Homoclinic Splitting without Trees

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ABSTRACT. We study a Hamiltonian describing a pendulum coupled with several anisochronous oscillators, giving a simple construction of unstable KAM tori and their stable and unstable manifolds for analytic perturbations.

When the coupling takes place through an even trigonometric polynomial in the angle variables, we extend analytically the solutions of the equations of motion, order by order in the perturbation parameter, to a large neighbourhood of the real line representing time. Subsequently, we devise an asymptotic expansion for the splitting (matrix) associated with a homoclinic point. This expansion consists of contributions that are manifestly exponentially small in the limit of vanishing gravity, by a shift-of-contour argument. Hence, we infer a similar upper bound for the splitting itself.

In particular, the derivation of the result does not call for a tree expansion with explicit cancellation mechanisms.

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In memoriam Pertti Lounesto — an inspiring teacher

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Preface

THIS thesis is the result of a research project that first started as an attempt to answer whether one could express the homoclinic splitting matrix in terms of "proper", or convergent, integrals of analytic functions over the real line, alone. The exponential smallness with respect to a forcing parameter could then be inferred from a shift-of-contour argument.

There pre-existed an impressive body of literature—mostly due to Gallavotti and his coworkers—resorting to regularized, "improper", integrals in overcoming convergence problems. The latter are caused by the asymptotically quasiperiodic behaviour of the homoclinic trajectory. In brief, my goal was to circumvent these KAM-type resonances, at least to some extent, in bounding the splitting.

Already in the second nontrivial order I ran into formidable analytical difficulties. Even if some progress was being made, it always seemed to be the result of technical trickery too case-specific to yield any argument applicable in general. After a good two years, I had to admit defeat, but was clueless as how to continue.

Accepting improper integrals as part of my formalism, advances began accumulating; expressed in terms of tree diagrams, certain terms of the perturbation expansion (also known as the Lindstedt series) could be grouped into expressions proportional to exponentially small integrals. This, of course, was little more than a special case of what had already been done by Gallavotti, *et al.* Still, the big picture somehow seemed buried under the multitude of trees.

Within a relatively short period of time, I realized that the key property in producing exponential smallness was the factorization of the aforementioned integrals, which is somehow related to Gallavotti's fruits. More importantly, I could encode this property into an asymptotic expansion of the splitting matrix, manifestly proportional to exponentially small quantities. In other words, I managed avoiding the earlier involved perturbative computations. Hence the title, Homoclinic Splitting without Trees, of the thesis.

Acknowledgements

Over the years, I have had many opportunities absorbing the knowledge of experts such as Giovanni Gallavotti, Guido Gentile, Vassili Gelfreich, Pierre Lochak, Mischa Rudnev, and Michela Procesi, to all of whom I now wish to express my gratitude.

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It is hard to imagine how I could have coped without the guidance of Antti Kupiainen. On several occasions I was able to rely on his technical skills and advance in my research after having got stuck on my own. He also spotted mistakes—sometimes silly, sometimes more serious—that I probably would have missed, thus setting me back on the right track.

Sincere thanks belong to my friends, who suitably distracted me from mathematics. My mother Sirkku and Jussi, father Antero and Elli, as well as my sisters Anna and Henna, were always there for me when needed. It is my pleasure to thank them all.

Annaflickan, jag älskar dig! Du har varit mitt största stöd under den senaste tiden.

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Mikko Stenlund
May 10, 2006

Part 1

Introductory Part

Chapter 1

Main Concepts and Results

We consider the Hamiltonian

$$\mathcal{H}(I, \phi, A, \psi) = \tfrac{1}{2}I^2 + g^2 \cos\phi + \tfrac{1}{2}A^2 - \lambda f(\phi, \psi) \tag{1.1}$$

of a pendulum coupled to d rotators, with $\phi \in \mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ and $I \in \mathbb{R}$ the coordinate and momentum of the pendulum, and $\psi \in \mathbb{T}^d := (\mathbb{S}^1)^d$ and $A \in \mathbb{R}^d$ the angles and actions of the rotators, respectively. The perturbation f is assumed to be real-valued and real-analytic in its arguments, and λ is a (small) real number, whereas the gravitational coupling constant g is taken to be positive. This Hamiltonian is sometimes called *the generalized Arnold model*.

The equations of motion are

$$\dot{\phi} = I, \quad \dot{\psi} = A, \quad \dot{I} = g^2 \sin\phi + \lambda\, \partial_\phi f, \quad \dot{A} = \lambda\, \partial_\psi f. \tag{1.2}$$

For the parameter value $\lambda = 0$, which is addressed as the unperturbed case, the pendulum and the rotators decouple. The former then has the separatrix flow $\phi : \mathbb{R} \to \mathbb{S}^1$ given by

$$\phi(t) = \Phi^0(e^{gt}),$$

where

$$\Phi^0(z) = 4 \arctan z.$$

By elementary trigonometry, this function possesses the symmetry property

$$\Phi^0(z) = 2\pi - \Phi^0(z^{-1}). \tag{1.3}$$

The phase space of the unperturbed pendulum looks as in Figure 1, where the separatrix—given by Φ^0—separates closed trajectories (libration) from open ones (rotation).

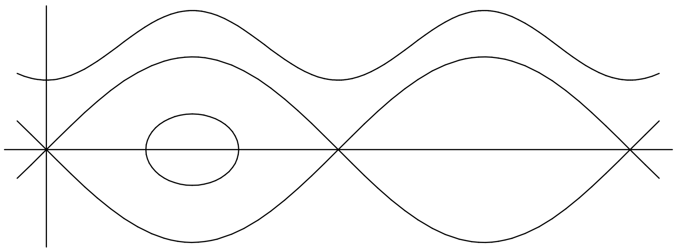


Figure 1. A (ϕ, I) plot showing the unperturbed pendulum separatrix that intersects the ϕ axis at integer multiples of 2π—the upright position of the pendulum.

On the other hand, $\psi : \mathbb{R} \to \mathbb{T}^d$ is quasiperiodic:

$$\psi(t) = \psi(0) + \omega t \pmod{2\pi},$$

such that the vector

$$\omega := A(0) \equiv A(t)$$

satisfies the Diophantine condition

$$|\omega \cdot q| > a\,|q|^{-\nu} \quad \text{for} \quad q \in \mathbb{Z}^d,\ q \neq 0, \tag{1.4}$$

with a and ν positive. Thus, at the instability point of the pendulum, the flow possesses the invariant tori

$$\mathcal{T}_0 := \Big\{ (\phi, \psi, I, A) = (0, \theta, 0, \omega) \;\Big|\; \theta \in T^d \Big\}$$

indexed by ω, with stable and unstable manifolds ($\mathcal{W}_0^s$ and $\mathcal{W}_0^u$, respectively) coinciding:

$$\mathcal{W}_0^{s,u} = \Big\{ (\phi, \psi, I, A) = \big(\Phi^0(z), \theta, gz\partial_z \Phi^0(z), \omega\big) \;\Big|\; z \in [0, \infty],\ \theta \in T^d \Big\}. \tag{1.5}$$

Remark 1.1. The constant g is the Lyapunov exponent for the unstable fixed point of the pendulum motion; in the limit $s \to -\infty$ two nearby initial angles $\phi(s)$ and $\phi(s + \delta s)$ separate at the exponential rate e^{gs}. As $\phi(t) = \Phi^0(e^{t/g^{-1}})$, the Lyapunov exponent fixes a natural time scale of g^{-1} units, characteristic of the pendulum motion in the unperturbed Hamiltonian system (1.1).

When the perturbation is switched on ($\lambda \neq 0$), our objective is to show that some of the invariant tori survive and have stable and unstable manifolds—or "whiskers" as Arnold has called them—that may not coincide anymore. We wish to prove bounds on their splitting.

1.1. A short interlude for literature

The study of "separatrix splitting" mentioned in the paragraph above is a subject with a long history, dating back to the Poincaré's classic *Les Méthodes nouvelles de la Mécanique céleste* [Poi93]. We refer the reader to the paper [GGM99] by Gallavotti, Gentile, and Mastropietro for a set of key references and interesting discussion.

In [GGM99], a three time scales problem arising in celestial mechanics is solved in that paper. There the pendulum is quasiperiodically and rapidly forced by *two* rotators having *totally different time scales* compared to that of the pendulum; one of the forcing frequencies tends to zero and the other to infinity. This has recently been simplified and generalized to a case of several rotators by Procesi, in [Pro03].

Rudiments of the powerful field theory techniques, adopted and developed by Gallavotti *et al.* in their works on the splitting problem, are well explained in [Gal98]. Here the influence of Eliasson's seminal work [Eli96] on the Lindstedt series should not be overlooked; it has to be considered the motivational impetus to introducing Feynman-like graphs for analyzing perturbation series in this line of science.

While the very general paper [RW98] by Rudnev and Wiggins contains an error in a crucial estimate, thus invalidating most of their proofs, a large part of it is certainly worth reading. In particular, readers valuing a pedagogical treatment will think highly of it. An erratum, reference [RW00], was later filed based on further work. The point in the latter, rather geometrical article, seems to be that *in suitable coordinates* the Fourier coefficients of the splitting obey so-called "quasiflat" estimates which directly lead to exponential smallness.

Moreover, Gelfreich gave an excellent introductory account on the splitting of separatrices at the XIII International Congress on Mathematical Physics (ICMP 2000) in London, [Gel01]. The exposition is accessible to the nonexpert, and we recommend it as a starting point to anyone intending to enter or familiarize oneself with the field. From there one should advance to [GL01], which covers more topics with more details.

The very recent and extensive memoir [LMS03] by Lochak, Marco, and Sauzin is written from the geometric point of view, adding more content to the concepts studied by the analysts. It also has a historical flavor to it, making it interesting and accessible to virtually anyone.

1.2. Main theorems

Our approach will be to construct the perturbed manifolds in a form similar to (1.5) as graphs of analytic functions over $[0, \infty] \times \mathbb{T}^d$. To see how this can be achieved, note that the unperturbed stable and unstable manifolds, $\mathcal{W}_0^s$ and $\mathcal{W}_0^u$, consist of trajectories

$$(\phi(t), \psi(t)) = (\Phi^0(e^{gt}), \omega t)$$

that at time $\pm\infty$ become quasiperiodic, as they wrap tighter and tighter around the invariant torus $\mathcal{T}_0$; indeed $(\phi(t), \psi(t)) \sim (0, \omega t)$ in the limit $t \to \pm\infty$.

Analogously, we will find the stable and unstable manifolds of the perturbed tori by looking for solutions of the form

$$(\phi(t), \psi(t)) = (\Phi(e^{\gamma t}, \omega t), \omega t + \Psi(e^{\gamma t}, \omega t)) = (0, \omega t) + (\Phi, \Psi)(e^{\gamma t}, \omega t) \tag{1.6}$$

with quasiperiodic behavior in *one* of the two limits $t \to \pm\infty$. In effect, we have to find functions $X : [0, \infty] \times \mathbb{T}^d \to \mathbb{R}^{d+1}$ specified by

$$X := (\Phi, \Psi),$$

and real-analytic in its variables $z \in [0, \infty]$ and $\theta \in \mathbb{T}^d$. Note especially that we anticipate the exponential rate ("Lyapunov exponent") $\gamma > 0$ to depend on λ, with $\gamma|_{\lambda=0} = g$.

Remark 1.2. One should not assume asymptotic quasiperiodicity in both of the limits $t \to \pm\infty$, as the unstable and stable manifolds, which we denote $\mathcal{W}_\lambda^u$ and $\mathcal{W}_\lambda^s$, are generically expected to depart for nonzero values of the perturbation parameter λ. Therefore, either the past *or* future asymptotic of a trajectory will evolve so as to ultimately reach the (deformed) invariant torus $\mathcal{T}_\lambda$. The separatrix in Figure 1 is thus transformed into something like the pair of curves in Figure 2.

Let us denote the total derivative d/dt by ∂_t and the complete angular gradient $(\partial_\phi, \partial_\psi)$ by ∂ for short. Substituting (1.6) into the equations of motion

$$\partial_t^2(\phi, \psi) = (\dot{I}, \dot{A}) = (g^2 \sin\phi, 0) + \lambda\,\partial f(\phi, \psi),$$

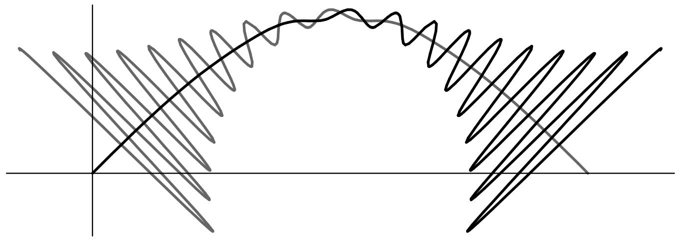


Figure 2. A typical (Poincaré section) illustration of what happens to the pendulum separatrix under perturbation. The origin has been shifted for convenience.

we get

$$(\omega\cdot\partial_\theta+\gamma e^{\gamma t}\partial_z)^2 X(e^{\gamma t},\omega t)=[(g^2\sin\Phi,0)+\lambda\,\partial f(X+(0,\theta))](e^{\gamma t},\omega t),$$

where θ stands for the canonical projection $[0,\infty]\times\mathbb{T}^d\to\mathbb{T}^d$.

Define the *z-dependent* partial differential operator

$$\mathcal{L}:=\omega\cdot\partial_\theta+\gamma z\partial_z,$$

and notice the characteristic identity

$$\mathcal{L}F(ze^{\gamma t},\theta+\omega t)=\partial_t F(ze^{\gamma t},\theta+\omega t) \tag{1.7}$$

for a differentiable map $(z,\theta)\mapsto F(z,\theta)$. Equation (1.7) simply reflects the time derivative nature of $\mathcal{L}$. In fact, if T is the "time-reversal map"

$$T(z,\theta)\equiv(z^{-1},-\theta), \tag{1.8}$$

then, by the chain rule,

$$\mathcal{L}(F\circ T)=-(\mathcal{L}F)\circ T. \tag{1.9}$$

Let us abbreviate

$$\Omega(X):=(g^2\sin\Phi,0)+\lambda\,\widetilde{\Omega}(X)\quad\text{with}\quad\widetilde{\Omega}(X):=\partial f(X+(0,\theta)). \tag{1.10}$$

As a consequence, we have reduced the equations of motion to the PDE

$$\mathcal{L}^2X=\Omega(X) \tag{1.11}$$

for the map $(z,\theta)\mapsto X(z,\theta)$ in a suitable Banach space of analytic functions, albeit its restriction to the set ("characteristic")

$$\big\{(z,\theta)=(e^{\gamma t},\omega t)\;\big|\;t\in\mathbb{R}\big\} \tag{1.12}$$

is what one is physically interested in. Our preference of working directly with the invariant manifolds, as opposed to individual trajectories traversing along them, motivates us encoding the time derivative in the operator $\mathcal{L}$. Nevertheless, it will be harmless—and indeed quite informative—for the reader to keep in mind that the objects we deal with originate from (1.12) and therefore have a direct physical interpretation.

Below, a key point will be introducing small *imaginary parts* to the (independent) arguments z and θ of X. This well-known method, corresponding to extending a dynamical system to "imaginary time", allows one to use the invincible armory of complex function theory.

The action variables trivially follow from the knowledge of $X(z,\theta)$:

$$(I(t), A(t)) = (0, \omega) + Y(e^{\gamma t}, \omega t), \quad Y := \mathcal{L}X.$$

The solutions X will provide a parametrization of the deformed tori and their stable and unstable manifolds. We find two kinds of solutions, $X^u(z,\theta)$ defined for $z \in [0, z_0] =: I^u$ and $X^s(z,\theta)$ defined for $z \in [z_0^{-1}, \infty] =: I^s$. Here, $z_0 > 1$. The tori will have the two parametrizations

$$\begin{aligned} \mathcal{T}_\lambda &= \left\{(\phi,\psi,I,A) = ((0,\theta) + X^u(0,\theta), (0,\omega) + Y^u(0,\theta)) \;\middle|\; \theta \in T^d\right\} \\ &= \left\{(\phi,\psi,I,A) = ((0,\theta) + X^s(\infty,\theta), (0,\omega) + Y^s(\infty,\theta)) \;\middle|\; \theta \in T^d\right\}, \end{aligned}$$

with stable and unstable manifolds

$$\mathcal{W}_\lambda^{s,u} = \left\{(\phi,\psi,I,A) = ((0,\theta) + X^{s,u}(z,\theta), (0,\omega) + Y^{s,u}(z,\theta)) \;\middle|\; z \in I^{s,u},\ \theta \in T^d\right\}.$$

In order to enable solving (1.11), we need to deal with quantities of the form $(\omega \cdot q)^{-1}$, $q \in \mathbb{Z}^d \setminus \{0\}$, stemming from the Fourier representation of the operator $\mathcal{L}$. Here the Diophantine property of the vector $\omega \in \mathbb{R}^d$ stated in (1.4) steps in. Since $\omega \equiv A|_{\lambda=0} = \dot{\psi}|_{\lambda=0}$, by rescaling time (and the actions, correspondingly) in the equations of motion (1.2), the constant a can be absorbed into g^2 and λ in the equations of motion, leaving the ratio λg^{-2} unchanged: $(g, \lambda) \mapsto (g/a, \lambda/a^2)$ [1]. Thus, we may as well take a to be 1 below, transforming the condition on ω into

$$|\omega \cdot q| > |q|^{-\nu} \quad \text{for} \quad q \in \mathbb{Z}^d \setminus \{0\}. \tag{1.13}$$

We will moreover consider λ small in a g-dependent fashion, taking

$$\epsilon := \lambda g^{-2} \tag{1.14}$$

small. This should be seen as an outreach towards the experimenter, albeit there is a technical wherefore: eventually we wish to study the limit $g \to 0$, which calls for such a choice. The domain we restrict ourselves to is given by

$$D := \left\{(\epsilon, g) \in \mathbb{C} \times \mathbb{R} \;\middle|\; |\epsilon| < \epsilon_0,\ 0 < g < g_0\right\}, \tag{1.15}$$

[1] This scaling is responsible for the usual requirement $\lambda = \mathcal{O}(a^2)$ for KAM tori.

for some positive values of ϵ_0 and g_0.

Finally, note that if $X = (\Phi, \Psi)$ solves (1.11) on some domain $D' \subset [0, \infty] \times \mathbb{T}^d$, then so does

$$X_{\alpha,\beta}(z, \theta) := X(\alpha z, \theta + \beta) + (0, \beta), \tag{1.16}$$

as long as $(\alpha z, \theta+\beta \pmod{2\pi}) \in D'$. The aforementioned invariance is a manifestation of the freedom of choosing initial conditions for (ϕ, ψ)—we may choose the origin of time and the configuration of the physical system there.

For $\epsilon = 0$, the solutions are obtained from

$$X^0(z, \theta) := (\Phi^0(z), 0) \tag{1.17}$$

using (1.16). In particular, $X^0(1, 0) = (\pi, 0)$. This will provide us with a natural way of fixing α and β below.

The first part of the present work is dedicated to the existence and analyticity properties of the invariant manifolds, and boils down to the statement that follows.

