|x|}\mathring{Z}_1^+(|\xi|) \\
|(-\partial_\xi^2 + k^2)^n\tilde{R}_{12}(x,\xi)| &< C\big(|\lambda^{2n+1}e^{-\mu(\tau)|x-\xi|}| \\
&\qquad + |\lambda e^{-\frac{1}{2}|x-\xi|}| + |\lambda^2 e^{-\mu(\tau)|x| - \frac{1}{4}|\xi|}|\big)
\end{aligned}
$$

where ω is bounded, $\mu(\tau) := \min\{c\,\mathrm{Im}\,\tau, 1\}$ for some $c > 0$ and $n \in \{0,1\}$. We derived this for small τ but by Theorem 4.1 it actually holds also for large τ on our integration path. Between small and large τ we have Theorems 2.9, 2.11 and Lemma 2.7. Everything is continuous on a compact interval, hence bounded. By choosing c appropriately $-\mu(\tau)$ can be used to estimate $\mathrm{Re}\,\mu_2/2$ from above and thus our estimate holds everywhere along the integration path.

As the explicit terms are analytic we can simply integrate them along the real axis ($e^{-\zeta(k,\tau)t}$ makes everything small as $|\tau| \to \infty$). Denote $t' := (1+2k^2)^2 t$. See Appendix A for details.

$$
\begin{aligned}
K_{00}^{\mathrm{rest}} &= -\mathring{Y}_1^+(x)e^{-(k^2+k^4)t}\int_{-\infty}^{\infty}\frac{2\tau}{k^2+\tau^2}e^{-\tau^2 t'}\left(e^{\mathrm{i}\tau|\xi|} + 2\mathrm{i}\tau\mathring{Z}_1^+(|\xi|)\right)\frac{\mathrm{d}\tau}{2\pi\mathrm{i}} \\
&= e^{(3k^4+4k^6)t}\mathring{Y}_1^+(x)\Bigg(f(k\sqrt{t'}, \tfrac{|\xi|}{2\sqrt{t'}}) - f(k\sqrt{t'}, -\tfrac{|\xi|}{2\sqrt{t'}}) \\
&\qquad + \left(4k\,\mathrm{z}(k\sqrt{t'}) - \frac{2e^{-k^2t'}}{\sqrt{\pi t'}}\right)\mathring{Z}_1^+(|\xi|)\Bigg)
\end{aligned}
$$

where f and z are defined by (A.3). The coefficient of the $\mathring{Z}_1^+$ was obtained from the requirement that $K_{00}^{\mathrm{rest}}(x,\xi)$ be continuously differentiable in ξ (because $\tilde{R}_{00}$ is).

$$
\begin{aligned}
K_{10}^{\mathrm{rest}} &= -e^{-(k^2+k^4)t}\int_{-\infty}^{\infty}e^{-\tau^2 t'}\left(e^{\mathrm{i}\tau|x+\xi|} - e^{\mathrm{i}\tau|x-\xi|}\right)\mathrm{d}\frac{\tau}{2\pi} \\
&= \frac{e^{-(k^2+k^4)t}}{\sqrt{4\pi t'}}\left(e^{-\frac{(x-\xi)^2}{4t'}} - e^{-\frac{(x+\xi)^2}{4t'}}\right)
\end{aligned}
\tag{5.3}
$$

when $x\xi > 0$ and zero otherwise.

$$\begin{aligned} K_{11}^{\text{rest}} &= e^{-(k^2+k^4)t}\,\text{sgn}(x(x-\xi))\mathring{Z}_1^+(|\xi|)\int_{-\infty}^{\infty} 2\tau e^{-\tau^2 t' + \mathrm{i}\tau|x|}\,\frac{\mathrm{d}\tau}{2\pi\mathrm{i}} \\ &= \text{sgn}(x-\xi)\frac{x}{\sqrt{4\pi}(t')^{\frac{3}{2}}}e^{-(k^2+k^4)t-\frac{x^2}{4t'}}\mathring{Z}_1^+(|\xi|). \end{aligned}$$

The rest can be estimated by integrating so that $\operatorname{Im}\tau > \frac{1}{2}k$.

$$\begin{aligned} (-\partial_\xi^2 + k^2)^n K_{01}^{\text{rest}} &= e^{-\frac{1}{2}k^2 t}\mathring{Y}_1^+(x) O(\frac{1}{t^{n+\frac{1}{2}}}e^{-\frac{1}{2}k|\xi|} + \frac{1}{t}e^{-|\xi|}) \\ |(-\partial_\xi^2 + k^2)^n K_{12}^{\text{rest}}| &< Ce^{-\frac{1}{2}k^2 t}\Bigg(\frac{1}{t^{n+1}}e^{-\frac{c}{2}k|x-\xi|} \\ &\qquad + \frac{1}{t}e^{-\frac{1}{2}|x-\xi|} + \frac{1}{t^{\frac{3}{2}}}e^{-\frac{c}{2}k|x|-\frac{1}{4}|\xi|}\Bigg). \end{aligned}$$

5.4 Everything at one swoop

When $k < 1/\sqrt{t}$ (in addition to $k < \epsilon$) a slightly different estimate is useful: instead of treating the pole separately we integrate around the whole spectrum at once. Everything goes essentially as in Section 5.3 except that now the integration path goes above $\mathrm{i}k$. If t is large enough we can raise the integration path by $2/\sqrt{t}$, otherwise we raise it by 2ϵ. This affects the $\tilde{R}_{00}$ integral:

$$\begin{aligned} K_{00}^{\text{all}} &= -\mathring{Y}_1^+(x)e^{-(k^2+k^4)t}\int_{-\infty+\frac{2\mathrm{i}}{\sqrt{t}}}^{\infty+\frac{2\mathrm{i}}{\sqrt{t}}} \frac{2\tau}{k^2+\tau^2}e^{-\tau^2 t'}\left(e^{\mathrm{i}\tau|\xi|} + 2\mathrm{i}\tau\mathring{Z}_1^+(|\xi|)\right)\frac{\mathrm{d}\tau}{2\pi\mathrm{i}} \\ &= e^{(3k^4+4k^6)t}\mathring{Y}_1^+(x)\Bigg(f(k\sqrt{t'}, \tfrac{|\xi|}{2\sqrt{t'}}) + f(-k\sqrt{t'}, \tfrac{|\xi|}{2\sqrt{t'}}) \\ &\qquad + \left(4k\,\mathrm{z}(k\sqrt{t'}) - 2k - \frac{2e^{-k^2 t'}}{\sqrt{\pi t'}}\right)\mathring{Z}_1^+(|\xi|)\Bigg) \end{aligned}$$

and the estimates for the remainders:

$$\begin{aligned} (-\partial_\xi^2 + k^2)^n K_{01}^{\text{all}} &= \mathring{Y}_1^+(x)O(\frac{1}{t^{n+\frac{1}{2}}}e^{-\frac{c|\xi|}{\sqrt{t}}} + \frac{1}{t}e^{-|\xi|}), \\ |(-\partial_\xi^2 + k^2)^n K_{12}^{\text{all}}| &< C\Bigg(\frac{1}{t^{n+1}}e^{-\frac{c|x-\xi|}{\sqrt{t}}} \\ &\qquad + \frac{1}{t}e^{-\frac{1}{2}|x-\xi|} + \frac{1}{t^{\frac{3}{2}}}e^{-\frac{c|x|}{\sqrt{t}}-\frac{1}{4}|\xi|}\Bigg) \end{aligned}$$

for some c and C. The other two terms are not affected as they are analytic in τ. K_{00}^{all} looks messy but

$$K_{00}^{\text{all}} = K_{00}^{\text{rest}} + e^{(3k^4+4k^6)t}\mathring{Y}_1^+(x)(e^{-k|\xi|} - 2k\mathring{Z}_1^+(|\xi|)),$$

which is what one would expect apart from the funny t dependency. But even that is compatible with K_0^{pole} when $k < c/\sqrt{t}$.