Theorem 1 (Whiskered tori). *Let f be real-analytic and ω satisfy the Diophantine condition* (1.13)*, and fix $g_0 > 0$. Then there exist a positive number ϵ_0 and a function $\gamma(\epsilon, g)$ on D, analytic in ϵ with $|\gamma - g| < Cg|\epsilon|$, such that equation* (1.11) *has a solution X^u which is analytic in ϵ as well as in (z, θ) in a neighbourhood of $[0, 1] \times \mathbb{T}^d$ and which satisfies*

$$X^u(1, 0) = (\pi, 0), \quad X^u(z, \theta) = X^0(z) + \mathcal{O}(\epsilon). \tag{1.18}$$

Similarly, there exists a solution $X^s(z, \theta) = X^0(z) + \mathcal{O}(\epsilon)$ which is an analytic function of $(z^{-1}, -\theta)$ in a neighbourhood of $[0, 1] \times \mathbb{T}^d$. The maps

$$W^{s,u}(z, \theta) = (X^{s,u}, Y^{s,u})(z, \theta) + ((0, \theta), (0, \omega)), \quad Y^{s,u} := \mathcal{L}X^{s,u}, \tag{1.19}$$

provide an analytic parametrization of the stable and unstable manifolds $\mathcal{W}_\lambda^{s,u}$ of the torus $\mathcal{T}_\lambda$.

Remark 1.3. The neighbourhoods above depend on the Diophantine exponent ν and the analyticity domain of f.

Remark 1.4. A uniqueness statement is not provided in Theorem 1; we do not prove the uniqueness of γ, although our construction does specify a single one. See Remark 3.6 below Theorem 3.4.

The second part of the project is more involved. Its content is summarized in the following theorem that discusses the splitting of the stable and unstable manifolds under perturbation. Of central interest in this context is the function

$$W^u - W^s = (X^u - X^s, Y^u - Y^s),$$

which describes the separation between a point on the manifold $\mathcal{W}_\lambda^u$ and its counterpart on $\mathcal{W}_\lambda^s$, for given values of the parameters z and θ.

For convenience, let us now assume that the perturbation is even:

$$f(\phi, \psi) = f(-\phi, -\psi).$$

Namely, if $(z, \theta) \mapsto X(z, \theta)$ solves equation (1.11), then, according to (1.9), so does $(z, \theta) \mapsto (2\pi, 0) - (X \circ T)(z, \theta)$. Consequently, by a simple time-reversal consideration (set $t \mapsto -t$ in (1.12)), the stable and unstable manifolds are related through

$$X^s = (2\pi, 0) - X^u \circ T. \tag{1.20}$$

In particular, as $T(1, 0) = (1, 0)$,

$$X^s(1, 0) = X^u(1, 0).$$

Moreover, the actions $Y^{s,u} = \mathcal{L}X^{s,u}$ satisfy

$$Y^s = Y^u \circ T, \tag{1.21}$$

yielding

$$Y^s(1, 0) = Y^u(1, 0).$$

In other words, a *homoclinic intersection* of the stable and the unstable manifolds $\mathcal{W}_\lambda^{s,u}$ occurs at $(z, \theta) = (1, 0)$, as their parametrizations (1.19) coincide at this *homoclinic point*.

Remark 1.5. Equation (1.20) is what remains of the symmetry $X^0 = (2\pi, 0) - X^0 \circ T$, which is just another way of writing (1.3), after the onset of perturbation. This is an instance of *spontaneous symmetry breaking*: The equations of motion, (1.11), remain unchanged under the transformation $X \mapsto (2\pi, 0) - X \circ T$, but the individual solutions do not respect this symmetry; $X^u \neq X^s = (2\pi, 0) - X^u \circ T$, if $\lambda \neq 0$.

In order to study the intersection more closely, say the relative attitudes of the manifolds there, we express the actions $Y^{s,u}$ as functions of the original angle variables $(\phi, \psi) = X^{s,u}(z, \theta) + (0, \theta)$ appearing in the Hamiltonian (1.1). To this end, let $F^{s,u} : (z, \theta) \mapsto (\phi, \psi)$ be the functions that mediate the above coordinate transformations, and write $Y^{s,u} = \bar{Y}^{s,u} \circ F^{s,u}$. At the same time we distinguish

$$\partial := (\partial_\phi, \partial_\psi) \quad \text{and} \quad D := (\partial_z, \partial_\theta)$$

for clarity. By the chain rule for Jacobian matrices,

$$DY^{s,u} = \left(\partial\bar{Y}^{s,u} \circ F^{s,u}\right) DF^{s,u}. \tag{1.22}$$

The invertibility of the matrix $DF^{s,u}$ is a consequence of Theorem 1. Indeed, knowing (1.18) and $X^0(z) = (4\arctan z, 0)$, it follows that

$$DF^{s,u}(z,\theta) = \begin{pmatrix} 4(1+z^2)^{-1} & 0 \\ 0 & \mathbb{1}_{d\times d} \end{pmatrix} + \mathcal{O}(\epsilon), \tag{1.23}$$

which renders the maps $F = F^{s,u}$ honest coordinate transformations if ϵ is small. As a matter of fact, block matrix inversion yields the explicit formula

$$(DF^{s,u})^{-1} = \begin{pmatrix} A^{-1} & -(\partial_z\Phi)^{-1}\partial_\theta\Phi\,\mathbf{B}^{-1} \\ -(\mathbb{1}+\partial_\theta\Psi)^{-1}\partial_z\Psi\,A^{-1} & \mathbf{B}^{-1} \end{pmatrix},$$

dropping some superscripts for notational reasons and given

$$A := \partial_z\Phi - \partial_\theta\Phi\,(\mathbb{1}+\partial_\theta\Psi)^{-1}\partial_z\Psi \quad \text{and} \quad \mathbf{B} := \mathbb{1}+\partial_\theta\Psi - \partial_z\Psi\,(\partial_z\Phi)^{-1}\partial_\theta\Phi.$$

Here Ψ is to be considered a column and $\partial_\theta\Phi$ a row vector, such that in $\mathbf{B}$ there appears a direct product whose components read $(\partial_z\Psi\,\partial_\theta\Phi)_{ij} = (\partial_z\Psi)_i\,(\partial_\theta\Phi)_j$.

At the homoclinic point, equation (1.20) implies that

$$F^{s,u}(1,0) = (\pi, 0) \quad \text{and} \quad DF^s(1,0) = DF^u(1,0) \tag{1.24}$$

hold. Casting in the obvious manner $\bar{Y}^{u,s} = (\bar{Y}^{u,s}_\Phi, \bar{Y}^{u,s}_\Psi)$, there are two natural objects related to the homoclinic point—namely, the *splitting matrices*:

$$\Upsilon := \partial_\theta(Y^u_\Psi - Y^s_\Psi)\,(1,0) \quad \text{and} \quad \widetilde{\Upsilon} := \partial_\psi\left(\bar{Y}^u_\Psi - \bar{Y}^s_\Psi\right)(\pi,0). \tag{1.25}$$

With the aid of (1.22), one has

$$\partial\left(\bar{Y}^u - \bar{Y}^s\right)(\pi,0) = \left[D(Y^u - Y^s)\ (DF^u)^{-1}\right](1,0).$$

Therefore, the matrices in (1.25) are related by

$$\widetilde{\Upsilon} = \Upsilon\,\mathbf{B}^{-1}(1,0) + uv^T, \tag{1.26}$$

the last term being the direct product of $u := \partial_z(Y^u_\Psi - Y^s_\Psi)(1,0)$ and $v := ((DF^u)^{-1}_{\Phi\Psi})^T(1,0) = -((\partial_z\Phi)^{-1}\partial_\theta\Phi\,\mathbf{B}^{-1})^T(1,0)$.

As $\mathbf{B} = 1 + \mathcal{O}(\epsilon)$, what (1.26) tells us is that $\widetilde{\Upsilon}$ and Υ differ by a close-to-identity transformation plus a rank-one correction. Hence, we infer the following:

PROPOSITION 1.6. *Suppose either* Υ *or* $\widetilde{\Upsilon}$ *is invertible with an inverse of* $\mathcal{O}(\epsilon^{-1})$. *Then, for small* ϵ, *also the other matrix is invertible, and*

$$\det \widetilde{\Upsilon} = (1 + \mathcal{O}(\epsilon)) \det \Upsilon.$$

Proof. If M is an invertible matrix and $\tilde{u} := M^{-1}u$,

$$\det (M + uv^T) = \det M \det (\mathbb{1} + \tilde{u}v^T) = (1 + v^T M^{-1} u) \det M,$$

where the last equality follows from the trace–log formula and using the identity $\operatorname{tr} (\tilde{u}v^T) = v^T \tilde{u}$ with the Taylor expansion of $\log(\mathbb{1} + \tilde{u}v^T)$. Last, in (1.26), both u and v are $\mathcal{O}(\epsilon)$, such that $v^T M^{-1} u = \mathcal{O}(\epsilon)$. □

In the sense of Proposition 1.6, studying Υ and $\widetilde{\Upsilon}$ are equivalent tasks. We choose Υ due to reasons to become clear later on, while other authors—see in particular [GGM99]—have favored $\widetilde{\Upsilon}$. Without going into the details, let us mention here, however, that it seems that the perturbation expansion of $\widetilde{\Upsilon}$ with respect to ϵ, for which purpose the Poincaré section $\{(\phi, \psi) = (0, \psi)\}$ is the natural one, results in somewhat tidier expressions than those due to Υ on the section $\{(z, \theta) = (1, \theta)\}$.

THEOREM 2 (Homoclinic splitting). *Denote* $\Delta = Y^u_\Psi - Y^s_\Psi$, *such that the splitting matrix reads* $\Upsilon = \partial_\theta \Delta(1, 0)$. *If* $\psi \mapsto f(\,\cdot\,, \psi)$ *is a trigonometric polynomial, then for each* $t \in \mathbb{R}$ *there exist positive constants* C, c, *and* p, *such that the exponentially small upper bound*

$$|\partial_\theta \Delta(e^{\gamma t}, \omega t)| \leq C |\epsilon| g^{-p} e^{-c g^{-1/(\nu+1)}},$$

where ν *is the Diophantine exponent, holds.*

Why study the $d \times d$ matrix Υ instead of the full Jacobi matrix $D(Y^u - Y^s)(1, 0)$ when measuring *transversality* of the homoclinic intersection $\mathcal{W}^s_\lambda \cap \mathcal{W}^u_\lambda$? Because the latter is singular. In order to see this, first notice that defining

$$\bar{f}(\phi, \psi) \equiv g^2 \cos \phi - \lambda\, f(\phi, \psi) \quad \text{and} \quad G(I, A) \equiv \tfrac{1}{2} I^2 + \tfrac{1}{2} A^2$$

allows us to write the restriction of the Hamiltonian to the unstable manifold as

$$\mathcal{H}^u(z, \theta) \equiv (\bar{f} \circ F^u + G \circ \mathcal{L} F^u)(z, \theta).$$

Moreover, the constancy of energy along trajectories, $\mathcal{L}\mathcal{H} = 0$, implies

$$\mathcal{H}^u(z, \theta) \equiv \mathcal{H}^u(z e^{\gamma t}, \theta + \omega t) \sim \mathcal{H}^u(0, \theta + \omega t) \equiv \mathcal{H}^u(0, 0) \qquad (t \to -\infty),$$

where the last equality follows from nonresonance of ω. Hence,

$$0 = D\mathcal{H}^u = (\partial G \circ \mathcal{L}F^u) D(\mathcal{L}F^u) + (\partial \bar{f} \circ F^u) DF^u.$$

Since $D(\mathcal{L}F^u) = DY^u$, subtracting the corresponding identity for $\mathcal{H}^s$ and recalling (1.24), we have

$$\left[D(Y^u - Y^s)(1,0)\right]^T V = 0, \qquad V := Y^{s,u}(1,0) + (0,\omega)^T,$$

which simply means that the splitting vanishes in the direction of the homoclinic trajectory. The author is grateful to Dr Mischa Rudnev for pointing this out. For further motivation, see item R1 of Appendix R and Section 8 in [GGM99].

Part 2

Invariant Manifolds as Analytic Graphs

Chapter 2

Perturbed Tori

Due to (1.20), we may concentrate on studying the unstable manifold, as the considerations presented below hold for the stable one by a straightforward change of notation. Throughout this chapter, and the ones that follow, the symbol C will stand for a generic constant that may vary from one place to another.

It turns out that solving (1.11) is easy except for $X(0,\theta)$ and $\partial_z X(0,\theta)$, *i.e.*, the invariant torus and the linearization of the unstable manifold around it. Namely, these problems involve the notorious small denominators of the Kolmogorov–Arnold–Moser (KAM) Theory.

The perturbed tori will be found by looking for solutions having the general form

$$\phi(t) = \Phi_0(\omega t), \quad \psi(t) = \omega t + \Psi_0(\omega t),$$

with $\Phi_0 : \mathbb{T}^d \to \mathbb{R}$ and $\Psi_0 : \mathbb{T}^d \to \mathbb{R}^d$ satisfying the "$t \to -\infty$ asymptotics"

$$\mathcal{D}^2\Phi_0(\theta) = g^2 \sin \Phi_0(\theta) + \lambda\, \partial_\phi f(\Phi_0(\theta), \theta + \Psi_0(\theta)) \tag{2.1}$$

$$\mathcal{D}^2\Psi_0(\theta) = \lambda\, \partial_\psi f(\Phi_0(\theta), \theta + \Psi_0(\theta)) \tag{2.2}$$

obtained from equation (1.11) by putting $z = 0$ and $\mathcal{D} = \omega \cdot \partial_\theta$. Note that if $X_0 = (\Phi_0, \Psi_0)$ is a solution to equations (2.1) and (2.2), then so is

$$\sigma_\beta X_0(\theta) := (\Phi_0(\theta + \beta), \Psi_0(\theta + \beta) + \beta)$$

for $\beta \in \mathbb{T}^d$.

This paragraph tries to briefly convince the reader of the (reasonable!) idea that studying the *asymptotics of a solution* to the general problem (1.11) naturally reduces to studying a *solution of the asymptotic problem* (2.1)–(2.2). Imagine for a while that we could solve the general equation (1.11) on $[0, z_0] \times \mathbb{T}^d$. Then, by the scaling invariance with respect to the variable z spelled out in (1.16), the function $X_{0,0}(z, \theta) \equiv X(0, \theta)$ will also be a solution. But plugging this "past time asymptotic" of X into (1.11), the equation itself reduces to its asymptotic form.

2.1. Spaces of analytic functions

Let us define the spaces we shall be working in. As linear subspaces of ℓ^1, the Banach spaces

$$\mathcal{B}^{\Phi}_{\sigma} := \Big\{ \Phi : \mathbb{T}^d \to \mathbb{C} \;\Big|\; \|\Phi\|_{\sigma} := \sum_{q \in \mathbb{Z}^d} |\hat{\Phi}(q)| e^{\sigma |q|} < \infty \Big\},$$

$$\mathcal{B}^{\Psi}_{\sigma} := \Big\{ \Psi : \mathbb{T}^d \to \mathbb{C}^d \;\Big|\; \|\Psi\|_{\sigma} := \sum_{q \in \mathbb{Z}^d} |\hat{\Psi}(q)| e^{\sigma |q|} < \infty \Big\},$$

for any $\sigma \geq 0$, have the advantage that Fourier analysis on their elements is convenient. Furthermore, we are trying to find a solution $X = (\Phi, \Psi)$ analytic on the torus, and, for a suitably small σ, such a function belongs to $\mathcal{B}^{\Phi}_{\sigma} \times \mathcal{B}^{\Psi}_{\sigma}$ because of the exponential decay of its Fourier coefficients; $|\hat{X}(q)| < Ce^{-\sigma|q|}$ with some positive constant C. Indeed, if $\sigma > 0$, the spaces above comprise precisely those functions on the torus that admit an analytic extension to the "strip" $|\Im\mathfrak{m}\, \theta| < \sigma$. We will occasionally write $\mathcal{B}_{\sigma}$ when referring to both $\mathcal{B}^{\Phi}_{\sigma}$ and $\mathcal{B}^{\Psi}_{\sigma}$, or when it makes no particular difference which of these two is in question.

Example 2.1 (Analytic extension of f). Consider the perturbation $(\phi, \psi) \mapsto f(\phi, \psi)$ in the Hamiltonian $\mathcal{H}$ given in (1.1). It is analytic on the compact set $\mathbb{S}^1 \times \mathbb{T}^d$, and by Abel's Lemma (multivariate power series converge on polydisks), it extends to an analytic map on a "strip" $|\Im\mathfrak{m}\, \phi|, |\Im\mathfrak{m}\, \psi| \leq \eta$ $(\eta > 0)$ around $\mathbb{S}^1 \times \mathbb{T}^d$.

Of course, as our analysis proceeds, f will appear all over the place. This, in turn, dictates the analyticity properties of a plethora of maps, in practice introducing the constraint $\sigma \leq \eta$ for the spaces $\mathcal{B}_{\sigma}$.

Notice the natural embeddings

$$\mathcal{B}_{\sigma+\alpha} \subset \mathcal{B}_{\sigma},$$

for $\alpha \geq 0$, due to the inequality

$$\|\cdot\|_{\sigma} \leq \|\cdot\|_{\sigma+\alpha}. \tag{2.3}$$

Consider the linear operator $\tau_\beta : \mathcal{B}_{\sigma+\alpha} \to \mathcal{B}_\sigma$ defined through setting $\widehat{\tau_\beta X}(q) = e^{iq\cdot\beta}\hat{X}(q)$, with $\beta \in \mathbb{C}^d$. It the imaginary part of β is small, τ_β is well-defined and bounded. Namely,

$$\|\tau_\beta X\|_\sigma = \sum_{q\in\mathbb{Z}^d} |\widehat{\tau_\beta X}(q)| e^{\sigma|q|} \leq \sum_{q\in\mathbb{Z}^d} e^{(|\Im\mathfrak{m}\,\beta|-\alpha)|q|} |\hat{X}(q)| e^{(\sigma+\alpha)|q|} \leq \|X\|_{\sigma+\alpha}$$

whenever $|\Im\mathfrak{m}\,\beta| \leq \alpha$, meaning $\|\tau_\beta\|_{\mathcal{L}(\mathcal{B}_{\sigma+\alpha};\mathcal{B}_\sigma)} \leq 1$. The realization of τ_β in terms of the variable θ is, of course, just the translation $\Psi(\theta) \mapsto \Psi(\theta+\beta)$. τ_β will serve as a useful device in encoding the real-analyticity of f as an algebraic property into the Fourier series of certain other functions. This is due to the the fact that exponential smallness of $|\widehat{X}(q)|$ in q implies real-analyticity of a function X on the torus, and vice versa.

We shall encounter n-linear maps from $\mathbb{C}^{d+1}$ into $\mathbb{C}$. Endowed with the norm

$$\|A\|_{\mathcal{L}(^n(\mathbb{C}^{d+1});\mathbb{C})} := \inf\left\{ M \geq 0 \;\middle|\; |A(z_1,\ldots,z_n)| \leq M|z_1|\ldots|z_n| \quad \forall\, z_i \in \mathbb{C}^{d+1} \right\}$$

they form the Banach space $\mathcal{L}(^n(\mathbb{C}^{d+1});\mathbb{C})$; see [Cha85].

2.2. Past and future asymptotics of the solution in the perturbed case

This section discusses the $t \to \pm\infty$ asymptotics of the general solution X. In these limits the motion settles onto the "distorted version" $\mathcal{T}_\lambda$ of the invariant torus $\mathcal{T}_0$ with the pendulum seizing to swing, but wiggling quasiperiodically about its unstable equilibrium.

THEOREM 2.2. *Under the assumptions of Theorem 1, there exist positive numbers r and ϵ_0 such that, for $(\epsilon, g) \in D$, equations* (2.1) *and* (2.2) *have a unique solution $X_0 = (\Phi_0, \Psi_0)$ in the class of those real-analytic functions of $\theta \in \mathbb{T}^d$ that satisfy $\|\Psi_0\|_{\ell^1} < r$ and $\langle\Psi_0\rangle = 0$ (zero average). The function X_0, defined on $\{|\Im\mathfrak{m}\,\theta| \leq \sigma\}\times D$ for some $\sigma > 0$, is analytic and uniformly bounded by $(C|\epsilon|, Cg^2|\epsilon|)$. Moreover, it is $\mathbb{R}\times\mathbb{R}^d$-valued on $\mathbb{T}^d$ for ϵ real. Any solution $X_0' = (\Phi_0', \Psi_0')$ with $\langle\Psi_0'\rangle = \beta \in \mathbb{R}^d$ and $\|\Psi_0' - \beta\|_{\ell^1} < r$ must be the one given by*

$$X_0'(\theta) \equiv X(\theta+\beta) + (0,\beta).$$

Remark 2.3. Remark 1.3 below Theorem 1 holds true. Recall that we have defined $\epsilon := \lambda g^{-2}$ in (1.14) and the domain D in (1.15). This is a version of the KAM Theorem. Notice that $X_0 \in \mathcal{B}_\sigma^\Phi \times \mathcal{B}_\sigma^\Psi$.

Proof. The proof is a reduction to the one given in [BGK99]. Here we systematically omit the subindex 0 of Φ_0, Ψ_0, and X_0. Let us concentrate on the pendulum part, equation (2.1), first. We expect Φ to be close to its unperturbed value, zero, and it pays to cancel the leading term of $g^2 \sin \Phi(\theta)$ on the right-hand side by subtracting $g^2\Phi(\theta)$ from both sides. We then have

$$(\mathcal{D}^2 - g^2)\Phi = U(\Phi, \Psi) =: U(X) \tag{2.4}$$

with

$$U(X)(\theta) := g^2(\sin \Phi(\theta) - \Phi(\theta)) + \lambda\, \partial_\phi f(\Phi(\theta), \theta + \Psi(\theta)). \tag{2.5}$$

Pay attention to the fact that $U(X)(\theta)$ depends locally on X—only through $X(\theta)$, that is. Abusing notation, we shall use $U(X)(\theta)$, $U(X, \theta)$, $U(X(\theta), \theta)$, *etc.*, in the same meaning, whichever is the most convenient form. Now, $U(\chi, \theta)$ is analytic in the vector argument $\chi = (\chi_\phi, \chi_\psi)$ in the region $|\chi_\phi|, |\chi_\psi| \leq \eta$, where $\eta > 0$ depends on the analyticity domain of f. The reader may consult Example 2.1 on page 18 for the definition of the number η.

Let us now write down the Fourier–Taylor expansion

$$\begin{aligned} U(X(\theta), \theta) &= \sum_{n=0}^{\infty} \frac{1}{n!} D^n U(0, \theta)\,(X(\theta), \ldots, X(\theta)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{q}=(q_1,\ldots,q_n) \\ q_i \in \mathbb{Z}^d}} e^{i\theta\cdot\sum_i q_i}\, D^n U(0, \theta)\,(\hat{X}(q_1), \ldots, \hat{X}(q_n)), \end{aligned} \tag{2.6}$$

where $D^n U(0, \theta) \in \mathcal{L}(^n(\mathbb{C}^{d+1}); \mathbb{C})$ is the nth Fréchet derivative of the map $U(\,\cdot\,, \theta) : \mathbb{C}^{d+1} \to \mathbb{C} : \chi \mapsto U(\chi, \theta)$.

The map $\theta \mapsto U_n(\theta) := \frac{1}{n!} D^n U(0, \theta)$ is analytic in the same domain as $\theta \mapsto U(0, \theta) = \lambda\, \partial_\phi f(0, \theta)$, *i.e.*, $|\Im\mathfrak{m}\, \theta| \leq \eta$. In particular, it is square-integrable—

$$\int_{\mathbb{T}^d} \|U_n(\theta)\|^2_{\mathcal{L}(^n(\mathbb{C}^{d+1});\mathbb{C})}\, d\theta < \infty$$

—and thus admits the Fourier representation $U_n(\theta) = \sum_{q\in\mathbb{Z}^d} e^{iq\cdot\theta} u_n(q)$ with coefficients

$$u_n(q) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-iq\cdot\theta}\, \frac{1}{n!} D^n U(0, \theta)\, d\theta \tag{2.7}$$

in $\mathcal{L}({}^n(\mathbb{C}^{d+1});\mathbb{C})$. Using this notation, we translate (2.6) into the Fourier language;

$$\hat{U}(\hat{X},q) := \widehat{U(X)}(q) = \sum_{n=0}^{\infty} \sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} u_n(q - \sum_{i=1}^{n} q_i)\,(\hat{X}(q_1),\ldots,\hat{X}(q_n)) \tag{2.8}$$

The representation of $\hat{U}(\hat{X},q)$ in equation (2.8) is a power series in $\hat{X}$, converging whenever $\hat{X}$ is sufficiently close to zero. Namely, we have

Lemma 2.4. *The multilinear maps $u_n(q)$ obey the bound*

$$\|u_n(q)\|_{\mathcal{L}({}^n(\mathbb{C}^{d+1});\mathbb{C})} \le Cg^2(r_0^3 + |\epsilon|)r_0^{-n}e^{-\rho|q|}, \tag{2.9}$$

where ρ and r_0 are any positive numbers satisfying $\rho + r_0 \le \eta$, $\eta > 0$ being the width of the analyticity domain of f in Example 2.1.

The proof of Lemma 2.4 is straightforward, but, for the sake of continuity, is given in Section 2.3 below.

Considering the closed origin-centered balls of radius $r < r_0/2$ in $\mathcal{B}_\sigma^\Phi$ and $\mathcal{B}_\sigma^\Psi$—$\bar{B}_{\sigma,r}^\Phi$ and $\bar{B}_{\sigma,r}^\Psi$, respectively—we next study $U_\beta : \bar{B}_{\sigma,r}^\Phi \times \bar{B}_{\sigma,r}^\Psi \to \mathcal{B}_\sigma^\Phi : (\Phi,\Psi) \mapsto \tau_\beta U(\tau_{-\beta}\Phi, \tau_{-\beta}\Psi)$. By equation (2.5),

$$U_\beta(\Phi(\theta),\Psi(\theta),\theta) = U(\Phi(\theta),\theta+\beta+\Psi(\theta)), \tag{2.10}$$

when $\beta \in \mathbb{R}^d$. The right-hand side is analytic in β, and extends to $|\mathfrak{Im}\,\beta| + \sigma + r < \eta$ through the same expression, leaving U_β analytic with respect to X.

More quantitatively, one checks using the bound (2.9) of Lemma 2.4 that the power series

$$\begin{aligned}\hat{U}_\beta(\hat{X})(q) &:= \tau_\beta \widehat{U(\tau_{-\beta}}X)(q) = e^{i\beta\cdot q}\,\hat{U}(\widehat{\tau_{-\beta}X},q)\\ &= \sum_{n=0}^{\infty}\sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} e^{i\beta\cdot(q-\sum_i q_i)}\,u_n(q-\sum_{i=0}^{n} q_i)\,(\hat{X}(q_1),\ldots,\hat{X}(q_n)),\end{aligned} \tag{2.11}$$

converges uniformly with respect to X and β, even if the latter has a small imaginary part. We now prove all this, remarking that the sole purpose of equation (2.10) was to clarify that, by introducing an imaginary β into $U_\beta(X)$, we are not force-feeding an imaginary argument to X, owing to the locality $U(X,\theta) = U(X(\theta),\theta)$ and contrary to what the expression $U_\beta(X) = \tau_\beta U(\tau_{-\beta}X)$ might at first sight suggest. $\|U_\beta(X)\|_\sigma$ obeys the upper bound

$$\sum_{n=0}^{\infty} \sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} \sum_{q\in\mathbb{Z}^d} e^{\sigma|q|} \Big| e^{i\beta\cdot(q-\sum_i q_i)}\, u_n(q - \sum_{i=0}^{n} q_i)\,(\hat{X}(q_1), \ldots, \hat{X}(q_n)) \Big|$$

$$\leq \sum_{n=0}^{\infty} \sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} \sum_{q\in\mathbb{Z}^d} e^{\sigma|q|} e^{|\Im\mathfrak{m}\,\beta||q-\sum_i q_i|} \Big\| u_n(q - \sum_{i=0}^{n} q_i) \Big\| \prod_{i=1}^{n} |\hat{X}(q_i)| e^{(\sigma-\sigma)|q_i|}$$

$$\leq \sum_{n=0}^{\infty} \sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} \sum_{q\in\mathbb{Z}^d} e^{(|\Im\mathfrak{m}\,\beta|+\sigma)|q-\sum_i q_i|} \Big\| u_n(q - \sum_{i=0}^{n} q_i) \Big\| \prod_{i=1}^{n} |\hat{X}(q_i)| e^{\sigma|q_i|}$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{q\in\mathbb{Z}^d} e^{(|\Im\mathfrak{m}\,\beta|+\sigma)|q|} \|u_n(q)\| \right) \sum_{\mathbf{q}\in(\mathbb{Z}^d)^n} \prod_{i=1}^{n} |\hat{X}(q_i)| e^{\sigma|q_i|}$$

$$\leq Cg^2(r_0^3 + |\epsilon|) \sum_{n=0}^{\infty} \sum_{q\in\mathbb{Z}^d} e^{(|\Im\mathfrak{m}\,\beta|+\sigma-\rho)|q|}\, r_0^{-n}\, \|X\|_\sigma^n \leq Cg^2(r_0^3 + |\epsilon|),$$

if we choose $|\Im\mathfrak{m}\,\beta| + \sigma < \rho$ and $r < r_0/2$, since

$$\|X\|_\sigma = \sum_{q\in\mathbb{Z}^d} \left(|\hat{\Phi}(q)|^2 + |\hat{\Psi}(q)|^2 \right)^{1/2} e^{\sigma|q|} \leq 2r.$$

Thus, fixing $r = r_0/3$, say, we obtain

$$\sup_{X\in\bar{B}^{\Phi}_{\sigma,r}\times\bar{B}^{\Psi}_{\sigma,r}} \|U_\beta(X)\|_\sigma \leq Cg^2(r^3 + |\epsilon|) \tag{2.12}$$

whenever

$$|\Im\mathfrak{m}\,\beta| + \sigma < \rho \leq \eta - 3r. \tag{2.13}$$

LEMMA 2.5. *Suppose* (2.13) *holds, and* $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$. *Then, for* r *and* ϵ_0 *small enough,*

$$(\mathcal{D}^2 - g^2)\Phi = U_\beta(\Phi, \Psi)$$

has a solution $\Phi_\beta(\Psi) \in \bar{B}^{\Phi}_{\sigma,r}$, *real-valued provided* β, ϵ, *and* Ψ *are, and there are no other solutions in the* ℓ^1*-ball* $\bar{B}^{\Phi}_{0,r} \supset \bar{B}^{\Phi}_{\sigma,r}$. *In fact,* $\Phi_\beta(\Psi) = \tau_\beta\Phi_0(\tau_{-\beta}\Psi)$. *The map* $\Psi \mapsto \Phi_\beta(\Psi)$ *is analytic on* $\bar{B}^{\Psi}_{\sigma,r}$. $\Phi_\beta(\Psi)$ *also depends analytically on* β *as well as on* $(\epsilon, g) \in D$ *(see* (1.15)*), and obeys the bound*

$$\|\Phi_\beta(\Psi)\|_\sigma \leq C|\epsilon| \tag{2.14}$$

uniformly in Ψ, β, *and* g.

Remark 2.6. The smallness condition is $C(r^3 + \epsilon_0) \leq r$, where C is the same constant as in (2.12) and contains the norm of the perturbation f.

The standard but lengthy proof of Lemma 2.5 may be found in Section 2.3.

Let us come back to equation (2.2), whose right-hand side may now be written solely in terms of $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$, amounting to

$$\mathcal{D}^2\Psi = V(\Psi) \tag{2.15}$$

with $V(\Psi)(\theta) \equiv \lambda\,\partial_\psi f(\Phi(\Psi)(\theta), \theta + \Psi(\theta))$. Consider then $V_\beta(\Psi) := \tau_\beta V(\tau_{-\beta}\Psi)$. By Lemma 2.5, it reads

$$V_\beta(\Psi)(\theta) \equiv V(\tau_{-\beta}\Psi)(\theta+\beta) \equiv \lambda\,\partial_\psi f(\Phi_\beta(\Psi)(\theta), \theta+\beta+\Psi)$$

and is analytic in the domain

$$\bar{B}^{\Psi}_{\sigma,r} \times D \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\} \times \{|\Im\mathfrak{m}\,\beta| < \rho - \sigma\} \tag{2.16}$$

with the uniform bound

$$\|V_\beta(\Psi)\|_\sigma \le \sup_{|\Im\mathfrak{m}\,\phi|,|\Im\mathfrak{m}\,\psi| \le \eta} |\lambda\,\partial_\psi f(\phi,\psi)| \le Cg^2|\epsilon|,$$

provided $C|\epsilon| \le \eta$ (see (2.14)).

Equation (2.15) is the variational equation corresponding to the action functional

$$S : \bar{B}^{\Psi}_{\sigma,r} \to \mathbb{R} : \Psi \mapsto S(\Psi) = \int_{\mathbb{T}^d} s(\Psi,\theta)\,d\theta$$

given by the integrand

$$s(\Psi,\theta) = \tfrac{1}{2}(\Phi\mathcal{D}^2\Phi + \Psi\cdot\mathcal{D}^2\Psi) + g^2\cos\Phi - \lambda f(\Phi, \theta+\Psi),$$

where $\Phi = \Phi(\Psi)$. S is invariant under the $\mathbb{T}^d$-action $\Psi(\theta) \mapsto \Psi_\beta(\theta) := \Psi(\theta+\beta)+\beta$, $\beta \in \mathbb{R}^d$. Hence, $\partial_\beta S(\Psi_\beta)|_{\beta=0} = 0$ yields the *Ward identity*

$$\int_{\mathbb{T}^d} \frac{\delta S(\Psi)}{\delta\Psi^i(\theta)}\,d\theta = \int_{\mathbb{T}^d} \Psi(\theta)\cdot\partial_{\theta^i}\frac{\delta S(\Psi)}{\delta\Psi(\theta)}\,d\theta \quad (i = 1,\ldots,d) \tag{2.17}$$

of the symmetry in the functional derivative notation. In fact,

$$\frac{\delta S(\Psi)}{\delta\Psi(\theta)} = (\mathcal{D}^2\Psi - V(\Psi))(\theta).$$

Integrating by parts three times one sees that

$$\int_{\mathbb{T}^d} \Psi(\theta)\cdot\partial_{\theta^i}\mathcal{D}^2\Psi(\theta)\,d\theta = -\int_{\mathbb{T}^d} \Psi(\theta)\cdot\partial_{\theta^i}\mathcal{D}^2\Psi(\theta)\,d\theta = 0.$$

The general identity (2.17) therefore reduces to the identity

$$\int_{\mathbb{T}^d} V^i(\Psi,\theta)\,d\theta = \int_{\mathbb{T}^d} \Psi(\theta)\cdot\partial_{\theta^i}V(\Psi,\theta)\,d\theta \tag{2.18}$$

for the map V.

In conclusion, we have the KAM-type small denominator problem (2.15) with $V_\beta(\Psi, \theta)$ analytic in the domain (2.16) and bounded there by $C|\lambda|$, together with the Ward identity (2.18) stemming from a translation symmetry of the action that generates the equation. Furthermore, $V_\beta(\Psi, \theta)$ is real-valued whenever β, ϵ, and Ψ are. For $0 < \sigma < \rho$—so that we may choose $\mathfrak{Im}\,\beta \neq 0$—this is precisely the setup in [BGK99], where the authors devise a method for dealing with such problems using a Renormalization approach.

The subtle analysis in [BGK99] yields a unique solution $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$ to (2.15) with zero average and analytic in $(\epsilon, g) \in D$. The inevitable loss of analyticity takes place in the domain of β. The map $\theta \mapsto \Psi(\theta)$ is $\mathbb{R}^d$-valued on the torus for real ϵ and satisfies $\|\Psi\|_\sigma \leq C|\lambda| = Cg^2|\epsilon|$. Here it could appear strange to the reader that Ψ should vanish in the limit $g \to 0$. But it is not, since we had to impose that $\lambda g^{-2} = \epsilon$ is small in order to guarantee that $V(\Psi)$—with $\Phi(\Psi)$ of the order of ϵ in it—is well-defined. In other words, when g is small, the coupling constant λ is also small.

Denote by Ψ_n, $n \in \mathbb{Z}_+$, the unique solution to (2.15) in the ball $\bar{B}^{\Psi}_{\sigma/n,r}$. Since $\bar{B}^{\Psi}_{\sigma,r} \subset \bar{B}^{\Psi}_{\sigma/n,r}$, Ψ has to coincide with Ψ_n. Hence, Ψ is the unique solution in

$$\bigcup_{n=1}^{\infty} \bar{B}^{\Psi}_{\sigma/n,r} \supset \left\{ \Psi : \mathbb{T}^d \to \mathbb{R}^d \;\middle|\; \Psi \text{ real-analytic and } \|\Psi\|_{\ell^1} < r \right\}.$$

Indeed, assuming the map $\theta \mapsto \Psi(\theta)$ is real-analytic, $\|\Psi\|_{\sigma/n} < \infty$ for some n, and we have that $\|\Psi\|_{\sigma/n} \searrow \|\Psi\|_0 \equiv \|\Psi\|_{\ell^1}$ as $n \to \infty$. Thus, if $\|\Psi\|_{\ell^1} < r$, we gather that $\|\Psi\|_{\sigma/n} < r$ for sufficiently large values of n.

Finally, let us demonstrate the translation property. Suppose then that $(\Phi_\beta, \bar{\Psi}_\beta) \in \bar{B}^{\Phi}_{\sigma,r} \times \bar{B}^{\Psi}_{\sigma,r}$ and that $(\Phi_\beta, \Psi_\beta) := (\Phi_\beta, \bar{\Psi}_\beta + \beta)$ solves the system (2.1)-(2.2). Then $(\Phi, \Psi) := (\tau_{-\beta}\Phi_\beta, \tau_{-\beta}\bar{\Psi}_\beta) \in \bar{B}^{\Phi}_{\sigma,r} \times \bar{B}^{\Psi}_{\sigma,r}$ solves it as well, and we must have

$$(\Phi_\beta, \Psi_\beta) = (\tau_\beta \Phi, \tau_\beta \Psi + \beta)$$

by Lemma 2.5. □

2.3. Proofs of Lemmata 2.4 and 2.5

In order to season Chapter 2 with its fair share of technical flavor, we now set about to validate the accessories used in the proof of Theorem 2.2.