5.5 Summary

Fortunately we shall not need all the hideous details of the previous sections—the contents of the present section will suffice. The following two theorems are now easy to see by collecting terms from the previous sections and using (A.4) for the K_{00} terms:

Theorem 5.1. *Let $K_0^{\text{rest}} := K_{00}^{\text{rest}} + K_{01}^{\text{rest}}$ and $K_0^{\text{all}} := K_{00}^{\text{all}} + K_{01}^{\text{all}}$. There exist positive ϵ and c such that for all $k < \epsilon$*

$$\begin{aligned} K_0^{\text{pole}} &= \mathring{Y}_1^+(x)(e^{-\frac{1}{3}k^3 t} + O(k(1+|\xi|)e^{-\frac{7}{24}k^3 t}), \\ K_0^{\text{rest}} &= \mathring{Y}_1^+(x)O(\frac{1}{\sqrt{t}}(1+|\xi|)e^{-\frac{1}{2}k^2 t}), \\ K_0^{\text{all}} &= \mathring{Y}_1^+(x)(1+O(\frac{1}{\sqrt{t}}(1+|\xi|))) \end{aligned}$$

and

$$\begin{aligned} |(-\partial_\xi^2 + k^2)K_0^{\text{pole}}| &< Ce^{-\frac{7}{24}k^3 t - |x|}(k^2 e^{-ck|\xi|} + ke^{-|\xi|}), \\ |(-\partial_\xi^2 + k^2)K_0^{\text{rest}}| &< Ce^{-\frac{1}{2}k^2 t - |x|}(\frac{1}{t}e^{-ck|\xi|} + \frac{1}{\sqrt{t}}e^{-|\xi|}), \\ |(-\partial_\xi^2 + k^2)K_0^{\text{all}}| &< Ce^{-|x|}(\frac{1}{t}e^{-c\frac{|\xi|}{\sqrt{t}}} + \frac{1}{\sqrt{t}}e^{-|\xi|}), \end{aligned}$$

where everything about K_0^{all} is said under the additional assumption $k < 1/\sqrt{t}$.

Furthermore, K_0^{pole}, K_0^{rest} and K_0^{all} are even as functions of ξ.

The first three estimates may look rather poor with their linear growth in ξ, considering that the actual kernels decrease. However, these estimates suffice to show that if

$$|h(\underline{\xi})| \le \frac{C}{(1+|(\underline{\xi})|)^{d+2}}$$

then

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-\mathrm{i}\vec{k}\cdot\vec{\xi}} K_0^{\mathrm{pole}} h \, \mathrm{d}\underline{\xi} &= \mathring{Y}_1^+(x)\Big(\int e^{-\frac{1}{3}k^3 t - \mathrm{i}\vec{k}\cdot\vec{\xi}} h \, \mathrm{d}\underline{\xi} + O(k e^{-\frac{7}{24}k^3 t})\Big) \\
&= \mathring{Y}_1^+(x)\Big(\int e^{-\frac{1}{3}k^3 t}(1 + O(\vec{k}\cdot\vec{\xi})) h \, \mathrm{d}\underline{\xi} + O(k e^{-\frac{7}{24}k^3 t})\Big) \\
&= \mathring{Y}_1^+(x)\Big(e^{-\frac{1}{3}k^3 t}\int h \, \mathrm{d}\underline{\xi} + O(\frac{1}{t^{1/3}} e^{-\frac{1}{4}k^3 t})\Big), \\
\int_{\mathbb{R}^d} e^{-\mathrm{i}\vec{k}\cdot\vec{\xi}} K_0^{\mathrm{rest}} h \, \mathrm{d}\underline{\xi} &= \mathring{Y}_1^+(x) O(\frac{1}{\sqrt{t}} e^{-\frac{1}{2}k^2 t}), \\
\int_{\mathbb{R}^d} e^{-\mathrm{i}\vec{k}\cdot\vec{\xi}} K_0^{\mathrm{all}} h \, \mathrm{d}\underline{\xi} &= \mathring{Y}_1^+(x)\Big(\int h \, \mathrm{d}\underline{\xi} + O(\frac{1}{\sqrt{t}})\Big).
\end{aligned}$$

Theorem 5.2. *Let $K_1^{\mathrm{rest}} := K_{10}^{\mathrm{rest}} + K_{11}^{\mathrm{rest}} + K_{12}^{\mathrm{rest}}$, $K_1^{\mathrm{all}} := K_{10}^{\mathrm{rest}} + K_{11}^{\mathrm{rest}} + K_{12}^{\mathrm{all}}$ and $K_1 := K_1^{\mathrm{pole}} + K_1^{\mathrm{rest}}$. There exist positive ϵ and c such that for all $k < \epsilon$*

$$\begin{aligned}
K_1 &= K_{10}^{\mathrm{rest}} + O(k^2 e^{-\frac{1}{4}k^3 t - ck|x-\xi|}) + O(\frac{1}{t} e^{-\frac{1}{2}k^2 t - ck|x-\xi|}) \\
K_1^{\mathrm{all}} &= K_{10}^{\mathrm{rest}} + O(\frac{1}{t} e^{-c\frac{|x-\xi|}{\sqrt{t}}})
\end{aligned}$$

and

$$\begin{aligned}
|(-\partial_\xi^2 + k^2)K_1| &\le C e^{-\frac{1}{4}k^3 t}(k^2 e^{-\frac{1}{2}|x-\xi|} + k^3 e^{-ck|x| - \frac{1}{4}|\xi|} + k^4 e^{-ck|x-\xi|}) \\
&\quad + C e^{-\frac{1}{2}k^2 t}\left(\frac{1}{t} e^{-\frac{1}{2}|x-\xi|} + \frac{1}{t} e^{-ck|x| - \frac{1}{4}|\xi|} + \frac{1}{t^{3/2}} e^{-ck|x-\xi|}\right) \\
|(-\partial_\xi^2 + k^2)K_1^{\mathrm{all}}| &\le C\left(\frac{1}{t} e^{-\frac{1}{2}|x-\xi|} + \frac{1}{t} e^{-c\frac{|x|}{\sqrt{t}} - \frac{1}{4}|\xi|} + \frac{1}{t^{3/2}} e^{-c\frac{|x-\xi|}{\sqrt{t}}}\right)
\end{aligned}$$

where everything about K_1^{all} is said under the additional assumption $k < 1/\sqrt{t}$. (K_{10}^{rest} is given by (5.3).)

In the previous paragraphs we analysed the case of small k. For larger values of k we have:

Theorem 5.3. *For any $\epsilon > 0$ there exist $C, c > 0$ such that for any $k > \epsilon$, $t > 1$ and $n \in \{0, 1\}$*

$$|(e^{tD_k H_k} D_k^n)(x, \xi)| < C e^{-\frac{1}{2}k^4 t - c|x-\xi|}$$

Proof. Theorem 4.2 takes care of very large k, say $k > r$. For $k \in [\epsilon, r]$ we use the Dunford–Cauchy integral

$$e^{tD_k H_k} = \int_\Gamma e^{-\zeta t}(\zeta - D_k H_k)^{-1} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}}$$

and estimate the resolvent. In Section 1.2 we saw that the spectrum has k^4 as a lower bound. Thus Γ can be chosen so that $\operatorname{Re}\zeta > \frac{1}{2}k^4$ and $\operatorname{Re}\mu_1 \leq \operatorname{Re}\mu_2 < -c < 0$ for some c. Asymptotically the path can be chosen to be $s \mapsto \frac{1}{2}k^4 + se^{\pm i\alpha}$ for some $\alpha < \pi/2$. The resolvent kernel is then bounded by $Ce^{-c|x-\xi|}$: for large $|\zeta|$ this follows from Theorem 4.1, otherwise just from (2.16) and continuity. $\square$

Chapter 6

The nonlinear analysis

6.1 Regularisation

We start by showing that due to the $-\triangle^2$ term solutions of (1.1) acquire arbitrarily many spatial derivatives in a short time. These are bounded in the following norm:

Definition 6.1. Define

$$\omega^r(\underline{x}) = \frac{1}{(1+|\underline{x}|)^r}.$$

Fix some $r > d+1$ and let X be the Banach space of continuous functions $\mathbb{R}^d \to \mathbb{C}$ with the norm

$$\|f\|_X := \sup_{\underline{x}\in\mathbb{R}^d} |f(\underline{x})|\omega^{-r}(\underline{x}). \tag{6.1}$$

Let $X^{(n)}$ be the space of n times continuously differentiable functions $\mathbb{R}^d \to \mathbb{C}$ with the norm

$$\|f\|_{X^{(n)}} := \max_{|\mu|\leq n} \|\partial^{\mu}_{\underline{x}} f\|_X$$

where μ is a multi-index.

The X norm was already used tacitly in the comment after Theorem 5.1, which also explains the bound on r.

Convolutions with exponentials are often needed:

Lemma 6.2. *Let $n > d$. There exists $C > 0$ such that for any $a, b > 0$*

$$\int_{\mathbb{R}^d} e^{-a|\underline{x}-\underline{\xi}|}\omega^n(b\underline{\xi})\,\mathrm{d}\underline{\xi} < \begin{cases} \frac{C}{a^d}\omega^n(b\underline{x}) & \textit{when } a \geq b \\ \frac{C}{a^d}\omega^n(b\underline{x}) + \frac{C}{b^d}\omega^n(a\underline{x}) & \textit{when } a < b \end{cases}$$

Proof. Divide the integration region into $S = \{\underline{\xi} : |\underline{\xi} - \underline{x}| < \frac{1}{2}|\underline{x}|\}$ and its complement:

$$\int_{\mathbb{R}^d} e^{-a|\underline{x}-\underline{\xi}|}\omega^n(b\underline{\xi})\,\mathrm{d}\underline{\xi} \\ < \omega^n(\frac{b}{2}\underline{x})\int_S e^{-a|\underline{x}-\underline{\xi}|}\,\mathrm{d}\underline{\xi} + e^{-\frac{a}{4}|\underline{x}|}\int_{\mathbb{R}^d\setminus S} e^{-\frac{a}{2}|\underline{x}-\underline{\xi}|}\omega^n(b\underline{\xi})\,\mathrm{d}\underline{\xi}.$$

□

Theorem 6.3. *Let η be the solution of* (1.4) *with $\eta(0) = \eta_0 \in X$. There exist $\delta_n > 0$ and $C_n > 0$ such that if $\|\eta_0\|_X < \delta_n$ then*

$$\|\eta(1)\|_{X^{(n)}} < C_n\|\eta_0\|_X.$$

Proof. The idea is to establish that if the initial condition is small enough in $X^{(n)}$ the solution gains another derivative in an arbitrarily short time: $\|\eta(t)\|_{X^{(n+1)}} < C_t\|\eta_0\|_{X^{(n)}}$.

η satisfies

$$\partial_t\eta = -\triangle^2\eta + \triangle N(\eta)$$

where $N(\eta) := (1+V)\eta + \frac{3}{2}\psi_0\eta^2 + \frac{1}{2}\eta^3$. This is equivalent to the integral equation

$$\eta(t) = e^{-t\triangle^2}\eta_0 + \int_0^t \triangle\, e^{-(t-s)\triangle^2}N(\eta(s))\,\mathrm{d}s \tag{6.2}$$

which can be differentiated:

$$\partial_{\underline{x}}^{\mu}\eta(t) = e^{-t\triangle^2}\partial_{\underline{x}}^{\mu}\eta_0 + \int_0^t \triangle\, e^{-(t-s)\triangle^2}\partial_{\underline{x}}^{\mu}N(\eta(s))\,\mathrm{d}s. \tag{6.3}$$

For a short enough time $\tau \le 1$ the right side of (6.2) is a contraction in a ball $\{\|\eta\|_{C([0,\tau],X^{(m)})} \le C\|\eta_0\|_{X^{(n)}}\}$. This is because N is Lipschitz continuous in any bounded subset of $X^{(n)}$ and the convolution kernel G of $e^{-t\triangle^2}$ satisfies

$$|\partial_{\underline{x}}^{\nu}G(t;\underline{x})| < \frac{C_\nu}{t^{\frac{|\nu|+d}{4}}}e^{-\frac{|\underline{x}|}{t^{1/4}}}.$$

By Lemma 6.2 ($t \le 1$)

$$\int_0^t\int_{\mathbb{R}^d}\frac{C}{(t-s)^{\frac{d+|\nu|}{4}}}e^{-\frac{|\underline{x}-\underline{\xi}|}{(t-s)^{1/4}}}\omega^r(\underline{\xi})\,\mathrm{d}\underline{\xi}\mathrm{d}s < C\int_0^t\frac{1}{(t-s)^{\frac{|\nu|}{4}}}\omega^r(\underline{x})\,\mathrm{d}s \\ < Ct^{1-\frac{|\nu|}{4}}\omega^r(\underline{x})$$

when $|\nu| < 4$. Thus (6.2) has a solution $\eta \in C([0,\tau], X^{(n)})$ with $\|\eta\| \leq C\|\eta_0\|_{X^{(n)}}$.

Differentiating (6.3) yet again yields

$$\partial_{\underline{x}_j}\partial_{\underline{x}}^{\mu}\eta(t) = \partial_{\underline{x}_j} e^{-t\,\triangle^2}\partial_{\underline{x}}^{\mu}\eta_0 + \int_0^t \partial_{\underline{x}_j} \triangle\, e^{-(t-s)\,\triangle^2}\partial_{\underline{x}}^{\mu} N(\eta(s))\,\mathrm{d}s.$$

Estimating the integrals as in the previous case we end up with

$$|\partial_{\underline{x}_j}\partial_{\underline{x}}^{\mu}\eta(t,\underline{x})| < C(\frac{1}{t^{1/4}}\|\partial_{\underline{x}}^{\mu}\eta_0\| + t^{1/4}\sup_{\nu\leq\mu}\|\partial_{\underline{x}}^{\nu}\eta\|)\omega^r(\underline{x}).$$

Now back to the theorem. Fix a time step $h = 1/\tilde{n}$ such that $h \leq \tau$ and $h \leq 1/n$. The previous argument shows that if $\|\eta(kh)\|_{X^{(k)}} < \delta$ then $\|\eta((k+1)h)\|_{X^{(k+1)}} < C_{h,\delta}\|\eta(kh)\|_{X^{(k)}}$ for $k < n$ and $\|\eta((k+1)h)\|_{X^{(n)}} < C_{h,\delta}\|\eta(kh)\|_{X^{(n)}}$ after that. Iterating this we find that if $\|\eta_0\|_X < C_{h,\delta}^{-\tilde{n}}\delta$ then $\|\eta(1)\|_{X^{(n)}} < C_{h,\delta}^{\tilde{n}}\|\eta_0\|_X$. $\square$

6.2 The norm

Now we introduce the rather complicated norm which will be used in the fixed point argument. We are back in the $(x,\vec{k})$ representation.