Proof of Lemma 2.4. Write $\|\cdot\| = \|\cdot\|_{\mathcal{L}(^n(\mathbb{C}^{d+1});\mathbb{C})}$ for short. From (2.7) and the Cauchy Integral Theorem,

$$\|u_n(q)\| = \Big\| \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-iq\cdot(\theta+i\beta)} \frac{1}{n!} D^n U(0, \theta + i\beta)\, d\theta \Big\|$$
$$\leq e^{q\cdot\beta} \frac{1}{n!} \sup_{\theta\in\mathbb{T}^d} \|D^n U(0, \theta + i\beta)\|,$$

for $\beta \in \mathbb{R}^d$ and $|\beta| \leq \eta$. Take $0 < \rho < \eta$ and choose $\beta = -\rho q/|q|$. We compute the standard norm of n-homogeneous polynomials,

$$\|D^n U(0, \theta + i\beta)\|_{\mathcal{P}^n(\mathbb{C}^{d+1};\mathbb{C})} := \sup_{|z|\leq 1} |D^n U(0, \theta + i\beta)\,(z, \ldots, z)|,$$

which, using the Cauchy Integral Formula, turns into

$$\sup_{|z|\leq 1} \Big| \frac{n!}{2\pi i} \oint_{\partial\mathbb{D}(0,r_0)} \frac{U(\zeta z, \theta + i\beta)\, d\zeta}{\zeta^{n+1}} \Big| \leq n!\, r_0^{-n} \sup_{|z|\leq 1} \sup_{\zeta\in\partial\mathbb{D}(0,r_0)} |U(\zeta z, \theta + i\beta)|.$$

Here $\mathbb{D}(0, r_0)$ is the origin-centered circle of radius r_0 in the complex plane, with the constraint $r_0 + \rho \leq \eta$. For $|z| \leq r$ and $|\Im\mathfrak{m}\, \theta| \leq \rho$ we estimate

$$|U(z, \theta)| \leq C g^2 (r_0^3 + |\epsilon|);$$

see equation (2.5). Here we have singled out $\lambda g^{-2} = \epsilon$, and C is independent of g. We stress that $U(z, \theta)$ simply stands for the expression obtained from the expression of $U(X, \theta)$ in (2.5) by replacing $X(\theta)$ by $z \in \mathbb{C}^{d+1}$.

Symmetric multilinear maps are fully determined by their diagonal—the corresponding homogeneous polynomial, that is—which is explicitly confirmed by the Polarization Formula stated and proven, *e.g.*, in the (horribly expensive) texts [Cha85, Din99]. Hence, in order to obtain the estimate in (2.9), we multiply the corresponding polynomial estimate by the factor $n^n/n!$, which, by Stirling's Formula, behaves asymptotically as $e^n/\sqrt{2\pi n}$ and may be absorbed into ρ and C. □

Proof of Lemma 2.5. The proof is a simple application of the Banach Fixed Point Theorem. We establish that, for a fixed $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$, the operator F given by the formula $F(\Phi) := (\mathcal{D}^2 - g^2)^{-1} U_\beta(\Phi, \Psi)$ maps $\bar{B}^{\Phi}_{\sigma,r}$ into itself and is a (strong) contraction there. The latter will hold, provided (2.13), when r and ϵ are taken small. The rest will follow from this.

We first observe that $(\mathcal{D}^2 - g^2)^{-1}$ is a linear operator bounded in norm by g^{-2}. Then, using the bound (2.12), compute

$$\|F(\Phi)\|_\sigma \leq g^{-2} \|U_\beta(\Phi, \Psi)\|_\sigma \leq C(r^3 + |\epsilon|) \leq r$$

for sufficiently small r and ϵ, which means that $F(\bar{B}^{\Phi}_{\sigma,r}) \subset \bar{B}^{\Phi}_{\sigma,r}$.

To prove contractiveness, take $\Phi_1, \Phi_2 \in \bar{B}^{\Phi}_{\sigma,r}$, and establish $\|F(\Phi_1) - F(\Phi_2)\|_\sigma \leq \mu \|\Phi_1 - \Phi_2\|_\sigma$ with $\mu = \mu(r, \epsilon) < 1$ for sufficiently small λ and r (*i.e.*, ρ). The actual computation resembles estimating the norm of U_β in the proof of Theorem 2.2, and is therefore omitted. The existence and uniqueness of the solution $\Phi(\Psi, \beta) \in \bar{B}^{\Phi}_{\sigma,r}$ now follow.

For β, ϵ, and Ψ real, F maps the closed subset of real-valued functions $\Phi \in \bar{B}^{\Phi}_{\sigma,r}$ into itself and is a contraction there, so $\Phi(\Psi, \beta)$ is real-valued by uniqueness.

The operator F depends analytically on the parameter Ψ in $\bar{B}^{\Psi}_{\sigma,r}$. Consider the sequence

$$0,\ F(0),\ F^2(0),\ F^3(0),\ \ldots$$

of successive substitutions. Each element $F^k(0)$ is analytic in $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$. Furthermore, the Banach Fixed Point Theorem guarantees that such a sequence converges to the fixed point $\Phi(\Psi, \beta)$ in geometric progression;

$$\|F^k(0) - \Phi(\Psi, \beta)\|_\sigma \leq \frac{\mu^n}{1-\mu} \|F(0)\|_\sigma < \frac{r\mu^n}{1-\mu}.$$

Consequently, $\Phi(\Psi, \beta)$ is the uniform limit of a sequence of analytic functions, and, as such, analytic itself. The same argument goes for $(\epsilon, g) \in D$ (see (1.15)), as well as for β in the domain specified by (2.13).

Let us denote $\Phi(\Psi) = \Phi(\Psi, 0)$. If $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$, then $\tau_{-\beta}\Psi \in \bar{B}^{\Psi}_{\sigma/2,r}$, such that $\Phi = \Phi(\tau_{-\beta}\Psi)$ is the unique element in $\bar{B}^{\Phi}_{\sigma/2,r}$ solving $\Phi = (\mathcal{D}^2 - g^2)^{-1} U(\Phi, \tau_{-\beta}\Psi)$. The diagonality of τ_β and $\mathcal{D}$ yields

$$\Phi_\beta(\Psi) = (\mathcal{D}^2 - g^2)^{-1} U_\beta(\Phi_\beta(\Psi), \Psi),$$

where $\Phi_\beta(\Psi) = \tau_\beta \Phi(\tau_{-\beta}\Psi) \in \bar{B}^{\Phi}_{0,r}$. But, in view of (2.3), the ball $\bar{B}^{\Psi}_{\sigma,r}$—containing Ψ—is a subset of $\bar{B}^{\Psi}_{0,r}$, and $\Phi = \Phi(\Psi, \beta)$ is the unique element in $\bar{B}^{\Phi}_{0,r}$ solving $\Phi = (\mathcal{D}^2 - g^2)^{-1} U_\beta(\Phi, \Psi)$. Thus, regarded as elements of $\bar{B}^{\Phi}_{0,r}$, the maps $\Phi(\Psi, \beta)$ and $\Phi_\beta(\Psi)$ coincide. On the other hand, $\Phi(\Psi, \beta) \in \bar{B}^{\Phi}_{\sigma,r}$, because $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$.

In conclusion, given $\Psi \in \bar{B}^{\Psi}_{\sigma,r}$, the map $\Phi = \Phi_\beta(\Psi) \in \bar{B}^{\Phi}_{\sigma,r}$ is the unique solution to the equation $\Phi = (\mathcal{D}^2 - g^2)^{-1} U_\beta(\Phi, \Psi)$ even in $\bar{B}^{\Phi}_{0,r} \supset \bar{B}^{\Phi}_{\sigma,r}$.

A priori, we know that $\|\Phi_\beta(\Psi)\|_\sigma = \|F(\Phi_\beta(\Psi))\|_\sigma \leq r$. On the other hand, we know that $\Phi_\beta(\Psi)|_{\epsilon=0} = 0$ by uniqueness, since $\Phi = 0$ solves the equation $\Phi = F(\Phi)|_{\epsilon=0} = (\mathcal{D}^2 - g^2)^{-1} g^2 (\sin\Phi - \Phi)$. Thus, $\Phi_\beta(\Psi)$ is of first order in $\epsilon = \lambda g^{-2}$ by analyticity, and uniformly bounded in Ψ and β, whence the estimate (2.14) follows. $\square$

Chapter 3

Lyapunov Exponent—Linearizing the Unstable Manifold

In this chapter we study the motion in the immediate vicinity of the invariant manifold ("torus") $\mathcal{T}_\lambda$ corresponding to the solution $X_0(\theta)$ of Theorem 2.2. To that end, suppose $X(z,\theta)$ is an analytic solution to equation (1.11) with $X(0,\theta) = X_0(\theta)$. Writing $X_1(\theta) := \partial_z X(0,\theta)$, the linearization X_1 should satisfy the equation

$$(\mathcal{D} + \gamma)^2 X_1 = D\Omega(X_0)X_1, \tag{3.1}$$

where $D\Omega(X_0)X_1$ is the Fréchet derivative of Ω at X_0, acting on X_1. This follows from operating by $\partial_z|_{z=0}$ on both sides of (1.11); the case of the left-hand side is straightforward elaboration, while that of the right-hand side is a consequence of the chain rule after observing that $\Omega(X)(z,\theta)$ depends on z only through X evaluated at (z,θ).

The trivial solution $X_1 \equiv 0$ to the linear equation (3.1) has physical relevance. Its existence merely reflects the fact that, once on it, the motion is "unwilling" to leave the invariant manifold $\mathcal{T}_\lambda$. In fact, if $X_1 = 0$, we must take $\Phi^0 = 0$ (instead of $4\arctan$) in order to stay on the separatrix of the pendulum, which—after some thought—amounts to $X = X_0$.

Note that (3.1) is a problem of "eigenvalue type"; recalling $\gamma|_{\epsilon=0} = g$, we will strive to choose $\gamma = \gamma(\epsilon, g)$ in a g-dependent neighbourhood, say

$$|\gamma - g| < g/2, \tag{3.2}$$

of its unperturbed value g, such that (3.1) has a nontrivial solution. That we succeed is the content of Theorem 3.4 below and indeed the main result of this chapter. Consequently, our γ will depend analytically on ϵ, nicely controlled by $|\gamma - g| < Cg|\epsilon|$.

The subtlety of proving Theorem 3.4 lies in solving a small denominator problem that emerges in the process. We go about dealing with this dilemma using a Renormalization Group method, treating such small denominators scale by scale. Here we follow the framework of [BGK99], albeit the technical setup below is very different—and fortunately much less complicated.

We wish to study the components of $X_1 = (\Phi_1, \Psi_1)$ separately on the left-hand side of (3.1). To do this, we cast the whole equation in matrix form by expressing the Fréchet derivative on the right-hand side in terms of the corresponding Jacobian matrix.

First, view the map $X \mapsto \Omega(X)$ as the map that takes (Φ, Ψ) to $(\Omega_\Phi(\Phi,\Psi), \Omega_\Psi(\Phi,\Psi))$ with the components $\Omega_\Phi(\Phi,\Psi) = g^2 \sin\Phi + \lambda\,\partial_\phi f(\Phi, \theta + \Psi)$ and $\Omega_\Psi(\Phi,\Psi) = \lambda\,\partial_\psi f(\Phi, \theta+\Psi)$. Then we split $D = (D_\Phi, D_\Psi)$, such that the component form of (3.1) reads

$$(\mathcal{D} + \gamma)^2 \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix} = \begin{pmatrix} g^2 \cos\Phi_0 + \lambda f_{\phi,\phi} & \lambda f_{\phi,\psi} \\ \lambda f_{\psi,\phi} & \lambda f_{\psi,\psi} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Psi_1 \end{pmatrix}. \tag{3.3}$$

In each entry, $f_{a,b}$ stands for the matrix $\lambda(\partial_b \partial_a f)(\Phi_0, \theta + \Psi_0)$.

Remark 3.1. The upper left corner of $D\Omega(X_0)$, expressing in (3.3) the "coupling" of the pendulum part Φ_1 to itself, dominates. Indeed, using the expansion $\cos\Phi_0 = 1 + \mathcal{O}(\Phi_0^2)$, the bound $\|\Phi_0\|_\sigma \le C|\epsilon|$ provided by Theorem 2.2, and the real-analyticity of f, we see that in

$$D\Omega(X_0) = \begin{pmatrix} D_\Phi\Omega_\Phi & D_\Psi\Omega_\Phi \\ D_\Phi\Omega_\Psi & D_\Psi\Omega_\Psi \end{pmatrix}(X_0) = \begin{pmatrix} g^2 & 0 \\ 0 & 0 \end{pmatrix} + g^2\epsilon A,$$

the matrix ϵA plays the role of a small perturbation.

From (3.3) we get for Ψ_1 the equation

$$\Psi_1 = \left[(\mathcal{D} + \gamma)^2 - \lambda f_{\psi,\psi}\right]^{-1} (\lambda f_{\psi,\phi}\Phi_1) =: J\Phi_1, \tag{3.4}$$

Here J is a well-defined bounded linear operator from $\mathcal{B}_\sigma^\Phi$ to $\mathcal{B}_\sigma^\Psi$, provided that ϵ_0 is small. Checking this is straightforward implementation of Neumann series and the fact that the operator $(\mathcal{D}+\gamma)^{-2}$ has the diagonal kernel

$$(\mathcal{D}+\gamma)^{-2}(p,q) = \delta_{p,q}(i\omega\cdot q+\gamma)^{-2}, \quad p,q\in\mathbb{Z}^d, \tag{3.5}$$

in the Fourier language. By going through the details, one obtains the bound

$$\|J\|_{\mathcal{L}(\mathcal{B}_\sigma^\Phi;\mathcal{B}_\sigma^\Psi)} \le C|\epsilon| \tag{3.6}$$

for the operator norm of J, after using (3.2).

Remark 3.2. The definition of J is an instance where demanding smallness of λg^{-2} is natural, indeed necessary, as was discussed in the context of introducing the scaled perturbation parameter $\epsilon = \lambda g^{-2}$ in (1.14).

Consequently, using (3.4), we get for Φ_1 the equation

$$[(\mathcal{D}+\gamma)^2 - g^2]\Phi_1 = g^2(\cos\Phi_0 - 1)\Phi_1 + \lambda f_{\phi,\phi}\Phi_1 + \lambda f_{\phi,\psi}J\Phi_1 =: H\Phi_1. \tag{3.7}$$

A word of motivation for subtracting $g^2\Phi_1$ from both sides above seems appropriate. $\Phi_0|_{\epsilon=0} = 0$ by Lemma 2.5. Therefore $H|_{\epsilon=0} = 0$, and $\Phi_1|_{\epsilon=0} = 4$ (recall that $\Phi^0(z) = 4\arctan z$) is a physically motivated nontrivial solution to (3.7). In other words, the differential operator $(\mathcal{D}+g)^2 - g^2$ is *singular*. On the other hand, when $\epsilon \ne 0$ is small, we know that Φ_0 remains close to zero, making the whole right-hand side in (3.7) small. We then hope to find a Lyapunov exponent γ, close to g, such that $(\mathcal{D}+\gamma)^2 - g^2 - H$ stays singular and the equation still admits a nontrivial solution close to the constant function 4.

It follows from (3.6) that the operator H appearing in (3.7), which lies in $\mathcal{L}(\mathcal{B}_\sigma^\Phi) \equiv \mathcal{L}(\mathcal{B}_\sigma^\Phi;\mathcal{B}_\sigma^\Phi)$, has the useful properties summarized below. The proof, which is hardly from The Book [1], comprises Section 3.1 and can be skipped by the reader without future damage.

LEMMA 3.3. *Denote the kernel of $H \in \mathcal{L}(\mathcal{B}_\sigma^\Phi)$ by $H(p,q)$, $(p,q) \in \mathbb{Z}^d\times\mathbb{Z}^d$. For $|\Im\mathfrak{m}\,\kappa| \le g/3$, there exists an operator $H(\kappa) \in \mathcal{L}(\mathcal{B}_\sigma^\Phi)$ related to H by*

$$(t_sH)(p,q) := H(p+s,q+s) = H(\omega\cdot s;p,q), \quad s\in\mathbb{Z}^d.$$

Let $0<\sigma'<\sigma$. The kernel $H(\kappa;p,q)$ is analytic on

$$\big\{(\kappa,\epsilon,g,\gamma)\ \big|\ |\Im\mathfrak{m}\,\kappa| \le g/3,\ (\epsilon,g)\in D,\ |\gamma-g|<g/2\big\}$$

[1] Paul Erdös spoke of The Book, in which God had written down the most elegant proofs for mathematical theorems. He himself doubted the existence of God, whom he called the Supreme Fascist and accused of hiding his socks, Hungarian passports, and the best equations. Liberally quoted from [Wik06].

and it satisfies the bound

$$|H(\kappa;p,q)| \le Cg^2|\epsilon|\, e^{-\sigma'|p-q|}$$

with $C = C(\sigma')$*. As for the* κ*-derivatives,*

$$|H^{(k)}(\kappa;p,q)| \le Ck!\,(g/3 - |\Im\mathfrak{m}\,\kappa|)^{-k} g^2|\epsilon|^2\, e^{-\sigma'|p-q|}, \quad k \ge 1.$$

Moreover,

$$\left|\frac{\partial}{\partial\gamma} H(0;0,0)\right| \le C|\epsilon|^2 g\, \frac{1}{1-2|\gamma-g|/g}.$$

3.1. Proof of Lemma 3.3

Proof. To simplify notations, we decompose

$$H = H_1 + H_2 \quad \text{with} \quad H_2 = \lambda f_{\phi,\psi} J.$$

Let Φ and Ψ be arbitrary functions in their "customary" function spaces $\mathcal{B}_\sigma^\Phi$ and $\mathcal{B}_\sigma^\Psi$, respectively.

H_1 acts as ordinary multiplication: $H_1\Phi(\theta) = H_1(\theta)\Phi(\theta)$ with $H_1(\theta) \in \mathbb{C}$. The map $\theta \mapsto H_1(\theta)$ is in $L^2(\mathbb{T}^d)$, due to its real-analyticity. Thus, Fourier transforms make perfect sense, and we write $\widehat{H}_1$ for the transform of the latter map. Denoting a kernel element of the operator H_1 by $H_1(p,q)$, we may write the identity

$$\sum_{q\in\mathbb{Z}^d} H_1(p,q)\hat{\Phi}(q) = \widehat{H_1\Phi}(p) = \sum_{q\in\mathbb{Z}^d} \widehat{H}_1(p-q)\hat{\Phi}(q), \quad p \in \mathbb{Z}^d,$$

or

$$H_1(p,q) \equiv \widehat{H}_1(p-q).$$

We gather that the all-important translation invariance

$$t_s H_1 = H_1 \tag{3.8}$$

holds, and that the kernel of H_1 satisfies

$$|H_1(p,q)| \le C|\lambda|\, e^{-\sigma|p-q|}, \quad p,q \in \mathbb{Z}^d.$$

Here $\sigma > 0$ is the width of the analyticity strip around the real $\mathbb{T}^d$ of the map $\theta \mapsto H_1(\theta)$, *i.e.*, of X_0. Since, by Theorem 2.2, X_0 is analytic with respect to $(\epsilon, g) \in D$, so is $H_1(p,q)$.

Observe that the expression defining J in (3.4) may be cast as

$$J\Phi = \left[\mathbb{1} - (\mathcal{D}+\gamma)^{-2}(\lambda f_{\psi,\psi})\right]^{-1} (\mathcal{D}+\gamma)^{-2}(\lambda f_{\psi,\phi}\Phi) = B\Lambda O\Phi,$$

where B, Λ, and O stand for $[\mathbb{1} - (\mathcal{D}+\gamma)^{-2}(\lambda f_{\psi,\psi})]^{-1}$, $(\mathcal{D}+\gamma)^{-2}$, and $\lambda f_{\psi,\phi}$, respectively. Assuming each index a and b in $f_{a,b}$ stands either for

ϕ or ψ, the reader should bear in mind that $f_{a,b}$ refers to the multiplication operator corresponding to the Jacobian matrix $(\partial_b \partial_a f)(\Phi_0, \theta + \Psi_0)$. The Fourier kernel of this multiplication operator reads

$$f_{a,b}(p,q) = \hat{f}_{a,b}(p-q), \tag{3.9}$$

whence the translation invariance identity

$$t_s f_{a,b} = f_{a,b} \tag{3.10}$$

readily follows.