Definition 6.4. Let $m > 2$. Assume $t \geq 1$ and let

$$k_t = \min\{k,1\} + \frac{1}{\sqrt{t}}. \tag{6.4}$$

Let $n > \frac{d+1}{3}$ and define the spaces X_t with the norm

$$\|f\|_t := \sup_{(x,\vec{k})\in\mathbb{R}^d} \frac{|f(x,\vec{k})|}{(\omega^m(x) + k_t\omega^m(k_t x))\omega^n(k^3 t)}.$$

From now on we shall denote ω^m simply by ω when there is no danger of confusion.

This needs some explaining. ω^m is related to norm of the space X in the previous section if we set $r = m + d - 1$. See also Theorem 5.1 and the comments after it. There are essentially two kinds of terms in the semigroup: nice functions of x and nice functions of kx or $x/\sqrt{t}$. The latter ones have k or $1/\sqrt{t}$ in front, which compensates for their poor decay. The denominator takes this into account. The time behaviour in the leading term of the semigroup is e^{-ck^3t}, which motivates the k^3t-dependent term. The limit on n will provide sufficient $\vec{k}$-integrability for the proof of Lemma 6.5.

Using the mixed Fourier representation involves replacing the powers of η in (1.5) by mixed convolutions. Continuing our abuse of notation we denote these by $*$:

$$(f*g)(x,\vec{k}) := \int_{\mathbb{R}^{d-1}} f(x,\vec{k}-\vec{p})g(x,\vec{p})\,\mathrm{d}\frac{\vec{p}}{2\pi}.$$

Lemma 6.5. *Let $f, g \in X_t$. Then*

$$\|f*g\|_t \leq \frac{C}{t^{\frac{d-1}{3}}}\,\|f\|_t\,\|g\|_t.$$

Proof. Using subscript t in the sense of (6.4) and $p := |\vec{p}|$

$$|(f*g)(x,\vec{k})| \leq \|f\|_t\|g\|_t \int_{\mathbb{R}^{d-1}} (\omega^m(x) + p_t\omega^m(p_t x))\omega^n(p^3t)\cdot$$
$$\cdot\,(\omega^m(x) + |\vec{k}-\vec{p}|_t\omega^m(|\vec{k}-\vec{p}|_t x))\omega^n(|\vec{k}-\vec{p}|^3t)\,\mathrm{d}\frac{\vec{p}}{2\pi}.$$

Expanding the product yields four terms. Three of these have at least one $\omega^m(x)$ and can be estimated by $C\omega^m(x)I$,

$$I := \int_{\mathbb{R}^{d-1}} \omega^n(p^3t)\omega^n(|\vec{k}-\vec{p}|^3t)\,\mathrm{d}\vec{p}.$$

Split the integration region into $S := \{\vec{p} : |\vec{p}-\vec{k}| < \frac{1}{2}|\vec{k}|\}$ and its complement. Then estimate

$$\begin{aligned}I &\leq \omega^n((\tfrac{1}{2}k)^3t)\left(\int_S \omega^n(|\vec{k}-\vec{p}|^3t)\,\mathrm{d}\vec{p} + \int_{\mathbb{R}^{d-1}\setminus S} \omega^n(p^3t)\,\mathrm{d}\vec{p}\right)\\ &\leq C\omega^n(k^3t)\int_{\mathbb{R}^{d-1}} \omega^n(p^3t)\,\mathrm{d}\vec{p} \leq \frac{C}{t^{\frac{d-1}{3}}}\omega^n(k^3t).\end{aligned}$$

The last term I' is more complicated:

$$\begin{aligned}I' &:= \int_{\mathbb{R}^{d-1}} p_t\omega^m(p_tx)\omega^n(p^3t)\,|\vec{k}-\vec{p}|_t\omega^m(|\vec{k}-\vec{p}|_tx)\omega^n(|\vec{k}-\vec{p}|^3t)\,\mathrm{d}\frac{\vec{p}}{2\pi}\\ &\leq Ck_t\omega^m(k_tx)\omega^n(k^3t)\int_S p_t\omega^m(|\vec{k}-\vec{p}|_tx)\omega^n(|\vec{k}-\vec{p}|^3t)\,\mathrm{d}\vec{p}\\ &\quad + C\omega^m(k_tx)\omega^n(k^3t)\int_{\mathbb{R}^{d-1}\setminus S} p_t|\vec{k}-\vec{p}|_t\omega^m(p_tx)\omega^n(p^3t)\,\mathrm{d}\vec{p}.\end{aligned}$$

The first term is of the appropriate form as the integral is bounded by $C/t^{\frac{d-1}{3}}$. We need to extract a k_t also from the second integral. Use $|\vec{k}-\vec{p}|_t \leq k_t + p_t$.

The term containing k_t is clear and it remains to estimate

$$\int_{\mathbb{R}^{d-1}\setminus S} p_t^2 \omega^m(p_t x)\omega^n(p^3 t)\,\mathrm{d}\vec{p} \le \int_{\mathbb{R}^{d-1}} (p+\frac{1}{\sqrt{t}})^2\omega^n(p^3 t)\,\mathrm{d}\vec{p}$$
$$\le \frac{C}{t^{\frac{d+1}{3}}} \le \frac{Ck_t}{t^{\frac{d-1}{3}}}.$$

□

6.3 The linear problem

From Theorem 6.3 it follows that if the initial data is in X, the solution at $t = 1$ will be in $X^{(3n)}$. We shall take $t = 1$ as the starting point for the contraction mapping argument. The contracted variable will be the difference between the solutions of the linearised problem and the nonlinear problem. First we need the following result on the linear problem:

Lemma 6.6. *Let $t \ge 1$. Then*

$$u := K(t-1)\hat{\eta}_1 = u_0 + u_1,$$
$$u_0(t;x,k) = Ae^{-\frac{1}{3}k^3 t}\mathring{Y}_1^+(x), \qquad A = \int_{\mathbb{R}^d}\eta_1(\underline{\xi})\,\mathrm{d}\underline{\xi},$$
$$\|u_1(t)\|_t < \frac{C}{t^{1/3}}\|\eta_1\|_{X^{(3n)}}.$$

Proof. Denote $s := t-1$. First assume $s > 1$ and $k < \epsilon$. According to Theorem 5.1 $K_0^{\text{pole}} + K_0^{\text{rest}}$ or K_0^{all}, depending whether $k > 1/\sqrt{s}$ or not, produces u_0 plus something which satisfies the estimate for u_1. By Theorem 5.2 and Lemma 6.2 the K_1 terms are easily found small and includible in u_1 except for K_{10}^{rest}, which requires somewhat more careful inspection.

Assume $x, \xi \ge 0$, $\epsilon < 1$. We shall use these two estimates derived from (5.3):

$$|K_{10}^{\text{rest}}(s)| \le \frac{C}{\sqrt{s'}}e^{-k^2 s-\frac{(x-\xi)^2}{4s'}} \le \frac{C}{\sqrt{s}}e^{-k^2 s-\frac{(x-\xi)^2}{36s}}$$
$$\le \frac{C}{\sqrt{s}}e^{-\frac{1}{2}k^2 s-\frac{1}{5}k_s|x-\xi|}, \tag{6.5}$$

$$|K_{10}^{\text{rest}}(s)| \le \frac{e^{-k^2 s}}{\sqrt{4\pi s'}}\left(e^{-\frac{(x-\xi)^2}{4s'}} - e^{-\frac{(x+\xi)^2}{4s'}}\right) \le \frac{1}{\sqrt{4\pi s'}}e^{-k^2 s-\frac{x^2+\xi^2}{4s'}}\left(e^{\frac{x\xi}{2s'}} - e^{-\frac{x\xi}{2s'}}\right)$$
$$\le \frac{x\xi}{\sqrt{4\pi}(s')^{3/2}}e^{-k^2 s-\frac{x^2+\xi^2}{4s'}+\frac{x\xi}{2s'}} \le \frac{x\xi}{\sqrt{4\pi}s^{3/2}}e^{-k^2 s-\frac{(x-\xi)^2}{4s'}}$$
$$\le \frac{C}{\sqrt{s}}k_s^2 x\xi e^{-\frac{1}{2}k^2 s-\frac{1}{5}k_s|x-\xi|} \tag{6.6}$$

where $s' := (1+2k^2)^2 s$. Split the integration region, use (6.5) for large ξ and (6.6) for small ξ:

$$\begin{aligned} |(K_{10}^{\text{rest}}(s)\hat{\eta}_1)(x)| \leq \frac{C\|\eta_1\|_X}{\sqrt{s}} e^{-\frac{1}{2}k^2 s} \Bigg(k_s^2 x \int_0^{1/k_s} & e^{-\frac{1}{5}k_s|x-\xi|} \xi \omega^m(\xi)\,\mathrm{d}\xi \\ + \int_{1/k_s}^{\infty} & e^{-\frac{1}{5}k_s|x-\xi|} \omega^m(\xi)\,\mathrm{d}\xi \Bigg). \end{aligned} \tag{6.7}$$

The first term can be estimated by

$$\frac{C\|\eta_1\|_X}{\sqrt{s}} k_s^2 x e^{-\frac{1}{2}k^2 s - \frac{1}{5}k_s x},$$

which can be absorbed into u_1. For the second term split the region further into $S := \{\xi \in [1/k_s, \infty] : |\xi - x| < x/2\}$ and its complement. Let us first integrate over S. S is empty unless $x > \frac{2}{3k_s}$, which assume. Then

$$\int_S e^{-\frac{1}{5}k_s|x-\xi|}\omega^m(\xi)\,\mathrm{d}\xi \leq \frac{C}{k_s}\omega^m(x) \leq \frac{Ck_s}{(k_s + k_s|x|)^2}\omega^{m-2}(x) \leq Ck_s\omega^m(k_s x).$$

For the complement of S we use

$$\int_{[\frac{1}{k_s},\infty]\setminus S} e^{-\frac{1}{5}k_s|x-\xi|}\omega^m(\xi)\,\mathrm{d}\xi \leq Ck_s e^{-\frac{1}{10}k_s x}\int \omega^{m-1}(\xi)\,\mathrm{d}\xi \leq Ck_s\omega^m(k_s x).$$

Thus the second term of (6.7) can also be absorbed into u_1.

For $k \geq \epsilon$ and $s > 1$ use Theorem 5.3 and Lemma 6.2:

$$|(K(s)\hat{\eta}_1)(x)| \leq C\|\eta_1\|_X e^{-\frac{1}{2}k^4 s}\int_{\mathbb{R}} e^{-c|x-\xi|}\omega^m(\xi)\,\mathrm{d}\xi \leq C\|\eta_1\|_X e^{-\frac{1}{2}k^4 s}\omega^m(x).$$

Even after multiplication by $\omega^{-n}(k^3 t) < C(k^4 t)^n$ this remains bounded by $C\|\eta_1\|_X e^{-\frac{1}{4}\epsilon^4 s}\omega^m(x)$ and can be absorbed into u_1. Similarly u_0 can be absorbed into u_1.

For $s \leq 1$ use Theorem 4.2 with $a = s^{-1/4}$:

$$\begin{aligned} |(K(s)\hat{\eta}_1)(x)| &\leq \frac{C\|\eta_1\|_{X^{(3n)}}\omega^n(k^3)}{s^{1/4}}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{s^{1/4}}}\omega^m(\xi)\,\mathrm{d}\xi \\ &\leq C\|\eta_1\|_{X^{(3n)}}\omega^n(k^3)\omega^m(x). \end{aligned}$$

Again this and u_0 can be absorbed into u_1. □

6.4 The contraction property

Next define

$$\nu := \hat{\eta} - u,$$
$$\nu_1 := \hat{\eta} - u_0 = \nu + u_1$$

where η solves (1.5), $\eta_1 = \eta(1)$, u and u_i are from Lemma 6.6. Rewrite (1.5) in terms of these:

$$\nu(t) = \mathcal{F}(\nu)(t) := -\int_1^t e^{-(t-s)D_k H_k} D_k N(s, \nu(s))\,\mathrm{d}s \tag{6.8}$$
$$N(s,\nu) := \tfrac{3}{2}\psi_0 \hat{\eta}^{*2} + \tfrac{1}{2}\hat{\eta}^{*3} = \tfrac{3}{2}\psi_0(u_0+\nu_1)^{*2} + \tfrac{1}{2}\hat{\eta}^{*3}$$

We will prove that this $\mathcal{F}$ is a contraction in the norm

$$\|\nu\| := \sup_{t\geq 1} t^{1/12}\|\nu(t)\|_t. \tag{6.9}$$

Assume $\|\eta_0\|_X < \delta$ and $\|\nu\| \leq \delta$, $\|\tilde{\nu}\| \leq \delta$ and $\|\tilde{\nu} - \nu\| \leq \tilde{\delta}$. We shall show that $\|\mathcal{F}(\nu)\| < C\delta^2$ and $\|\mathcal{F}(\tilde{\nu}) - \mathcal{F}(\nu)\| < C\delta\tilde{\delta}$ for some C independent of δ and $\tilde{\delta}$. δ can then be taken small enough to assure that $\mathcal{F}$ maps the ball of radius δ into itself and contracts. All instances of C in the sequel will be independent of δ and $\tilde{\delta}$.

By Theorem 6.3 and Lemma 6.6 $\|u_0\|_t \leq C\delta$ and $\|u_1\|_t \leq C\delta/t^{1/3}$. Thus $\|\nu_1\| \leq C\delta$ and, as $\hat{\eta} = u_0 + u_1 + \nu$, $\|\hat{\eta}(t)\|_t \leq C\delta$. By Lemma 6.5

$$\|N\|_t \leq \tfrac{3}{2}\|\hat{\eta}^{*2}\|_t + \tfrac{1}{2}\|\hat{\eta}^{*3}\|_t < \frac{C\delta^2}{t^{\frac{d-1}{3}}}. \tag{6.10}$$

A straightforward computation also yields

$$\|N(\tilde{\nu}) - N(\nu)\|_t < \frac{C\delta\tilde{\delta}}{t^{\frac{d-1}{3}+\frac{1}{12}}}. \tag{6.11}$$

When $t \geq 2$ we treat $t - s \leq 1$ separately using Theorem 4.2 and the rest using the results of Chapter 5 and Lemma 6.6. When $t - s > 1$ and $k \leq \epsilon$ we split the semigroup kernel into $K_0 + K_1$, where

$$K_0 := \begin{cases} K_0^{\mathrm{all}} & \text{when } k < \frac{1}{\sqrt{t}}, \\ K_0^{\mathrm{pole}} + K_0^{\mathrm{rest}} & \text{otherwise.} \end{cases}$$

The worst term would appear to be $K_0 D_k(\psi_0 u_0^{*2})$ but fortunately the function is odd and thus annihilated by $K_0 D_k$. (6.8) is thus split as

$$\begin{aligned}
\mathcal{F} &= \mathcal{A}(t) + \mathcal{B}(t) + \mathcal{C}(t) \\
\mathcal{A}(t) &= -\int_1^{t-1} K_0(t-s) D_k N(s, \nu(s))\,\mathrm{d}s \\
&= -\int_1^{t-1} K_0(t-s) D_k\left(\tfrac{3}{2}\psi_0(2u_0+\nu_1) * \nu_1 + \tfrac{1}{2}\hat{\eta}^{*3}\right)\mathrm{d}s \\
\mathcal{B}(t) &= -\int_1^{t-1} K_1(t-s) D_k N(s, \nu(s))\,\mathrm{d}s \\
\mathcal{C}(t) &= -\int_{t-1}^{t} K(t-s) D_k N(s, \nu(s))\,\mathrm{d}s.
\end{aligned}$$