Denoting $\Lambda(q) \equiv \Lambda(q,q) \equiv (i\omega \cdot q + \gamma)^{-2}$, we are interested in the kernel

$$J(p,q) = \sum_{r \in \mathbb{Z}^d} B(p,r)\Lambda(r)O(r,q), \quad p,q \in \mathbb{Z}^d, \tag{3.11}$$

of J. We shall also need the "shifted version" of $\Lambda(q)$,

$$\Lambda(\kappa; q) := (i\omega \cdot q + i\kappa + \gamma)^{-2}, \quad \kappa \in \mathbb{C}. \tag{3.12}$$

It is related to $\Lambda(q)$ by the property

$$t_s \Lambda(q) = \Lambda(\omega \cdot s; q). \tag{3.13}$$

Further, $\Lambda(\kappa; q)$ is analytic on $\{\kappa \mid |\Im\mathfrak{m}\, \kappa| \le g/3\} \times \{\gamma \mid |\gamma - g| < g/2\}$ and satisfies

$$|\Lambda(\kappa; q)| \le 36 g^{-2}. \tag{3.14}$$

Equation (3.14) also means that the operator $\Lambda(\kappa)$ corresponding to the kernel in (3.12) belongs to $\mathcal{L}(\mathcal{B}_\sigma)$ with $\|\Lambda(\kappa)\|_{\sigma;\sigma} \le 36g^{-2}$. As for the rest of $J = B\Lambda O$, interpreting $f_{a,b}$ as multiplication and the subsequent inequality

$$\|f_{a,b}\|_{\sigma;\sigma} = \sup_{p \in \mathbb{Z}^d} \sum_{q \in \mathbb{Z}^d} |f_{a,b}(q,p)| e^{\sigma(|q|-|p|)} \le \sup_{p \in \mathbb{Z}^d} \sum_{q \in \mathbb{Z}^d} |\hat{f}_{a,b}(q-p)| e^{\sigma|q-p|} = \|f_{a,b}\|_\sigma$$

show that $B, O \in \mathcal{L}(\mathcal{B}_\sigma)$.

As in the case of H_1, O acts as multiplication by a real-analytic function whose modulus is bounded by $C|\lambda|$. Thus, we estimate

$$|O(p,q)| \le C|\lambda|\, e^{-\sigma|p-q|} \quad \text{and} \quad |\Lambda(p)O(p,q)| \le C|\epsilon|\, e^{-\sigma|p-q|}. \tag{3.15}$$

Bounding the kernel of B calls for the Neumann series

$$B = \sum_{k=0}^{\infty} B_k, \quad \text{with} \quad B_k := (\lambda\, \Lambda f_{\psi,\psi})^k . \tag{3.16}$$

Then

$$\widehat{B\Psi}(p) = \sum_{k=0}^{\infty} \widehat{B_k\Psi}(p) = \sum_{q \in \mathbb{Z}^d} \sum_{k=0}^{\infty} B_k(p,q)\hat{\Psi}(q). \tag{3.17}$$

The equalities here need an argument, since both involve changing the order of integration or, summation. We go about giving such arguments with the aid of Fubini's Theorem. First, $\|B_k\|_{\sigma;\sigma} \le (C|\epsilon|)^k$ yields

$$\int_{\mathbb{T}^d} \sum_{k=0}^{\infty} \left|B_k\Psi(\theta)e^{-ip\cdot\theta}\right| d\theta \le \int_{\mathbb{T}^d} \sum_{k=0}^{\infty} |B_k\Psi(\theta)|\, d\theta$$
$$\le \int_{\mathbb{T}^d} \sum_{k=0}^{\infty} \sum_{q\in\mathbb{Z}^d} \left|\widehat{B_k\Psi}(q)\right| d\theta \le \int_{\mathbb{T}^d} \sum_{k=0}^{\infty} \|B_k\Psi\|_\sigma \, d\theta < \infty,$$

such that the first equality in (3.17) holds true. As for the second equality, one easily obtains $B_k(p,q) \le (C|\epsilon|)^k$, and therefore

$$\sum_{k=0}^{\infty} \sum_{q\in\mathbb{Z}^d} |B_k(p,q)||\hat{\Psi}(q)| \le \|\Psi\|_\sigma \left(\frac{1}{1-C|\epsilon|}\right) \sum_{q\in\mathbb{Z}^d} e^{-\sigma|q|} < \infty.$$

The expression of B_k contains $k-1$ products of the operator $\lambda\,\Lambda f_{\psi,\psi}$ with itself, which appear as convolutions in terms of Fourier transforms. Explicitly, *cf.* (3.9),

$$B_k(p,q) = \lambda^k \sum_{\substack{q_i\in\mathbb{Z}^d \\ i=1,\dots,k-1}} \Lambda(p)\hat{f}_{\psi,\psi}(p-q_1)\,\Lambda(q_1)\hat{f}_{\psi,\psi}(q_1-q_2)\cdots$$
$$\cdots\Lambda(q_{k-1})\hat{f}_{\psi,\psi}(q_{k-1}-q). \quad (3.18)$$

Using the bound $|\Lambda(p)\hat{f}_{\psi,\psi}(q)| \le Cg^{-2}\, e^{-\sigma|q|}$ together with

$$e^{-\sigma(|p-q_1|+\cdots+|q_{k-1}-q|)} \le e^{-\sigma'|p-q|} e^{-(\sigma-\sigma')(|p-q_1|+\cdots+|q_{k-1}-q|)} \quad (3.19)$$

for $0 < \sigma' < \sigma$, we see that

$$|B_k(p,q)| \le \left(Cg^{-2}|\lambda|\right)^k e^{-\sigma'|p-q|} \sum_{\substack{q_i\in\mathbb{Z}^d \\ i=1,\dots,k-1}} e^{-(\sigma-\sigma')(|p-q_1|+\cdots+|q_{k-1}-q|)}$$
$$\le (C|\epsilon|)^k\, e^{-\sigma'|p-q|} \left(\sum_{r\in\mathbb{Z}^d} e^{-(\sigma-\sigma')|r|}\right)^k \le (C|\epsilon|)^k\, e^{-\sigma'|p-q|}.$$

Thus, choosing ϵ appropriately small we make the geometric series arising in (3.17) convergent and obtain

$$|B(p,q)| \le C\, e^{-\sigma'|p-q|}, \quad p,q \in \mathbb{Z}^d.$$

Using the latter with (3.15) in (3.11) leads to

$$|J(p,q)| \le C|\epsilon|\, e^{-\sigma'|p-q|}, \quad p,q \in \mathbb{Z}^d.$$

Finally, $H_2(p,q) = \lambda \sum_{r\in\mathbb{Z}^d} \hat{f}_{\phi,\psi}(p-r)J(r,q)$ implies

$$|H_2(p,q)| \le Cg^2|\epsilon|^2 e^{-\sigma'|p-q|}, \quad p,q \in \mathbb{Z}^d. \tag{3.20}$$

Exploiting (3.10), we compute

$$t_s H_2 = \lambda\, t_s(f_{\phi,\psi}J) = \lambda\, f_{\phi,\psi}\, t_s J = \lambda\, f_{\phi,\psi}\,(t_s B)(t_s\Lambda)O. \tag{3.21}$$

What about $t_s B$? Let us compute it with the aid of (3.16). First of all,

$$t_s B_k = \lambda^k\, t_s(\Lambda f_{\psi,\psi})^k = \lambda^k\,[(t_s\Lambda)f_{\psi,\psi}]^k = \lambda^k\,[\Lambda(\omega\cdot s)f_{\psi,\psi}]^k$$

making use of (3.13). Looking at (3.18), this implies that s appears in the expression of $(t_s B_k)(p,q)$ only in the factors of the form $\Lambda(r+s) = (i\omega\cdot(r+s)+\gamma)^{-2}$. Thus, $(t_s B_k)(p,q)$ depends on s only through $\omega\cdot s$. Moreover, the dependence on $\omega\cdot s$ is analytic in a neighbourhood of the real line, as is clarified in the paragraph below.

Consider the shifted quantity

$$B_k(\kappa;p,q) := \lambda^k \sum_{\substack{q_i\in\mathbb{Z}^d\\ i=1,\dots,k-1}} \Lambda(\kappa;p)\hat{f}_{\psi,\psi}(p-q_1)\,\Lambda(\kappa;q_1)\hat{f}_{\psi,\psi}(q_1-q_2)\cdots$$
$$\cdots\Lambda(\kappa;q_{k-1})\hat{f}_{\psi,\psi}(q_{k-1}-q),$$

which for $\kappa = \omega\cdot s$ becomes $(t_s B_k)(p,q)$. The summand above is analytic on

$$D_g := \{\epsilon \mid |\epsilon| < \epsilon_0\} \times \{\kappa \mid |\Im\mathfrak{m}\,\kappa| \le g/3\} \times \{\gamma \mid |\gamma - g| < g/2\},$$

and the sum converges uniformly, as is readily observed after recalling the bound (3.14) on $\Lambda(\kappa;q)$ and looking at the estimation of $|B_k(p,q)|$ on page 32. Thus, $B_k(\kappa;p,q)$ is analytic. But the Neumann series $\sum_{k=0}^{\infty} B_k(\kappa;p,q)$ also converges uniformly, making the limit $B(\kappa;p,q)$ analytic on D_g. Evidently,

$$(t_s B)(p,q) = B(\omega\cdot s;p,q).$$

We now extend the definition of H_2 by

$$H_2(\kappa;p,q) := \lambda \sum_{q_1,q_2\in\mathbb{Z}^d} \hat{f}_{\phi,\psi}(p-q_1)B(\kappa;q_1,q_2)\Lambda(\kappa;q_2)O(q_2,q),$$

motivated by equation (3.21). Using (3.14), a straightforward computation shows that also $H_2(\kappa;p,q)$ obeys an estimate of the form (3.20) and that the convergence of the sum over q_1 and q_2 above is uniform on D_g. Hence, $H_2(\kappa; p, q)$ is an analytic function on the latter region. Further,

$$(t_s H_2)(p, q) = H_2(\omega \cdot s; p, q).$$

Recalling the translation invariance (3.8) of H_1, the operator $H = H_1 + H_2$ inherits all the features of H_2 discussed in the previous paragraph.

The bound on the derivative $H^{(k)}(\kappa; p, q)$ is achieved by a Cauchy estimate. To that end, one observes

$$H'(\kappa) = H_2'(\kappa)$$

and uses the bound (3.20) on D_g. Similarly, $\partial H/\partial\gamma = \partial H_2/\partial\gamma$, and we get the bound on $\partial H(0; 0, 0)/\partial\gamma$ appearing in the formulation of the lemma.

All the estimates in the present proof are independent of the actual value of g, as long as $0 < g < g_0$, except for explicit appearances of g in them. That is to say, they hold on $\bigcup_{0<g<g_0} D_g = \big\{(\kappa, \epsilon, g, \gamma) \bigm| |\Im\mathfrak{m}\, \kappa| \leq g/3,\ (\epsilon, g) \in D,\ |\gamma - g| < g/2\big\}$. □

3.2. Local invariant manifolds: rudiments of renormalization

We now proceed to stating the main theorem of this chapter, discussing the linearization X_1. The proof of our result is based on an engrossing Renormalization Group (RG) technique whose elements we lay down here.

The main purpose of this section is to convey the central RG ideas employed to the reader. Indeed, only after a rather heuristic discussion will all the necessary technical aspects be sorted out with full rigor in Sections 3.3–3.4. The latter step crucially involves making use of the elementary observations of Lemma 3.3.

THEOREM 3.4. *Under the assumptions of Theorem 1, there exist a number ϵ_0 and a map $\gamma = \gamma(\epsilon, g)$ on D, analytic in ϵ, with $|\gamma - g| \leq Cg|\epsilon|$, such that equation* (3.1) *has a nontrivial solution X_1 which is*

(1) *analytic in $|\epsilon| < \epsilon_0$ and*

(2) *analytic in θ in a complex neighbourhood $\mathcal{U}$ of $\mathbb{T}^d$,*

and satisfies the physical constraint

$$\Phi_1\big|_{\epsilon=0} = 4 \equiv \langle\Phi_1\rangle.$$

Furthermore, it is real-valued if ϵ and θ are real, and

$$\sup_{\theta\in\mathcal{U}} |\Psi_1(\theta)| \le C|\epsilon| \quad \textit{and} \quad \sup_{\theta\in\mathcal{U}} |\Phi_1(\theta) - 4| \le Cg|\epsilon|.$$

The map γ is independent of $\langle\Psi_0\rangle$. If X_1 and X_1' correspond to X_0 and X_0' of Theorem 2.2, respectively, with $\langle\Psi_0\rangle = 0$ and $\langle\Psi_0'\rangle = \beta \in \mathbb{R}^d$, then

$$X_1'(\theta) \equiv X_1(\theta + \beta).$$

Remark 3.5. We chose the normalization 4, because the function $X^0(z,\theta) = (4\arctan z, 0)$ is the physically motivated unperturbed solution (separatrix) and $\arctan z = z + \mathcal{O}(z^3)$. Observe that $\Psi_1|_{\epsilon=0} = 0$ is automatic by (3.1) and $\gamma|_{\epsilon=0} = g > 0$.

Remark 3.6. The pair (γ, X_1) of Theorem 3.4 is unique in the sense that it is the only one making *our construction* work, which is manifested by Lemma 3.13 below. However, we do not prove its uniqueness as far as the properties spelled out above are concerned.

Let us commence sketching the backbone of Theorem 3.4 by recalling equation (3.7):

$$[(\mathcal{D} + \gamma)^2 - g^2]\Phi_1 = H\Phi_1.$$

We expand the square and obtain

$$(\mathcal{D}^2 + 2\gamma\mathcal{D})\Phi_1 = (H + g^2 - \gamma^2)\Phi_1. \tag{3.22}$$

For $\epsilon = 0$ it is known that

$$\Phi_1(\theta)|_{\epsilon=0} = \partial_z\Phi(0,\theta)|_{\epsilon=0} \equiv \partial_z\Phi^0(0) = 4.$$

This suggests that, for $\epsilon \neq 0$ but small, the solution Φ_1 should have values close to the unperturbed value 4. Due to the linearity of (3.22) such a solution may be normalized as $\langle\Phi_1\rangle = 4$. Thus, we set

$$\Phi_1(\theta) = 4 + \xi(\theta), \tag{3.23}$$

where we *demand* the function $\xi : \mathbb{T}^d \to \mathbb{R}$ to vanish on the average, *i.e.*,

$$\hat{\xi}(0) = 0. \tag{3.24}$$

Plugging (3.23) into (3.22) results in

$$(\mathcal{D}^2 + 2\gamma\mathcal{D})\xi = \pi_0(\xi + 4), \quad \text{where} \quad \pi_0 := H + g^2 - \gamma^2.$$

After switching into Fourier representation, this reads

$$\hat{\xi}(q) = G(q)\Big[\sum_{p\in\mathbb{Z}^d} \pi_0(q,p)\hat{\xi}(p) + \hat{\rho}_0(q)\Big] \quad \text{if } q \in \mathbb{Z}^d \setminus \{0\}, \tag{3.25}$$

$$0 = \sum_{p\in\mathbb{Z}^d} \pi_0(0,p)\hat{\xi}(p) + \hat{\rho}_0(0), \tag{3.26}$$

where ρ_0 is a function defined through its Fourier transform by setting

$$\hat{\rho}_0(q) := 4\pi_0(q,0). \tag{3.27}$$

The symbol $G(q)$ stands for the diagonal element $G(q,q)$ of the operator G whose Fourier kernel is given by

$$G(p,q) := \delta_{p,q} \begin{cases} \left(2i\gamma\,\omega\cdot q - (\omega\cdot q)^2\right)^{-1} & \text{if } q \in \mathbb{Z}^d \setminus \{0\}, \\ 0 & \text{if } q = 0. \end{cases} \tag{3.28}$$

The matter of the fact is that, in terms of our new notations, any solution ξ of

$$\xi = G(\pi_0\xi + \rho_0) \tag{3.29}$$

also solves (3.25); only the zero mode constraint (3.24) has been included here. After finding such a ξ, we go on to show that it is a solution to (3.26), as well.

As is apparent from the definition of G, this problem involves arbitrarily small denominators $\omega \cdot q$. Our strategy is to recursively decompose G into parts, each of which corresponds to denominators up to a given order of magnitude. We then end up solving "partial problems" of (3.29) scale by scale, and show that these solutions converge to a true solution of (3.29) as the recursion proceeds and smaller and smaller denominators become dealt with.

Leaving the all-important scaling parameter $\aleph \in\,]0,1[$ to be decided later[2], we shall need the entire functions

$$\chi_n : \mathbb{C} \to \mathbb{C} : \chi_n(\kappa) = \begin{cases} e^{-(\aleph^{-n}\kappa)^6} & \text{if } n \in \mathbb{Z}_+, \\ 1 & \text{if } n = 0. \end{cases}$$

Their importance lies in the fact that the sequence $(\chi_n - \chi_{n+1})_{n\in\mathbb{N}}$ of functions is an analytic partition of unity on $\mathbb{R}\setminus\{0\}$; on this set

$$0 \le \sum_{n=0}^{N-1} (\chi_n - \chi_{n+1}) = 1 - \chi_N \nearrow 1 \quad \text{pointwise, as} \quad N \to \infty.$$

Some of the first members of the sequence appear plotted in Figure 1.

[2]Aleph, $\aleph$, is the first letter in the Hebrew alphabet.

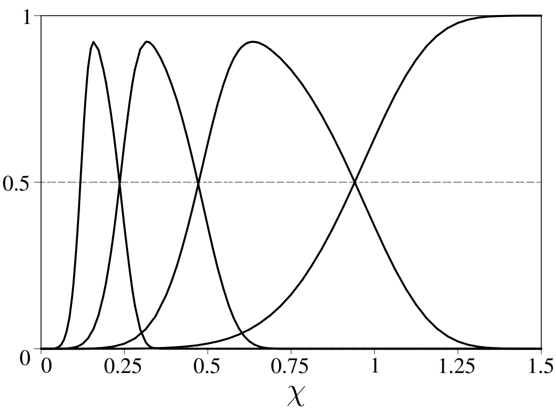


Figure 1. Graphs of $\chi_n - \chi_{n+1}$ with $n = 0, 1, 2, 3$, and $\aleph = \frac{1}{2}$. The maxima are located roughly at $\aleph^n$.