Let us start with $\mathcal{C}(t)$. By Theorem 4.2

$$|(K(t-s)D_k)(x,\xi)| < \frac{C}{(t-s)^{\frac{3}{4}}} e^{-|x-\xi|}$$

as $t - s \le 1$. Hence and by (6.10)

$$\begin{aligned}
\mathcal{C}(t) &\le C\delta^2 \int_{t-1}^{t} \frac{1}{(t-s)^{\frac{3}{4}} s^{\frac{d-1}{3}}} \int_{\mathbb{R}} e^{-|x-\xi|}(\omega^m(\xi) + k_s\omega^m(k_s\xi))\omega^n(k^3 s)\,\mathrm{d}\xi\mathrm{d}s \\
&\le C\delta^2 \int_{t-1}^{t} \frac{1}{(t-s)^{\frac{3}{4}} s^{\frac{d-1}{3}}} (\omega^m(x) + k_s\omega^m(k_s x))\omega^n(k^3 s)\,\mathrm{d}s \\
&\le \frac{C\delta^2}{t^{\frac{d-1}{3}}} (\omega^m(x) + k_t\omega^m(k_t x))\omega^n(k^3 t),
\end{aligned}$$

where Lemma 6.2 and $s \in [t-1, t]$ were used. Thus $\|\mathcal{C}(t)\|_t \le C\delta^2/t^{\frac{d-1}{3}}$. Similarly we get $\|\mathcal{C}(t,\tilde{\nu}) - \mathcal{C}(t,\nu)\|_t \le C\delta\tilde{\delta}/t^{\frac{d-1}{3}}$ by starting from (6.11) and throwing away the extra $1/12$.

When estimating $\mathcal{A}$ and $\mathcal{B}$ these integrals often appear:

Lemma 6.7. *Assume $p \le q$ and $t > 2$.*

$$\int_1^{t-1} \frac{\mathrm{d}s}{(t-s)^p s^q} < \begin{cases} \frac{C}{t^{p+q-1}} & \text{when } q < 1 \\ \frac{C\log t}{t^p} & \text{when } q = 1 \\ \frac{C}{t^p} & \text{when } q > 1. \end{cases}$$

Proof. Split the integration region:

$$\int_1^{t-1} \frac{\mathrm{d}s}{(t-s)^p s^q} \le \left(\frac{2}{t}\right)^p \int_1^{t/2} \frac{\mathrm{d}s}{s^q} + \left(\frac{2}{t}\right)^q \int_{t/2}^{t-1} \frac{\mathrm{d}s}{(t-s)^p}$$
$$\le \left(\frac{2}{t}\right)^p \int_1^{t/2} \frac{\mathrm{d}s}{s^q} + \left(\frac{2}{t}\right)^q \int_1^{t/2} \frac{\mathrm{d}s}{s^p}.$$

□

For $\mathcal{A}$ and $\mathcal{B}$ we have several cases depending on the value of k. Let us start with $\mathcal{B}$ when $k \le \min\{\epsilon, 1/\sqrt{t}\}$. Using Theorem 5.2 and (6.10) we find

$$|\mathcal{B}(t)| \le C\delta^2(I_1 + I_2 + I_3)$$
$$I_1 = \int_1^{t-1} \int_{\mathbb{R}} \frac{1}{t-s} e^{-\frac{1}{2}|x-\xi|} \frac{1}{s^{\frac{d-1}{3}}} (\omega(\xi) + k_s\omega(k_s\xi))\, \mathrm{d}\xi \mathrm{d}s$$
$$I_2 = \int_1^{t-1} \int_{\mathbb{R}} \frac{1}{t-s} e^{-c\frac{|x|}{\sqrt{t-s}} - \frac{1}{4}|\xi|} \frac{1}{s^{\frac{d-1}{3}}} (\omega(\xi) + k_s\omega(k_s\xi))\, \mathrm{d}\xi \mathrm{d}s$$
$$I_3 = \int_1^{t-1} \int_{\mathbb{R}} \frac{1}{(t-s)^{3/2}} e^{-c\frac{|x-\xi|}{\sqrt{t-s}}} \frac{1}{s^{\frac{d-1}{3}}} (\omega(\xi) + k_s\omega(k_s\xi))\, \mathrm{d}\xi \mathrm{d}s.$$

Similarly, using (6.11) and throwing away the 1/12 we get $|\mathcal{B}(t,\tilde{\nu}) - \mathcal{B}(t,\nu)| \le C\delta\tilde{\delta}(I_1 + I_2 + I_3)$.

By Lemma 6.2, $1/\sqrt{s} < k_s < 2/\sqrt{s}$ and $\omega(x/\sqrt{s}) \le \omega(x/\sqrt{t})$

$$I_1 \le C \int_1^{t-1} \frac{1}{(t-s)s^{\frac{d-1}{3}}} \Big(\omega(x) + \frac{1}{\sqrt{s}}\omega(\frac{x}{\sqrt{s}})\Big)\, \mathrm{d}s$$
$$\le \frac{C\log t}{t^{\frac{1}{3}}} \big(\omega(x) + k_t\omega(k_t x)\big).$$

For I_2 we use simply

$$I_2 \le C \int_1^{t-1} \frac{\mathrm{d}s}{(t-s)s^{\frac{d-1}{3}}} \omega(\frac{x}{\sqrt{t}}) \le \frac{C\log t}{t^{\frac{d-1}{3}}} \omega(\frac{x}{\sqrt{t}})$$

assuming $d \le 4$. Unfortunately this is no good when $d = 2$. For $d \ge 3$ we have $I_2 \le C\log t/t^{1/6} k_t\omega(k_t x)$ (even when $d > 4$).

With I_3 we need to use Lemma 6.2 again, then estimate $\omega(x/\sqrt{s})$ and $\omega(x/\sqrt{t-s})$ by $\omega(x/\sqrt{t})$:

$$I_3 \le \int_1^{t-1} \frac{C}{(t-s)^{\frac{3}{2}} s^{\frac{d-1}{3}}} \Big(\sqrt{t-s}\,\omega(x) + \omega(\frac{x}{\sqrt{t-s}}) + \sqrt{\frac{t-s}{s}}\,\omega(\frac{x}{\sqrt{s}})\Big)$$
$$\le \frac{C}{t^{\frac{d-1}{3}}} \Big(\omega(x)\log t + \omega(\frac{x}{\sqrt{t}})\Big) + \frac{C\log t}{t^{5/6}} \omega(\frac{x}{\sqrt{t}})$$

when $d \le 4$. This will do in three and four dimensions but when $d = 2$ the second term in parentheses is not small enough (for $d > 4$ we get $I_3 \le \frac{C}{t}(\omega(x) + \omega(x/\sqrt{t}))$).