Let us now introduce the diagonal operators G_n and Γ_n, $n \in \mathbb{N}$, defined by

$$G_n(q) := \chi_n(\omega \cdot q) G(q) \quad \text{and} \quad \Gamma_n := G_n - G_{n+1},$$

respectively. Observe that $G_0 = G$ and $G_n(0) = 0$. The point here is that in $\Gamma_n(q)$ the functions $\chi_n(\omega \cdot q) - \chi_{n+1}(\omega \cdot q)$ act as cutoffs for the values of $\omega \cdot q$. Each Γ_n deals with the denominators $\omega \cdot q$ that are roughly of order $\aleph^n$ and, intuitively,

$$\Gamma_{<n} := \sum_{k=0}^{n-1} \Gamma_k = G - G_n \tag{3.30}$$

gets closer and closer to G as n tends to infinity. Instead of the full equation (3.29), consider the easier, approximate problem

$$x_n = \Gamma_{<n}(\pi_0 x_n + \rho_0), \tag{3.31}$$

obtained by replacing G with $\Gamma_{<n}$. It is easier since $\Gamma_{<n}$ discards the most dangerous ones of the small denominators. However, its solution should become a better and better approximation of the solution of (3.29) with increasing n.

Having $G_0 = G_1 + \Gamma_0$, we decompose $\xi = \xi_1 + \eta_0$ and assume that $\eta_0 = \eta_0(\xi_1)$ solves the "large denominator problem"

$$\eta_0 = \Gamma_0(\pi_0(\xi_1 + \eta_0) + \rho_0). \tag{3.32}$$

Then, solving the original problem (3.29) for ξ amounts to solving

$$\xi_1 = G_1(\pi_0(\xi_1 + \eta_0) + \rho_0) \tag{3.33}$$

for ξ_1.

Assuming $\mathbb{1} - \Gamma_0\pi_0$ is invertible[3] (we shall prove it is, once the necessary Banach spaces have been defined), we can extract η_0 out of (3.32) and get

$$\eta_0 = (\mathbb{1} - \Gamma_0\pi_0)^{-1}\Gamma_0(\pi_0\xi_1 + \rho_0). \tag{3.34}$$

Therefore, (3.33) transforms into

$$\xi_1 = G_1(\mathbb{1} - \pi_0\Gamma_0)^{-1}(\pi_0\xi_1 + \rho_0)$$

with the aid of the identities

$$\pi_0(\mathbb{1} - \Gamma_0\pi_0)^{-1} = (\mathbb{1} - \pi_0\Gamma_0)^{-1}\pi_0$$

and

$$(\mathbb{1} - \pi_0\Gamma_0)^{-1}\pi_0\Gamma_0 = (\mathbb{1} - \pi_0\Gamma_0)^{-1} - \mathbb{1}.$$

Thus, defining the new objects

$$\pi_1 := (\mathbb{1} - \pi_0\Gamma_0)^{-1}\pi_0 \quad \text{and} \quad \rho_1 := (\mathbb{1} - \pi_0\Gamma_0)^{-1}\rho_0,$$

we obtain

$$\eta_0 = \Gamma_0(\pi_1\xi_1 + \rho_1)$$

and

$$\xi_1 = G_1(\pi_1\xi_1 + \rho_1). \tag{3.35}$$

Indeed, equation (3.35) has *precisely the same form* as the original problem (3.29) formulated in terms of ξ. Now, relaxing the assumption that η_0 be *a priori* known, suppose we are able to solve (3.35), and take (3.34) as the definition of η_0, instead. Then the solution of the full problem is recovered using the simple relation

$$\xi = \xi_1 + \eta_0 = (\mathbb{1} - \Gamma_0\pi_0)^{-1}(\xi_1 + \Gamma_0\rho_0).$$

Owing to the aforementioned formal covariance between equations (3.29) and (3.35), we may iterate the construction above. Thus, in general, solving

$$\xi_{n+1} = G_{n+1}(\pi_{n+1}\xi_{n+1} + \rho_{n+1}) \tag{3.36}$$

for ξ_{n+1} with the definitions

$$\pi_{n+1} := (\mathbb{1} - \pi_n\Gamma_n)^{-1}\pi_n, \tag{3.37}$$

$$\rho_{n+1} := (\mathbb{1} - \pi_n\Gamma_n)^{-1}\rho_n, \tag{3.38}$$

$$\eta_n := \Gamma_n(\pi_{n+1}\xi_{n+1} + \rho_{n+1}), \tag{3.39}$$

produces $\xi_n = \xi_{n+1} + \eta_n$, or

$$\xi_n = (\mathbb{1} - \Gamma_n\pi_n)^{-1}(\xi_{n+1} + \Gamma_n\rho_n) \tag{3.40}$$

[3] Think of Γ_0 as comprising only large denominators and π_0 being proportional to ϵ.

for the solution of $\xi_n = G_n(\pi_n \xi_n + \rho_n)$.

Equations (3.40) and (3.38) reveal through

$$\pi_n \xi_n + \rho_n = \pi_n \left[(\mathbb{1} - \Gamma_n \pi_n)^{-1} \xi_{n+1} + \Gamma_n \rho_{n+1} \right] + (\mathbb{1} - \pi_n \Gamma_n) \rho_{n+1}$$

the recursion invariance

$$\pi_0 \xi_0 + \rho_0 = \pi_1 \xi_1 + \rho_1 = \cdots = \pi_n \xi_n + \rho_n = \cdots \tag{3.41}$$

in our construction.

Let us tidy up the notation by giving the definitions

$$v_n(y) \equiv \pi_n y + \rho_n \quad \text{and} \quad f_n := \mathbb{1} + \Gamma_{<n} v_n \quad \text{with} \quad \Gamma_{<0} = 0. \tag{3.42}$$

In particular, (3.41) takes the form $v_n(\xi_n) = v_0(\xi_0)$. We also set

$$\Xi_n(y) \equiv (\mathbb{1} - \Gamma_n \pi_n)^{-1} (y + \Gamma_n \rho_n), \tag{3.43}$$

such that (3.40) reads $\xi_n = \Xi_n(\xi_{n+1})$, and (3.41) reduces to

$$v_{n+1} = v_n \circ \Xi_n. \tag{3.44}$$

Of course, this is nothing but a convenient way of writing

$$v_{n+1} = (\mathbb{1} - \pi_n \Gamma_n)^{-1} v_n.$$

Notice also that Ξ_n is formally invertible.

One easily verifies

$$\Xi_n = \mathbb{1} + \Gamma_n v_{n+1}. \tag{3.45}$$

As a consequence,

$$f_n(\Xi_n(y)) = \Xi_n(y) + \Gamma_{<n} v_n(\Xi_n(y)) = y + \Gamma_n v_{n+1}(y) + \Gamma_{<n} v_n(\Xi_n(y)).$$

Applying (3.44) on the last term yields

$$f_{n+1} = f_n \circ \Xi_n. \tag{3.46}$$

Since $f_0 = \mathbb{1}$, we have the cumulative formula

$$f_n = \Xi_0 \circ \Xi_1 \circ \cdots \circ \Xi_{n-1}. \tag{3.47}$$

Hence, a similar expansion of (3.44) implies

$$v_n = v_0 \circ f_n.$$

Inserting here the definition of f_n, we get

$$v_n = v_0 \circ (\mathbb{1} + \Gamma_{<n} v_n). \tag{3.48}$$

PROPOSITION 3.7. *Let $\xi_n = \Xi_n(\xi_{n+1})$. If ξ_{n+1} satisfies $\xi_{n+1} = G_{n+1}(\pi_{n+1}\xi_{n+1} + \rho_{n+1})$, then ξ_n satisfies $\xi_n = G_n(\pi_n \xi_n + \rho_n)$, and vice versa.*

Proof. Suppose $\xi_{n+1} = G_{n+1}v_{n+1}(\xi_{n+1})$. By $G_n = G_{n+1} + \Gamma_n$ and (3.44),

$$G_n v_n \circ \Xi_n = G_{n+1}v_{n+1} - \mathbb{1} + \mathbb{1} + \Gamma_n v_{n+1}.$$

But, with the aid of (3.45), this transforms into

$$(G_n v_n - \mathbb{1}) \circ \Xi_n = G_{n+1}v_{n+1} - \mathbb{1}.$$

As Ξ_n is invertible with $\xi_n = \Xi_n(\xi_{n+1})$, the identity above proves the claimed formal equivalence of the small denominator problems (3.36), or $G_n v_n(\xi_n) = \xi_n$, with differing indices n. □

Recalling (3.47), we immediately arrive at

Corollary 3.8. *If* $\xi_n = G_n(\pi_n \xi_n + \rho_n)$, *then*

$$\xi_0 := f_n(\xi_n) = \xi_n + \Gamma_{<n} v_n(\xi_n)$$

solves the complete problem: $\xi_0 = G_0(\pi_0 \xi_0 + \rho_0)$.

Remark 3.9. The solution ξ_0 appearing in the corollary comprises two terms having clear interpretations. The first term, ξ_n, solves the small denominator problem, namely $\xi_n = G_n(\pi_n \xi_n + \rho_n)$, at the nth step. The second term, $\Gamma_{<n} v_n(\xi_n)$, on the other hand, consists of the sum

$$\eta_{<n}(\xi_n) := \sum_{k=0}^{n-1} \eta_k(\xi_{k+1}) \quad \text{with} \quad \xi_{k+1} = (\Xi_{k+1} \circ \cdots \circ \Xi_{n-1})(\xi_n),$$

where $\eta_k = \eta_k(\xi_{k+1})$ solves the large denominator problem that can be written as $\eta_k = \Gamma_k v_k(\xi_{k+1} + \eta_k)$.

Finally, we make a crucial observation. If we operate on (3.48) by $\Gamma_{<n}$ from the left and set

$$x_n := f_n(0) = \Gamma_{<n} v_n(0), \tag{3.49}$$

we solve the approximate problem (3.31):

$$x_n = \Gamma_{<n}(\pi_0 x_n + \rho_0).$$

We shall demonstrate that the approximate solutions x_n form a Cauchy sequence in a simple Banach space, and that their limit

$$\xi := \lim_{n\to\infty} x_n \tag{3.50}$$

solves the original equation (3.29).

We beg the reader's pardon as we pass to the following sketchy paragraph whose sole purpose is motivational. It could be titled "RG heuristics", and may be skipped without future damage.

Think of an abstract map $\mathcal{R}_n$ that takes (π_n, ρ_n, G_n) to $(\pi_{n+1}, \rho_{n+1}, G_{n+1})$. The recursion scheme

$$\xi = G(\pi_0\xi + \rho_0) \overset{\mathcal{R}_0}{\mapsto} \xi_1 = G_1(\pi_1\xi_1 + \rho_1) \overset{\mathcal{R}_1}{\mapsto} \cdots \overset{\mathcal{R}_{n-1}}{\mapsto} \xi_n = G_n(\pi_n\xi_n + \rho_n) \overset{\mathcal{R}_n}{\mapsto} \cdots$$

is called *renormalization* of the problem, and $\mathcal{R}_n$ is the corresponding renormalization transformation. Then, in view of Proposition 3.7, it remains for one to demonstrate that this process "converges", in order to be able to solve the original equation $\xi = G(\pi_0\xi + \rho_0)$. That is to say, one wishes that the *renormalization flow* of the triplet (π_0, ρ_0, G_0), $(\pi_n, \rho_n, G_n) = (\prod_{k=0}^{n-1} \mathcal{R}_k)(\pi_0, \rho_0, G_0)$, in a sense tends to a fixed point (π^*, ρ^*, G^*) of some limiting operator "$\mathcal{R}_\infty = \lim_{k\to\infty} \mathcal{R}_k$" as $n \to \infty$, and that the equation

$$\xi^* = G^*(\pi^*\xi^* + \rho^*) \tag{3.51}$$

is well-defined and solvable.

In our case $G^*\rho^* = 0$, such that the equation is linear and possesses the trivial solution $\xi^* = 0$. Corollary 3.8 then throws light on why (3.50) should solve (3.29); $f_n(\xi_n)$ solves it, and ξ_n approaches zero with increasing n. Therefore, it is fair to expect that also $\lim_{n\to\infty} f_n(0)$ is a solution.

Remark 3.10. The name "Renormalization Group", coined by physicists, is rather misleading. In particular, the transformations $\mathcal{R}_n$ are virtually never invertible.

3.3. Banach spaces

Technically speaking, we need to control the renormalization flow (3.37)–(3.39) by estimating the kernel elements of Γ_n and π_n, for the operators $\mathbb{1} - \pi_n\Gamma_n$ and $\mathbb{1} - \Gamma_n\pi_n$ had better be invertible between suitable spaces. Such Banach spaces will be defined in this section.

We begin by analyzing the properties of the operators Γ_n. *A priori*, one expects the most significant contribution to arise from such q's that $\omega \cdot q = \mathcal{O}(\aleph^n)$, due to the cutoff $\chi_n - \chi_{n+1}$ in the definition of these operators. Therefore, (3.28) implies

$$|\Gamma_n(q)| = \mathcal{O}(g^{-1}\aleph^{-n}). \tag{3.52}$$

Next we shall concentrate on providing a rigorous and more detailed estimate on such kernel elements.

Consider the entire function

$$s(z) := \begin{cases} (1 - e^{-z})/z & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

For a complex number $z = x + iy$,

$$\left|\frac{1 - e^{-z}}{z}\right|^2 = |z|^{-2}(1 - 2e^{-x}\cos y + e^{-2x}). \tag{3.53}$$

It is therefore clear that the maximum of $|s(z)|$ in any region $x_0 \geq x$ is achieved on the real line, at $z = x_0$; in order to maximize $|s(z)|$ one has to minimize the real part of z.

Assuming $n \geq 1$,

$$\chi_n(\kappa) - \chi_{n+1}(\kappa) = e^{-(\aleph^{-n}\kappa)^6}\left[1 - e^{-(\aleph^{-n}\kappa)^6(\aleph^{-6}-1)}\right].$$

Dealing separately with the case $n = 0$, we get

$$\chi_n(\kappa) - \chi_{n+1}(\kappa) = \begin{cases} e^{-(\aleph^{-n}\kappa)^6}(\aleph^{-n}\kappa)^6(\aleph^{-6}-1)\, s((\aleph^{-n}\kappa)^6(\aleph^{-6}-1)) & \text{if } n \geq 1, \\ (\aleph^{-1}\kappa)^6\, s((\aleph^{-1}\kappa)^6) & \text{if } n = 0. \end{cases} \tag{3.54}$$

We consider these functions in the strips $|\mathfrak{Im}\,\kappa| < \aleph^n b$, where b is a constant. Following the dogma stated below (3.53) we intend to estimate $\mathfrak{Re}\,(\aleph^{-n}\kappa)^6$ from below in order to bound the absolute value of (3.54) from above. Let us denote $z = \aleph^{-n}\kappa = |z|e^{i\varphi}$ for a while, such that $|\mathfrak{Im}\,z| < b$, or

$$|\sin\varphi| < b/|z|, \tag{3.55}$$

becomes the relevant constraint.

First, the entire functions e^{-z^6} and $s(cz^6)$, with $c \in \{1, \aleph^{-6} - 1\}$, are bounded in any disk $|z| < r$ and $e^{-\frac{1}{2}|z|^6}$ is bounded away from zero there. In particular, we gather that there exists a constant $C > 0$, which is monotonically increasing in r, such that

$$\left|e^{-z^6}\right| \leq Ce^{-\frac{1}{2}|z|^6} \quad \text{and} \quad |cz^6 s(cz^6)| \leq C|z|^6 \quad \text{for} \quad |z| < r.$$

Outside the disk $|z| < 6b$, (3.55) tells us that $\sin\varphi$ must lie on the interval $[-1/6, 1/6] + 2k\pi$ for integer k. Therefore, $\cos 6\varphi \geq 1/2$ and

$$\mathfrak{Re}\, z^6 = |z|^6 \cos 6\varphi \geq |z|^6/2.$$

Consequently,

$$\left|e^{-z^6}\right| = e^{-\mathfrak{Re}\, z^6} \leq e^{-\frac{1}{2}|z|^6} \quad \text{for} \quad |z| \geq 6b,$$

and, for any nonnegative number c, the bound

$$|cz^6 s(cz^6)| \le \left|1 - e^{-cz^6}\right| \le 1 + e^{-c\,\mathfrak{Re}\,z^6} \le 2$$

holds true.

Putting things together, we choose $r = 6b$ and obtain

$$\left|e^{-(\aleph^{-n}\kappa)^6}\right| \le Ce^{-\frac{1}{2}|\aleph^{-n}\kappa|^6} \quad \text{for} \quad |\mathfrak{Im}\,\kappa| < \aleph^n b.$$

Hence, (3.54) implies the bound

$$|\chi_n(\kappa) - \chi_{n+1}(\kappa)| \le C|\aleph^{-n}\kappa|^\ell \begin{cases} e^{-\frac{1}{2}|\aleph^{-n}\kappa|^6} & \text{if } n \ge 1, \\ 1 & \text{if } n = 0, \end{cases} \tag{3.56}$$

for $\ell = 0, 1, \ldots, 6$, where $|\mathfrak{Im}\,\kappa| < \aleph^n b$ and the (new) constant C only depends on b—in a monotonically increasing manner.

We turn to estimating the kernel of the diagonal operators Γ_n. Pay attention to the fact that $G(q)$, which was defined in (3.28), only depends on q through $\omega\cdot q$. Therefore, it is handy to introduce the analytic function

$$\iota : \mathbb{C} \setminus \{0, 2i\gamma\} \to \mathbb{C} : \iota(\kappa) = (2i\gamma\kappa - \kappa^2)^{-1}.$$

In particular, $\iota(\omega \cdot q) = G(q)$ for $q \ne 0$. This motivates the further definition

$$\Gamma_n(\kappa; p, q) := \delta_{p,q} \begin{cases} [\chi_n(\omega \cdot q + \kappa) - \chi_{n+1}(\omega \cdot q + \kappa)]\iota(\omega \cdot q + \kappa) & \text{if } \omega \cdot q + \kappa \ne 0, \\ 0 & \text{if } \omega \cdot q + \kappa = 0. \end{cases}$$

The importance of the resulting operator $\Gamma_n(\kappa)$ is based on the possibility of employing complex analysis along with the "variable" $\omega \cdot q$:

$$\Gamma_n(q, q) = \Gamma_n(0; q, q) = \Gamma_n(\omega \cdot q; 0, 0).$$

Following earlier conventions, we shall often write $\Gamma_n(\kappa; q)$ instead of the complete $\Gamma_n(\kappa; q, q)$.

Notice that the map $\kappa \mapsto \kappa\iota(\kappa)$ has an analytic continuation to $\mathbb{C} \setminus \{2i\gamma\}$, which we identify with the map itself. With this in mind, we compute

$$\inf |2i\gamma - (\omega \cdot q + \kappa)| = g/2,$$

the infimum being taken over $|\mathfrak{Im}\,\kappa| \le g/2$ and $|\gamma - g| < g/2$, whereby

$$|(\omega \cdot q + \kappa)\,\iota(\omega \cdot q + \kappa)| \le 2g^{-1} \tag{3.57}$$

for such κ and γ.

We can bound the nonzero kernel elements of $\Gamma_n(\kappa)$, within $|\Im\mathfrak{m}\,\kappa| < \aleph^n b$, by

$$|\Gamma_n(\kappa;q)| = \aleph^{-n} \frac{|\chi_n(\omega\cdot q+\kappa) - \chi_{n+1}(\omega\cdot q+\kappa)|}{|\aleph^{-n}(\omega\cdot q+\kappa)|} \, |(\omega\cdot q+\kappa)\,\iota(\omega\cdot q+\kappa)|$$
$$\le C_\Gamma\, g^{-1}\aleph^{-n}\min(1, |\aleph^{-n}(\omega\cdot q+\kappa)|^5) \begin{cases} e^{-\frac{1}{2}|\aleph^{-n}(\omega\cdot q+\kappa)|^6} & \text{if } n\ge 1,\\ 1 & \text{if } n=0, \end{cases} \tag{3.58}$$

making use of (3.56) and (3.57). Here we have imposed the condition $b \le g/2$ on b. In particular, we have confirmed the earlier heuristic estimate (3.52).