Let us then proceed with $\mathcal{A}$, still assuming $k \le \min\{\epsilon, 1/\sqrt{t}\}$. First bound the argument using Lemma 6.5:

$$\|\tfrac{3}{2}\psi_0(2u_0 + \nu_1) * \nu_1 + \tfrac{1}{2}\hat{\eta}^{*3})\|_s \le \frac{C\delta^2}{s^{\frac{d-1}{3}+\frac{1}{12}}}. \tag{6.12}$$

Denote the exponent by $q := \frac{d-1}{3} + \frac{1}{12}$. Then by Theorem 5.1

$$\begin{aligned}
|\mathcal{A}(t)| \le\ & C\delta^2 e^{-|x|} \int_1^{t-1} \int_{\mathbb{R}} \Big(\frac{1}{t-s} e^{-c\frac{|\xi|}{\sqrt{t-s}}} + \frac{1}{\sqrt{t-s}} e^{-|\xi|}\Big)\cdot \\
&\qquad \cdot \frac{1}{s^q}\Big(\omega(\xi) + \frac{1}{\sqrt{s}}\omega(\frac{\xi}{\sqrt{s}})\Big)\, \mathrm{d}\xi \mathrm{d}s \\
\le\ & C\delta^2 e^{-|x|} \int_1^{t-1} \frac{1}{s^q}\Big(\frac{1}{\sqrt{t-s}} + \frac{1}{(t-s)\sqrt{s}} \int_{\mathbb{R}} e^{-c\frac{|\xi|}{\sqrt{t-s}}} \omega(\frac{\xi}{\sqrt{s}})\, \mathrm{d}\xi\Big)\, \mathrm{d}s \\
\le\ & C\delta^2 \omega(x) \int_1^{t-1} \frac{\mathrm{d}s}{s^q\sqrt{t-s}} \le \frac{C\delta^2}{t^{1/6}}\omega(x)
\end{aligned}$$

where $d \ge 3$ was used on the last line. If $d = 2$, however, we get a completely useless estimate that grows with t. $|\mathcal{A}(t, \tilde{\nu}) - \mathcal{A}(t, \nu)| \le C\delta\tilde{\delta}\omega(x)/t^{1/6}$ can be derived similarly by starting from (6.11) instead of (6.12).

Next we shall deal with $k \in (1/\sqrt{t}, \epsilon)$ where the estimates of Theorems 5.1 and 5.2 are more complicated. Let us start with $\mathcal{B}$ again. It is bounded by the sum of

$$\begin{aligned}
C\delta^2 \int_1^{t-1} \frac{e^{-\frac{1}{2}k^2(t-s)}}{t-s} \int_{\mathbb{R}} \Big(e^{-\frac{1}{2}|x-\xi|} + e^{-ck|x|-\frac{1}{4}|\xi|} + \frac{1}{\sqrt{t-s}} e^{-ck|x-\xi|}\Big)\cdot \\
\cdot \frac{1}{s^{\frac{d-1}{3}}}(\omega(\xi) + k_s\omega(k_s\xi))\omega^n(k^3 s)\, \mathrm{d}\xi \mathrm{d}s
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
C\delta^2 \int_1^{t-1} e^{-\frac{1}{4}k^3(t-s)} \int_{\mathbb{R}} \Big(k^2 e^{-\frac{1}{2}|x-\xi|} + k^3 e^{-ck|x|-\frac{1}{4}|\xi|} + k^4 e^{-ck|x-\xi|}\Big)\cdot \\
\cdot \frac{1}{s^{\frac{d-1}{3}}}(\omega(\xi) + k_s\omega(k_s\xi))\omega^n(k^3 s)\, \mathrm{d}\xi \mathrm{d}s.
\end{aligned} \tag{6.14}$$

The first integral is rather like the $k < 1/\sqrt{t}$ case. Use Lemma 6.2,

$$e^{-\frac{1}{2}k^2(t-s)}\omega^n(k^3 s) \le C\omega^n(k^3 t)$$

and assume again $d \leq 4$:

$$(6.13) \leq C\delta^2\omega^n(k^3t)\int_1^{t-1}\frac{1}{s^{\frac{d-1}{3}}(t-s)}\Big(\omega(x)+\omega(kx) + \frac{1}{k\sqrt{t-s}}(\omega(x)+k_s\omega(k_sx))\Big)\,\mathrm{d}s$$
$$\leq C\delta^2\omega^n(k^3t)\Big(\frac{\log t}{t^{\frac{d-1}{3}}}(\omega(x)+\omega(k_tx)) + \frac{1}{t^{\frac{d-1}{3}}}(\sqrt{t}\omega(x)+\omega(k_tx))\Big)$$

(for $d > 4$ replace $\frac{d-1}{3}$ by 1). Again this only works when $d \geq 3$, in which case we get

$$(6.13) \leq C\delta^2\omega^n(k^3t)\frac{\log t}{t^{1/6}}(\omega(x)+k_t\omega(k_tx)).$$

Then the other term:

$$(6.14) \leq C\delta^2\int_1^{t-1}e^{-\frac{1}{4}k^3(t-s)}\frac{\omega^n(k^3s)}{s^{\frac{d-1}{3}}}\Big(k^2\omega(x)+k^3\omega(kx)+k^2k_s\omega(k_sx)\Big)\,\mathrm{d}s$$
$$\leq C\delta^2\omega^n(k^3t)\int_1^{t-1}\frac{1}{s^{\frac{d-1}{3}}(t-s)^{\frac{2}{3}}}(\omega(x)+k_s\omega(k_tx))\,\mathrm{d}s$$
$$\leq C\delta^2\omega^n(k^3t)\frac{1}{t^{1/6}}(\omega(x)+k_t\omega(k_tx))$$

when $d \geq 3$. Here we estimated

$$e^{-\frac{1}{4}k^3(t-s)}k^2\omega^n(k^3s) \leq \frac{C\omega^n(k^3t)}{(t-s)^{2/3}}.$$

Next we estimate $\mathcal{A}$ when $k \in (1/\sqrt{t}, \epsilon)$. We split it into two parts as we did with $\mathcal{B}$: by Theorem 5.1 and (6.12) $\mathcal{A}$ is bounded by the sum of

$$C\delta^2e^{-|x|}\int_1^{t-1}e^{-\frac{1}{2}k^2(t-s)}\int_{\mathbb{R}}\Big(\frac{1}{t-s}e^{-ck|\xi|}+\frac{1}{\sqrt{t-s}}e^{-|\xi|}\Big)\cdot \frac{\omega^n(k^3s)}{s^q}(\omega(\xi)+k_s\omega(k_s\xi))\,\mathrm{d}\xi\mathrm{d}s \quad (6.15)$$

and

$$C\delta^2e^{-|x|}\int_1^{t-1}e^{-\frac{1}{4}k^3(t-s)}\int_{\mathbb{R}}\Big(k^2e^{-ck|\xi|}+ke^{-|\xi|}\Big)\cdot \frac{\omega^n(k^3s)}{s^q}(\omega(\xi)+k_s\omega(k_s\xi))\,\mathrm{d}\xi\mathrm{d}s \quad (6.16)$$

where $q = \frac{d-1}{3} + \frac{1}{12}$. The first term is bounded by

$$(6.15) \leq C\delta^2 \omega(x)\omega^n(k^3 t) \int_1^{t-1} \frac{\mathrm{d}s}{s^q \sqrt{t-s}} \leq \frac{C\delta^2}{t^{1/6}} \omega(x)\omega^n(k^3 t)$$

when $d \geq 3$. With the second term we again need to exchange a k for $1/(t-s)^{1/3}$:

$$(6.16) \leq C\delta^2 \omega(x)\omega^n(k^3 t) \int_1^{t-1} \frac{\mathrm{d}s}{s^q (t-s)^{1/3}} \leq \frac{C\delta^2}{t^{1/12}} \omega(x)\omega^n(k^3 t)$$

when $d \geq 3$.

The differences $\mathcal{B}(t, \tilde{\nu}) - \mathcal{B}(t, \nu)$ and $\mathcal{A}(t, \tilde{\nu}) - \mathcal{A}(t, \nu)$ are again treated by using (6.11) instead of (6.10) or (6.12), which results in estimates similar to the above, with δ^2 replaced by $\delta\tilde{\delta}$.