Now to the spaces promised. Ultimately the solution of (3.29), namely ξ (and therefore Φ_1) will live in the space $\mathcal{B}^{\Phi}_{\alpha^*} \subset \ell^1(\mathbb{Z}^d;\mathbb{C})$ for a sufficiently small width α^* of the analyticity strip—see Section 2.1. The following weights will come in handy:

$$w_n(q) := \begin{cases} e^{\aleph^{-n}|\omega\cdot q|} & \text{if } n\ge 1,\\ 1 & \text{if } n=0. \end{cases} \tag{3.59}$$

We extend these to negative indices by setting

$$w_{-n}(q) \equiv w_n(q)^{-1} \qquad (n\in\mathbb{Z}).$$

Definition 3.11 (Spaces h_n). For $n\in\mathbb{Z}$, let

$$\|\xi\|_n := \sum_{q\in\mathbb{Z}^d} |\hat{\xi}(q)| w_n(q).$$

These norms induce the Banach spaces h_n. Observe that h_0 is the space $\ell^1(\mathbb{Z}^d;\mathbb{C})$ with the usual unweighted norm $\|\xi\|_0 := \sum_{q\in\mathbb{Z}^d}|\hat{\xi}(q)|$.

Notice that our weights satisfy

$$w_{n+1}(q)^{\aleph} = w_n(q) \quad \text{and} \quad w_n(q)\ge 1 \qquad (n\ge 1). \tag{3.60}$$

The spaces at hand thus realize the embedding hierarchy

$$h_{n+1}\subset h_n \qquad (n\in\mathbb{Z})$$

due to the trivial inequalities

$$\|\cdot\|_n \le \|\cdot\|_{n+1} \qquad (n\in\mathbb{Z}) \tag{3.61}$$

for the corresponding norms.

Operator norms $\|\cdot\|_{\mathcal{L}(h_n;h_m)}$ between such spaces h_n and h_m will be denoted by $\|\cdot\|_{n;m}$ for short. We actually have,

$$\|L\|_{n;m} = \sup_{q\in\mathbb{Z}^d} \sum_{p\in\mathbb{Z}^d} |L(p,q)| w_m(p) w_{-n}(q) \qquad (m,n\in\mathbb{Z}). \tag{3.62}$$

Either from this or from (3.61) by the Schwarz inequality one infers

$$\|\cdot\|_{n+1;m} \leq \|\cdot\|_{n;m} \leq \|\cdot\|_{n;m+1} \qquad (m,n\in\mathbb{Z})$$

such that the operator spaces satisfy

$$\mathcal{L}(h_n;h_{m+1}) \subset \mathcal{L}(h_n;h_m) \subset \mathcal{L}(h_{n+1};h_m) \qquad (m,n\in\mathbb{Z}).$$

Moreover, the Schwarz inequality implies the useful bounds

$$\|L_1L_2\|_{n;m} \leq \|L_1\|_{l;m}\|L_2\|_{n;l} \qquad (l,m,n\in\mathbb{Z}). \tag{3.63}$$

From now on, n will always assume *nonnegative* values. Define the domain

$$D_n := \left\{\kappa\in\mathbb{C}\,\middle|\,|\kappa| < \aleph^n b\right\}, \tag{3.64}$$

recalling that $b \leq g/2$. Then (3.58) easily validates the bounds

$$\|\Gamma_n(\kappa)\|_{-n;n} \leq C_\Gamma\, g^{-1}\aleph^{-n} \tag{3.65}$$

for $\kappa\in D_n$, where the (new) constant C_Γ is independent of κ and g as long as $g < g_0$. This shows, in particular, that

$$\Gamma_n(\kappa) \in \mathcal{L}(h_{-n};h_n) \subset \mathcal{L}(h_0;h_0).$$

Remark 3.12. The weights $w_n(q)$ arise as follows. The diagonal kernel of Γ_n is strongly concentrated around small denominators $\omega\cdot q$ of order $\aleph^n$; for large $\omega\cdot q$ the value of $\Gamma_n(q)$ is very close to zero, *but not quite equal to zero* as opposed to the ideal case of sharp cutoffs. Therefore, in an expression such as $\widehat{\Gamma_n\xi}(q) = \Gamma_n(q)\hat{\xi}(q)$ we cannot quite let $|\hat{\xi}(q)|$ be *arbitrarily* large for large values of $\omega\cdot q$. This "tail" can be of the order of $w_n(q) = e^{\aleph^{-n}|\omega\cdot q|}$, say, which amounts to $\xi\in h_{-n}$.

It has to be emphasized that having the same power of $\aleph^{-n}$ and $|\omega\cdot q|$ in $w_n(q)$ is crucial, which can be read off from (3.58). This way $\omega\cdot q$ "scales" as $\aleph^n$ in all estimates in the nth step of the iteration.

The motivation for introducing the spaces h_n, on the other hand, comes from the fact that in the recursion (3.37) the domain of π_n will shrink. So, in the norms $\|\cdot\|_n$ we incorporate a weight that increases as n grows. It is a matter of convenience to use the inverse of the weight $w_n(q)^{-1}$ appearing in $\|\cdot\|_{-n}$.

3.4. Renormalization made rigorous: estimates and the Lyapunov exponent

The rest of this chapter is devoted to demonstrating that the renormalization flow of π_n in (3.37) is controlled in the norms $\|\cdot\|_{n;-n}$ such that the products $\|\pi_n\|_{n;-n}\|\Gamma_n\|_{-n;n}$ are small, so as to make the recursion formulae (3.37)–(3.39) well-defined through Neumann series. Recalling (3.65), the task roughly amounts to making sure that $\|\pi_n\|_{n;-n}$ decays at least as rapidly as $\aleph^n$ with increasing n.

According to Lemma 3.3, $\pi_0 = H + g^2 - \gamma^2 \in \mathcal{L}(\mathcal{B}_\sigma^\Phi)$ can be written as

$$\pi_0(p,q) = p_0(\omega\cdot q)\delta_{p,q} + \tilde{\pi}_0(p,q),$$

where $\tilde{\pi}_0$ vanishes on the diagonal, and in the first term

$$p_0(\kappa) := \delta_0 + \bar{p}_0(\kappa), \qquad \bar{p}_0(0) = 0,$$

depends analytically on κ, as long as $|\mathfrak{Im}\,\kappa| \le g/3$; explicitly $\delta_0 = H(0;0,0) + g^2 - \gamma^2$ and $\bar{p}_0(\kappa) = H(\kappa;0,0) - H(0;0,0)$.

Similarly, we split π_n into its diagonal and off-diagonal parts:

$$\pi_n(p,q) = p_n(\omega\cdot q)\delta_{p,q} + \tilde{\pi}_n(p,q), \qquad \tilde{\pi}_n(q,q) = 0,$$

with

$$p_n(\kappa) = \delta_n + \bar{p}_n(\kappa), \qquad \bar{p}_n(0) = 0.$$

The possibility of doing this follows from the computation

$$t_s\pi_0 = t_s H + \delta = H(\omega\cdot s) + \delta =: \pi_0(\omega\cdot s)$$

and its recursive consequence

$$\begin{aligned} t_s\pi_{n+1} &= \left(\mathbb{1} - (t_s\pi_n)(t_s\Gamma_n)\right)^{-1} t_s\pi_n \\ &= \left(\mathbb{1} - \pi_n(\omega\cdot s)\,\Gamma_n(\omega\cdot s)\right)^{-1}\pi_n(\omega\cdot s) \\ &=: \pi_{n+1}(\omega\cdot s). \end{aligned}$$

Motivated by the form of the s-dependence in the expressions above, let us inductively define the maps

$$\pi_{n+1,\beta}(\kappa) := \left(\mathbb{1} - \pi_{n\beta}(\kappa)\,\Gamma_n(\kappa)\right)^{-1}\pi_{n\beta}(\kappa), \qquad \kappa\in D_n, \quad |\mathfrak{Im}\,\beta| < \alpha_n, \tag{3.66}$$

starting at

$$\pi_{0\beta}(\kappa) := P_0(\kappa) + \tilde{\pi}_{0\beta}(\kappa), \qquad \kappa\in D_0, \quad |\mathfrak{Im}\,\beta| < \alpha_0,$$

by setting $b \leq g/3$ in (3.64). Here

$$P_0(\kappa; p, q) := p_0(\kappa + \omega \cdot q)\delta_{p,q},$$
$$\tilde{\pi}_{0\beta}(\kappa; p, q) := e^{i\beta\cdot(p-q)} H(\kappa; p, q)(1 - \delta_{p,q}),$$

and, with σ' coming from Lemma 3.3,

$$\alpha_{n+1} := \left(1 - \frac{4}{(n+3)^2}\right)\alpha_n, \qquad \alpha_0 < \sigma'. \tag{3.67}$$

In particular, Eric Weisstein's World of Mathematics [Wei] tells us that

$$\alpha_n \searrow \alpha_0 \cdot \prod_{k=3}^{\infty}\left(1 - \frac{4}{k^2}\right) = \frac{\alpha_0}{6} > 0 \qquad \text{as} \qquad n \to \infty. \tag{3.68}$$

As far as notation is concerned, we may omit β if it equals zero: $\pi_n(\kappa) \equiv \pi_{n0}(\kappa)$, and so forth. By a straightforward induction argument,

$$\pi_{n\beta}(\kappa; p, q) := e^{i\beta\cdot(p-q)}\pi_n(\kappa; p, q),$$

such that β does not enter the diagonal of $\pi_{n\beta}$. Of course,

$$|\pi_{n\beta}(\kappa; p, q)| = e^{-\Im\mathfrak{m}\,\beta\cdot(p-q)}|\pi_n(\kappa; p, q)|. \tag{3.69}$$

For clarity, set

$$P_n(\kappa) := \delta_n \mathbb{1} + \bar{P}_n(\kappa) \quad \text{with} \quad \bar{P}_n(\kappa; p, q) := \bar{p}_n(\kappa + \omega \cdot q)\delta_{p,q},$$

so that we may express the operator $\pi_{n\beta}(\kappa)$ itself, without reference to its kernel, as

$$\pi_{n\beta}(\kappa) = P_n(\kappa) + \tilde{\pi}_{n\beta}(\kappa) = \delta_n + \bar{P}_n(\kappa) + \tilde{\pi}_{n\beta}(\kappa), \qquad \delta_n \equiv \delta_n \mathbb{1},$$

for short. This decomposition satisfies

$$\|\pi_{n\beta}(\kappa)\|_{n;-n} \leq |\delta_n| + \|\bar{P}_n(\kappa)\|_{n;-n} + \|\tilde{\pi}_{n\beta}(\kappa)\|_{n;-n}. \tag{3.70}$$

It will turn out that the sum in (3.70) is finite if $\kappa \in D_n$ and $|\Im\mathfrak{m}\,\beta| < \alpha_n$—indeed very small, as we are trying to prove—meaning that $\pi_{n\beta}(\kappa) \in \mathcal{L}(h_n; h_{-n})$.

The crux of analyzing the renormalization flow is the following lemma, for which we provide an inductive proof later on in this section. The reader is advised to take the result as granted for now.

Lemma 3.13 (Modified Lyapunov exponent controls the flow). *Set $b = g/3$ and $\aleph = \min(\frac{1}{8}, b^2)$. There exist constants $c_\gamma > 0$, $C > 0$, $c > 0$, $\mu > 1$, and a unique Lyapunov exponent γ satisfying*

$$|\gamma - g| < c_\gamma |\epsilon| g \tag{3.71}$$

such that, for any $n \in \mathbb{N}$, the bounds

$$\|\tilde{\pi}_{n\beta}(\kappa)\|_{n;-n} \leq C|\epsilon|g \begin{cases} g & \text{if } n = 0, \\ \aleph^n e^{-c\mu^n} & \text{if } n \geq 1, \end{cases} \tag{3.72}$$

$$\|\bar{P}_n(\kappa)\|_{n;-n} \leq C|\epsilon|g \begin{cases} g & \text{if } n = 0, \\ \aleph^n & \text{if } n \geq 1, \end{cases} \tag{3.73}$$

$$|\delta_n| \leq C|\epsilon|g \begin{cases} g & \text{if } n = 0, \\ \aleph^{2n} & \text{if } n \geq 1, \end{cases} \tag{3.74}$$

hold true for $(\epsilon, g) \in D$, $\kappa \in D_n$ and $|\mathfrak{Im}\,\beta| < \alpha_n$. Moreover, c is bounded away from zero and $\mu \to \infty$ in the limit $g \to 0$.

Remark 3.14. The factor $\aleph^{2n}$ in (3.74) is rather arbitrary, and its precise form is inessential. In fact, the super-exponentially small bound on $\tilde{\pi}_{n\beta}(\kappa)$ enables proving decay faster than p^{-n} with *any* p.

Remark 3.15. The sole purpose of introducing the complex variable κ is to go about proving the bound (3.73) on the *diagonal* part of π_n. We use analyticity in κ and restrict the latter to a domain of ever decreasing size.

The *possibility* of including the complex parameter β in the analysis, on the other hand, facilitates proving exponential decay of $\pi_n(\kappa; p, q)$ in the quantity $|p - q|$. This is sufficiently rapid for obtaining the bound (3.72) on the *off-diagonal* part of π_n. Also the analyticity strip of β around $\mathbb{R}$ is taken narrower and narrower upon iteration, but no narrower than a certain limit. Therefore the choice (3.67) for the numbers $\alpha_n > 0$ was made, as they have the positive infimum of $\alpha_0/6$ units spelled out in (3.68). The only restriction here being $\alpha_0 < \sigma'$ due to Lemma 3.3.

Corollary 3.16. *The bounds of Lemma 3.13 imply*

$$\|\pi_{n\beta}(\kappa)\|_{n;-n} \leq C|\epsilon|g \begin{cases} g & \text{if } n = 0 \\ \aleph^n & \text{if } n \geq 1, \end{cases}$$

The caveat to get around in the proof of Lemma 3.13 is that δ_n is reluctant to go to zero along the recursion. To change the state of affairs, we fine-tune the Lyapunov exponent γ such that also $\delta_n \to 0$ as $n \to \infty$. As stated in the lemma, there turns out to exist precisely one such value of γ. This is what ultimately enables us to prove the convergence of our renormalization scheme, consequently validating Theorem 3.4 discussing the linearized solution X_1.

Let us look at the flow (3.66) more closely, observing that we may formally split

$$\begin{aligned}\big(\mathbb{1} - \pi_{n\beta}(\kappa)\Gamma_n(\kappa)\big)^{-1} &= \big(\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa) - \tilde{\pi}_{n\beta}(\kappa)\Gamma_n(\kappa)\big)^{-1} \\ &= \big[\mathbb{1} - (\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1}\,\tilde{\pi}_{n\beta}(\kappa)\Gamma_n(\kappa)\big]^{-1}(\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1} \\ &= (\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1} + r_{n\beta}(\kappa).\end{aligned}$$

The remainder $r_{n\beta}(\kappa)$ reads explicitly

$$r_{n\beta}(\kappa) := (\mathbb{1} - \pi_{n\beta}(\kappa)\Gamma_n(\kappa))^{-1}\ \tilde{\pi}_{n\beta}(\kappa)\Gamma_n(\kappa)\ (\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1}.$$

In fact, this quantity is asymptotically *very small* in $\mathcal{L}(h_{-n}; h_{-n})$ due to the explicit factor $\tilde{\pi}_{n\beta}$; given the bounds (3.72)–(3.74) for some particular value of n,

$$\|r_{n\beta}(\kappa)\|_{-n;-n} \leq C|\epsilon|e^{-c\mu^n}. \tag{3.75}$$

Continuing abstractly, (3.66) becomes

$$\pi_{n+1,\beta}(\kappa) = (\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1}\,P_n(\kappa) + s_n(\kappa) + \tilde{s}_{n\beta}(\kappa),$$

where $s_n(\kappa)$ is the diagonal and $\tilde{s}_{n\beta}(\kappa)$ the off-diagonal part of the small remainder term $r_{n\beta}(\kappa)\pi_{n\beta}(\kappa)$, respectively. Therefore, the diagonal $P_n(\kappa)$—containing the problematic δ_n—and the off-diagonal $\tilde{\pi}_{n\beta}(\kappa)$ iterate according to the rules

$$\begin{cases} P_{n+1}(\kappa) = (\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1}\,P_n(\kappa) + s_n(\kappa), \\ \tilde{\pi}_{n+1,\beta}(\kappa) = \tilde{s}_{n\beta}(\kappa). \end{cases} \tag{3.76}$$

Notice that $s_n(\kappa)$ is indeed free of β. Namely, we learned earlier that the diagonal operators $P_n(\kappa)$ are independent of β, so that $s_n(\kappa)$ must also be.

By construction, $\delta_n = \pi_{n\beta}(0;0,0) = P_n(0;0)$ for each n, such that the diagonality of $(\mathbb{1} - P_n(0)\Gamma_n(0))^{-1}$ with $\Gamma_n(0;0) = 0$ implies that changes in δ_n upon iteration only arise from the small term s_n in (3.76):

$$\delta_{n+1} = \delta_n + d_n, \qquad d_n := s_n(0;0). \tag{3.77}$$

But $r_{n\beta}(0;0,0) = 0$, again because $\Gamma_n(0;0) = 0$, such that

$$d_n = s_n(0;0) = (r_n\pi_n)(0;0,0) = (r_n\tilde{\pi}_n)(0;0,0). \tag{3.78}$$

We remind the reader of our convention of dropping one of the kernel indices of diagonal operators. For instance, $s_n(\kappa;q) \equiv s_n(\kappa;q,q)$.

Occasionally it is convenient to have also the flow of $\bar{P}_{n+1}(\kappa)$ extracted from (3.76) in explicit form; it reads

$$\bar{P}_{n+1}(\kappa) = \bar{P}_n(\kappa) + P_n(\kappa)\Gamma_n(\kappa)(\mathbb{1} - P_n(\kappa)\Gamma_n(\kappa))^{-1}P_n(\kappa) + (s_n(\kappa) - d_n). \tag{3.79}$$

Proof of Lemma 3.13. Here we finally prove that the bounds (3.72)–(3.74), such that (3.76)—and indeed everything above—becomes not only formally justified. To this end, we proceed by induction. As iterating (3.72) and (3.73) is rather easy, the proof boils down to *choosing the value of our free parameter*, the Lyapunov exponent γ, so as to guarantee that δ_n satisfies (3.74) at each step.

(i) Case $n = 0$**.** Consider κ restricted to D_0 with $b \leq g/3$. Lemma 3.3 and $\bar{P}_0(\kappa; q) = H(\kappa; q, q) - H(0; 0, 0)$ readily imply

$$\|\bar{P}_0(\kappa)\|_{0;0} \leq C_0|\lambda|.$$

Furthermore, increasing C_0 and employing (3.69) with $|\mathfrak{Im}\,\beta| < \alpha_0 < \sigma'$,

$$\|\tilde{\pi}_{0\beta}(\kappa)\|_{0;0} \leq C_0|\lambda|.$$

The leading Taylor coefficient $\bar{p}_0'(0) = H'(0; 0, 0)$ and the corresponding remainder of the function $\bar{p}_0 = H(\,\cdot\,; 0, 0) - H(0; 0, 0)$ satisfy

$$|\bar{p}_0'(0)| \leq \tfrac{1}{4}C_0|\epsilon|^2 g \quad \text{and} \quad |\bar{p}_0(\kappa) - \bar{p}_0'(0)\kappa| \leq \tfrac{1}{2}C_0|\epsilon|^2|\kappa|^2,$$

taking C_0 large enough.