When $k > \epsilon$ we use Theorem 5.3 and (6.10):

$$\begin{aligned}
&\left| \int_1^{t-1} K(t-s) D_k N(s, \nu(s))\, \mathrm{d}s \right| \\
&\qquad \leq C\delta^2 \int_1^{t-1} e^{-\frac{1}{2}k^4(t-s)} \frac{\omega^n(k^3 s)}{s^{\frac{d-1}{3}}} \int_{\mathbb{R}} e^{-c|x-\xi|} (\omega(\xi) + k_s \omega(k_s \xi))\, \mathrm{d}\xi \mathrm{d}s \\
&\qquad \leq C\delta^2 \omega^n(k^3 t) \int_1^{t-1} \frac{e^{-\frac{1}{4}k^4(t-s)}}{s^{\frac{d-1}{3}}} (\omega(x) + k_s \omega(k_s x))\, \mathrm{d}s \\
&\qquad \leq \frac{C}{t^{\frac{d-1}{3}}} \delta^2 \omega^n(k^3 t)(\omega(x) + k_t \omega(k_t x))
\end{aligned}$$

and similarly for the difference $\mathcal{F}(\tilde{\nu}) - \mathcal{F}(\nu)$.

Finally, when $t \leq 2$ we estimate like we did with $\mathcal{C}$ earlier. Thus we have established:

Lemma 6.8. *Assume $\|\eta_0\| < \delta$ for a small enough δ. Then $\mathcal{F} : \bar{B}_\delta \to \bar{B}_\delta$, given by* (6.8) *is a contraction in the norm* (6.9). *$\bar{B}_\delta := \{\nu : \|\nu\| \leq \delta\}$.*

6.5 Wrapping it up

When $\|\eta_0\|_X < \delta$ for a small enough δ it follows from the contraction mapping principle that (6.8) has a unique solution ν satisfying $\|\nu(t)\|_t \leq \delta/t^{1/12}$. Rolling back our variable changes we find that this is equivalent to (1.5) having a unique solution for which $\hat{\eta} = u + \nu$, provided we get from $t = 0$ to $t = 1$ somehow. Theorem 6.3 takes care of that and also guarantees sufficient differentiability for η to solve (1.4). By Lemma 6.6 $\|u_1\|_t \leq C\delta/t^{1/3}$, thus

$\nu_1 = \nu + u_1$ is small in the t norm. Eventually we see that (1.1) has a unique solution

$$\psi = \psi_0 + \check{u}_0 + \check{\nu}_1$$

where $\check{}$ was used to denote the inverse transform of (1.7). From the t norm estimate it follows that $|\check{\nu}_1(t, \underline{x})| < C\delta / t^{\frac{d+1}{3} + \frac{1}{12}}$. By Lemma 6.6

$$\check{u}_0(t, \underline{x}) = \frac{A}{2} \partial_x \psi_0(x) \phi(t, \vec{x})$$

where $A = \int_{\mathbb{R}^d} \eta(1, \underline{x}) \, \mathrm{d}\underline{x} = \int_{\mathbb{R}^d} \eta_0(\underline{x}) \, \mathrm{d}\underline{x}$, as the integral is preserved by (1.4). Thus we have proved Theorem 1.1.

Appendix A

Another integral

The following integral is needed in Chapter 5:

$$
\begin{aligned}
-\int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \frac{2\tau}{k^2+\tau^2} e^{-\tau^2 t+\mathrm{i}\tau|\xi|} \frac{\mathrm{d}\tau}{2\pi\mathrm{i}} \\
= \int_{-\infty}^{\infty} \frac{2\mathrm{i}(\tau+\mathrm{i}\alpha)}{k^2+(\tau+\mathrm{i}\alpha)^2} e^{-(\tau+\mathrm{i}\alpha)^2 t+\mathrm{i}\tau|\xi|-\alpha|\xi|} \frac{\mathrm{d}\tau}{2\pi}. \quad \text{(A.1)}
\end{aligned}
$$

The second line looks like the inverse Fourier transform of a product, so let us try to transform the factors separately.

$$
\int_{-\infty}^{\infty} \frac{2\mathrm{i}(\tau+\mathrm{i}\alpha)}{k^2+(\tau+\mathrm{i}\alpha)^2} e^{\mathrm{i}\tau\xi} \frac{\mathrm{d}\tau}{2\pi} = \int_{-\infty+\mathrm{i}\alpha}^{\infty+\mathrm{i}\alpha} \frac{-2\tau}{k^2+\tau^2} e^{\mathrm{i}\tau\xi} \frac{\mathrm{d}\tau}{2\pi\mathrm{i}} e^{\alpha\xi}. \quad \text{(A.2)}
$$

The trick here is to notice that the integration path can be raised or lowered arbitrarily as long as any residues are taken into account. The exponential factor can thus be used to make the integral small as long as $\xi \neq 0$. If $\xi = 0$ the integral diverges but has a principal value, which is zero when $\alpha = 0$ due to the integrand being odd.

There are two cases, which interest us: $\alpha > k$ and $\alpha = 0$. In the first case the integration goes above both poles $\pm\mathrm{i}k$ and we get

$$
\text{(A.2)} = \begin{cases} 0 & \text{when } \xi > 0, \\ 2\cosh(k\xi)e^{\alpha\xi} & \text{when } \xi < 0. \end{cases}
$$

If $\alpha = 0$ we integrate between the poles to get

$$
\text{(A.2)} = \begin{cases} -e^{-k\xi} & \text{when } \xi > 0, \\ e^{k\xi} & \text{when } \xi < 0. \end{cases}
$$

The other factor is familiar enough:

$$\int_{-\infty}^{\infty} e^{-(\tau+\mathrm{i}\alpha)^2 t+\mathrm{i}\tau\xi}\,\frac{\mathrm{d}\tau}{2\pi} = \frac{1}{\sqrt{4\pi t}} e^{\frac{-\xi^2}{4t}+\alpha\xi}.$$

Putting all this together

$$(\mathrm{A.1}) = \begin{cases} \int_{-\infty}^{0} \frac{2}{\sqrt{4\pi t}} e^{-\frac{(|\xi|-\eta)^2}{4t}} \cosh(k\eta)\,\mathrm{d}\eta & \text{when } \alpha > k, \\ \int_{-\infty}^{\infty} \frac{-1}{\sqrt{4\pi t}} e^{-\frac{(|\xi|-\eta)^2}{4t}} \operatorname{sgn}(\eta) e^{-k|\eta|}\,\mathrm{d}\eta & \text{when } \alpha = 0. \end{cases}$$

These convolutions can be computed rather explicitly:

$$(\mathrm{A.1}) = \begin{cases} e^{k^2 t}\big(f(k\sqrt{t}, \frac{|\xi|}{2\sqrt{t}}) + f(-k\sqrt{t}, \frac{|\xi|}{2\sqrt{t}})\big) & \text{when } \alpha > k, \\ e^{k^2 t}\big(f(k\sqrt{t}, \frac{|\xi|}{2\sqrt{t}}) - f(k\sqrt{t}, -\frac{|\xi|}{2\sqrt{t}})\big) & \text{when } \alpha = 0, \end{cases}$$

where

$$f(x,y) := e^{2xy}\,\mathrm{z}(x+y), \tag{A.3a}$$

$$\mathrm{z}(x) := \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} e^{-r^2}\,\mathrm{d}r. \tag{A.3b}$$

Of course the difference between the $\alpha > k$ case and the $\alpha = 0$ case is only the residue $e^{k^2 t - k|\xi|}$ at $\mathrm{i}k$, which can also be seen using $f(-x,y) = e^{-2xy} - f(x,-y)$. We also have these estimates:

$$|\partial_y^{n+1} f(x,y) - (2x)^{n+1} f(x,y)| < C_n(1 + x^n + |y|^n) e^{-x^2-y^2} \tag{A.4a}$$

and when $x > 0$

$$|f(x,y)| \le e^{-x^2-\min\{|xy|,y^2\}} \le e^{-\frac{3}{4}x^2-|xy|}, \tag{A.4b}$$

$$|f(x,y) - f(x,-y)| < C(1+x)|y|e^{-x^2}. \tag{A.4c}$$

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