Assume that γ lies in the open g-centered disk of radius $c|\epsilon|g$:

$$\gamma \in I_\gamma := \mathbb{D}(g, c_\gamma|\epsilon|g). \tag{3.80}$$

Recall that $\delta_0 = \epsilon g^2 u(\epsilon, g, \gamma) + g^2 - \gamma^2$, where $\epsilon g^2 u(\epsilon, g, \gamma) = H(0; 0, 0)$. If $\delta_0(\gamma_1) = \delta_0(\gamma_2)$ and we denote $\gamma_i = g(1 + x_i)$, the Mean-Value Theorem yields

$$|\gamma_1 - \gamma_2| \leq \tfrac{1}{2}\big(|x_1 + x_2| + |\epsilon|g\|\partial_\gamma u\|_\infty\big)|\gamma_1 - \gamma_2|.$$

By Lemma 3.3, $\|\partial_\gamma u\|_\infty \leq C|\epsilon|g^{-1}/(1 - 2c_\gamma|\epsilon|)$, and $|x_1 + x_2| < 2c_\gamma|\epsilon|$. For a sufficiently small $|\epsilon|$, we gather $\gamma_1 = \gamma_2$, such that $\gamma \mapsto \delta_0$ is one-to-one on I_γ. Moreover, the image of the disk I_γ contains the disk $\mathbb{D}\big(0, (2c_\gamma - c_\gamma^2|\epsilon| - \|u\|_\infty)|\epsilon|g^2\big)$. Thus, for a sufficiently large value of c_γ and small value of ϵ, there exists a *closed* set $J_0 \subset I_\gamma$ which $\gamma \mapsto \delta_0$ maps analytically and *bijectively* onto the closed disk

$$I_0 := \bar{\mathbb{D}}(0, C_0|\epsilon|g^2).$$

We are about to prove below that a correct choice of γ leads to

$$\delta_n \in I_n := \bar{\mathbb{D}}(0, C_0|\epsilon|g\,\aleph^{2n})$$

for each and every $n \in \mathbb{Z}_+$.

(ii) Induction step: hypotheses. Fix $n \in \mathbb{N}$. Suppose

$$\|\tilde{\pi}_{n\beta}(\kappa)\|_{n;-n} \leq C_n|\epsilon|g\,\aleph^n \begin{cases} g & \text{if } n = 0, \\ e^{-c\mu^n} & \text{if } n \geq 1, \end{cases}$$

for some constants $c > 0$ and $\mu > 1$—to be fixed later—and

$$\|\bar{P}_n(\kappa)\|_{n;-n} \leq C_n|\epsilon|g\,\aleph^n \begin{cases} g & \text{if } n = 0, \\ 1 & \text{if } n \geq 1, \end{cases}$$

hold true for $|\epsilon| < \epsilon_n$, $|\mathfrak{Im}\,\beta| < \alpha_n$, and $\kappa \in D_n$. Suppose there exists a closed set $J_n \subset I_\gamma$ and a bijective analytic map $\Delta_n : J_n \longrightarrow I_n : \gamma \mapsto \delta_n$.

Further, let the kernel elements of these operators be analytic in D_n and continuous in the closure $\bar{D}_n$. Also the estimates

$$|\bar{p}_n'(0)| \leq \left(1 - \frac{1}{n+2}\right)\frac{1}{2}C_n g \begin{cases} |\epsilon|^2 & \text{if } n = 0, \\ |\epsilon|^{3/2} & \text{if } n \geq 1, \end{cases} \tag{3.81}$$

and

$$|\bar{p}_n(\kappa) - \bar{p}_n'(0)\kappa| \leq \frac{1}{2}C_n\aleph^{-n}|\kappa|^2 \begin{cases} |\epsilon|^2 & \text{if } n = 0, \\ |\epsilon|^{3/2} & \text{if } n \geq 1, \end{cases}, \tag{3.82}$$

which facilitate dealing with the Taylor expansion of $\bar{p}_n$, are supposed to be satisfied.

In particular, it follows from (3.70), $b \leq g/3$, and the inductive hypotheses that

$$|p_n(\kappa)| \leq B_n C_n|\epsilon|g\,\aleph^n \quad \text{with} \quad B_n := \begin{cases} b|\epsilon| + g & \text{if } n = 0, \\ b|\epsilon|^{1/2} + \aleph^n & \text{if } n \geq 1, \end{cases} \tag{3.83}$$

and

$$\|\pi_{n\beta}(\kappa)\|_{n;-n} \leq A_n C_n|\epsilon|g\,\aleph^n, \tag{3.84}$$

where

$$A_n := \begin{cases} g & \text{if } n = 0, \\ 1 + \aleph^n + e^{-c\mu^n} & \text{if } n \geq 1. \end{cases} \tag{3.85}$$

The strategy is to iterate the above hypotheses and prove that, in the bitter end, C_n and ϵ_n can be chosen in an n-independent fashion, *uniformly in g*.

(ii a) The off-diagonal $\tilde{\pi}_{n+1,\beta}(\kappa)$. If $\tilde{\beta} \in \mathbb{C}^d$, then

$$|\tilde{\pi}_{n+1,\tilde{\beta}}(\kappa; p, q)|\, e^{-\mathfrak{Im}\,(\beta-\tilde{\beta})\cdot(p-q)} w_n(p)^{-1} w_n(q)^{-1} \leq \|\tilde{\pi}_{n+1,\beta}(\kappa)\|_{n;-n}.$$

But with a modification of (3.75),

$$\|r_{n\beta}(\kappa)\|_{-n;-n} \le 4C_\Gamma C_n|\epsilon|\tilde{B}_n \quad \text{where} \quad \tilde{B}_n := \begin{cases} g & \text{if } n = 0, \\ e^{-c\mu^n} & \text{if } n \ge 1, \end{cases} \tag{3.86}$$

such that

$$\|\tilde{\pi}_{n+1,\beta}(\kappa)\|_{n;-n} \le \|r_{n\beta}(\kappa)\pi_{n\beta}(\kappa)\|_{n;-n} \le 2C_n|\epsilon|g\,\aleph^n\tilde{B}_n$$

both provided ϵ meets the condition

$$|\epsilon| \le \epsilon_{n+1} := \max\left(\epsilon_n, \tfrac{1}{2}(A_nC_nC_\Gamma)^{-1}\right). \tag{3.87}$$

Hence, if $|\mathfrak{Im}\,\beta| < \alpha_n$,

$$|\tilde{\pi}_{n+1,\tilde{\beta}}(\kappa;p,q)| \le 2C_n|\epsilon|g\,\aleph^n\tilde{B}_n \cdot (1-\delta_{p,q})\,e^{\mathfrak{Im}\,(\beta-\tilde{\beta})\cdot(p-q)}w_n(p)w_n(q).$$

Now assume $|\mathfrak{Im}\,\tilde{\beta}| < \alpha_{n+1}$ and, fixing p and q, take

$$\beta = \tilde{\beta} + i(\alpha_n - \alpha_{n+1})\frac{p-q}{|p-q|}.$$

Obviously $|\mathfrak{Im}\,\beta| < \alpha_n$. What we get this way is

$$|\tilde{\pi}_{n+1,\tilde{\beta}}(\kappa;p,q)| \le 2C_n|\epsilon|g\,\aleph^n\tilde{B}_n \cdot (1-\delta_{p,q})\,e^{-(\alpha_n-\alpha_{n+1})|p-q|}w_n(p)w_n(q)$$

for each pair $(p,q) \in \mathbb{Z}^d \times \mathbb{Z}^d$. Thus, from the expression (3.62) for the norm,

$$\begin{aligned} &\|\tilde{\pi}_{n+1,\tilde{\beta}}(\kappa)\|_{n+1;-(n+1)} \\ &\qquad \le 2C_n|\epsilon|g\,\aleph^n\tilde{B}_n \sup_{q\in\mathbb{Z}^d} \sum_{p\in\mathbb{Z}^d\setminus\{q\}} e^{-4(n+3)^{-2}\alpha_n|p-q|}\frac{w_n(p)w_n(q)}{w_{n+1}(p)w_{n+1}(q)} \\ &\qquad \le 2C_n|\epsilon|g\,\aleph^n\tilde{B}_n \sum_{p\in\mathbb{Z}^d\setminus\{0\}} e^{-4(n+3)^{-2}\alpha_n|p|}w_{n+1}(p)^{-(1-\aleph)}. \end{aligned} \tag{3.88}$$

After (3.60), the second inequality follows from shifting p to $p+q$. We control the above bound by treating the cases $|\omega\cdot p| \le \aleph^{(n+1)/2}$ and $|\omega\cdot p| > \aleph^{(n+1)/2}$ separately. In fact, if $|\omega\cdot p| \le \aleph^{(n+1)/2}$, then $|p| > \aleph^{-(n+1)/2\nu}$ follows from (1.13), and

$$e^{-4(n+3)^{-2}\alpha_n|p|} < e^{-2n^{-2}\alpha_n|p|}\cdot e^{-2(n+1)^{-2}\alpha_n\aleph^{-(n+1)/2\nu}}, \quad w_{n+1}(p)^{-(1-\aleph)} < 1,$$

whereas

$$|\omega\cdot p| > \aleph^{(n+1)/2} \quad \Longrightarrow \quad w_{n+1}(p)^{-(1-\aleph)} < e^{-(1-\aleph)\aleph^{-(n+1)/2}}.$$

Since $\alpha_n > \alpha_0/6$ by (3.68) and, for $a > 1$ and $m > 0$,

$$m^{-2}a^m \geq \frac{e^2}{4}(\ln a)^2,$$

we have

$$e^{-2(n+1)^{-2}\alpha_n \aleph^{-(n+1)/2\nu}} \leq e^{-\frac{1}{12}e^2\alpha_0 \ln(\aleph^{-1/4\nu})\,\aleph^{-(n+1)/4\nu}}$$

The remaining d-dimensional geometric series satisfies

$$\sum_{p\in\mathbb{Z}^d\setminus\{0\}} e^{-2(n+1)^{-2}\alpha_0|p|} \leq C(d)\left(\frac{(n+1)^2}{\alpha_0}\right)^d.$$

Hence, we infer that if $|\Im\mathfrak{m}\,\beta| < \alpha_{n+1}$ and $\kappa \in D_n$, then

$$\|\tilde{\pi}_{n+1,\beta}(\kappa)\|_{n+1;-(n+1)} \leq C_{n+1}|\epsilon|g\,\aleph^{n+1}e^{-c\mu^{n+1}},$$

where we finally pin down the values of the previously free parameters $c := \frac{1}{2}\min\left(\frac{1}{12}e^2\alpha_0\ln(\aleph^{-1/4\nu}), 1-\aleph\right) > 0$ and $\mu := \aleph^{-1/\max(4\nu,2)} > 1$, and take

$$C_{n+1} \geq 2C(d)\,\aleph^{-1}e^{-c\mu^{n+1}}\,\tilde{B}_n\left(\frac{(n+1)^2}{\alpha_0}\right)^d C_n. \tag{3.89}$$

(ii b.1) The non-constant part $\bar{P}_{n+1}(\kappa)$ of the diagonal. If $\kappa \in D_{n+1}$ and $|\omega\cdot q| < \aleph^n(1-\aleph)b$, then $\kappa + \omega\cdot q \in D_n$. So, by (3.60),

$$\begin{aligned}
\|\bar{P}_{n+1}(\kappa)\|_{n+1;-(n+1)} &= \sup_{q\in\mathbb{Z}^d}|\bar{P}_{n+1}(\kappa;q)|w_{n+1}(q)^{-2}\\
&\leq \max\left\{\sup_{|\omega\cdot q|<\aleph^n(1-\aleph)b}\frac{|\bar{p}_{n+1}(\kappa+\omega\cdot q)|}{w_{n+1}(q)^2},\ \sup_{|\omega\cdot q|\geq\aleph^n(1-\aleph)b}|\bar{P}_{n+1}(\kappa;q)|w_n(q)^{-2/\aleph}\right\}\\
&\leq \max\left\{\sup_{|\omega\cdot q|<\aleph^n(1-\aleph)b}\frac{|\bar{p}_{n+1}(\kappa+\omega\cdot q)|}{w_{n+1}(q)^2},\ e^{-2b\aleph^{-1}(1-\aleph)^2}\|\bar{P}_{n+1}(\kappa)\|_{n;-n}\right\}.
\end{aligned}$$

But we know that the relations $\|\bar{P}_{n+1}(\kappa)\|_{n;-n} \leq \|P_{n+1}(\kappa)\|_{n;-n} + |\delta_{n+1}|$ and $|\delta_{n+1}| = |P_{n+1}(0;0)| \leq \|P_{n+1}(0)\|_{n;-n}$ hold. Moreover, (3.84) and (3.66) yield

$$\|P_{n+1}(\kappa)\|_{n;-n} \leq \|\pi_{n+1,\beta}(\kappa)\|_{n;-n} \leq 2\|\pi_{n\beta}(\kappa)\|_{n;-n} \leq 2A_nC_n|\epsilon|g\,\aleph^n,$$

assuming (3.87) and $\kappa \in D_n \supset D_{n+1}$ hold. Observe that, for positive x and p, $x^{-1}e^{-x^{-p}/(ep)} \leq 1$. Consequently, if we demand that

$$\aleph \leq \min\left(\tfrac{1}{8}, b^2\right), \tag{3.90}$$

say, and

$$C_{n+1} \geq A_nC_n, \tag{3.91}$$

it remains to be proven that

$$\sup_{\substack{\kappa\in D_{n+1}\\ |\omega\cdot q|<\aleph^n(1-\aleph)b}} \frac{|\bar{p}_{n+1}(\kappa+\omega\cdot q)|}{w_{n+1}(q)^2} \le C_{n+1}|\epsilon|g\,\aleph^{n+1}. \tag{3.92}$$

Notice that the rather arbitrary (3.90) imposes an interrelation between $\aleph$ and g, which is needed in the limit $b \le g/3 \to 0$; since we cannot take b large, we have to take $\aleph = o(b)$ in order to guarantee $e^{-2b\aleph^{-1}(1-\aleph)^2} \le \aleph/4$ above.

In order to verify (3.92), we use the recursion formula

$$\bar{p}_{n+1} - \bar{p}_n \;=\; p_n\,\gamma_n\,a_n\,p_n + s_n(\,\cdot\,;0) - s_n(0;0) \tag{3.93}$$

subject to

$$a_n := (1-p_n\gamma_n)^{-1} \quad \text{and} \quad \gamma_n(\kappa) := \Gamma_n(\kappa;0),$$

which is an advocate of (3.79). The bound (3.58) yields

$$|\gamma_n(\kappa)| \le C_\Gamma g^{-1}\aleph^{-n}|\aleph^{-n}\kappa|^5 \qquad (\kappa\in D_n). \tag{3.94}$$

By virtue of $|s_n(\kappa;0)| \le \|r_n(\kappa)\pi_n(\kappa)\|_{n;-n}$, (3.86) gives

$$|s_n(\kappa;0)| \le 4C_n^2 C_\Gamma|\epsilon|^2 \begin{cases} g^3 & \text{if } n=0,\\ A_n\,g\,\aleph^n e^{-c\mu^n} & \text{if } n\ge 1,\end{cases} \tag{3.95}$$

in D_n.

(ii b.2) The Taylor expansion of $\bar{p}_{n+1}(\kappa) \equiv \bar{P}_{n+1}(\kappa;0)$. Let us abbreviate

$$\sigma_n(\kappa) \equiv \bar{p}_n(\kappa) - \bar{p}_n'(0)\kappa,$$

for each natural number n. The objective is to show that the estimates

$$|\bar{p}_{n+1}'(0)| \le \left(1-\frac{1}{n+3}\right)\frac{C_{n+1}|\epsilon|^{3/2}g}{2}, \tag{3.96}$$

i.e., the iterate of (3.81), and

$$\sup_{\kappa\in D_n}|(\sigma_{n+1}-\sigma_n)(\kappa)| \le C_{n+1}|\epsilon|^{7/4}g\,\aleph^{n+1} \tag{3.97}$$

hold. Indeed, with the aid of such bounds together with (3.82), (3.92) follows from

$$\sup_{x\ge 0}\,(x+|\kappa|)^k e^{-\alpha x} = \left(\frac{k}{\alpha}\right)^k e^{\alpha|\kappa|-k} \qquad (\alpha>0)$$

for $k=1,2$ and ϵ suitably small. Moreover, the Cauchy estimate

$$|\sigma_{n+1}(\kappa)| \le |\sigma_n(\kappa)| + b^{-2}|\kappa|^2\,\frac{\aleph^{-2n}}{1-\aleph}\,\sup_{\zeta\in D_n}|(\sigma_{n+1}-\sigma_n)(\zeta)| \qquad (\kappa\in D_{n+1}),$$

implies that also (3.82) gets successfully iterated.

The bound in (3.94) implies

$$\gamma_n(0) = \gamma_n'(0) = 0,$$

such that $\bar{p}_{n+1}'(0) = \bar{p}_n'(0) + s_n'(0;0)$ according to (3.93), and hence

$$\bar{p}_{n+1}'(0) - \bar{p}_n'(0) = \frac{1}{2\pi i}\oint_{\partial D_n} \frac{s_n(\zeta;0)}{\zeta^2}\, d\zeta.$$

Thus, resorting to (3.95),

$$|\bar{p}_{n+1}'(0) - \bar{p}_n'(0)| \le \aleph^{-n} b^{-1} \sup_{\kappa \in D_n} |s_n(\kappa, 0)| \le \frac{C_{n+1}|\epsilon|^{3/2} g}{2(n+2)(n+3)},$$

if the constant C_{n+1} satisfies

$$C_{n+1} \ge 8(n+2)(n+3) C_\Gamma b^{-1} A_n \tilde{B}_n |\epsilon|^{1/2} C_n^2. \tag{3.98}$$

The bound (3.96) now follows, assuming also $C_n \le C_{n+1}$.

We still need to demonstrate (3.97). This will be provided by (3.93), since then

$$\sigma_{n+1}(\kappa) - \sigma_n(\kappa) = (p_n\, \gamma_n\, a_n\, p_n)(\kappa) + s_n(\kappa;0) - \sum_{l=0}^{1} s_n^{(l)}(0;0)\frac{\kappa^l}{l!},$$

such that (3.83), (3.94) and (3.95) yield (3.97) if

$$C_{n+1} \ge 4 C_\Gamma \aleph^{-1}\big(B_n^2 b^5 + 6 A_n \tilde{B}_n\big)|\epsilon|^{1/4} C_n^2 \tag{3.99}$$

Intuition behind (ii b.1–2). Due to the super-exponential decay of $s_n(\kappa;0)$ in (3.95) and the strong induction hypothesis $|\delta_n| \le C_0|\epsilon| g\, \aleph^{2n}$ on the constant part of p_n, the flow of the remainder $\bar{p}_n = p_n - \delta_n$ reads roughly

$$\bar{p}_{n+1} \approx (1 - \bar{p}_n \gamma_n)^{-1} \bar{p}_n, \tag{3.100}$$

by (3.76). Hence, the *a priori* bound $|(1-\bar{p}_n\gamma_n)^{-1}| \le 1 + C|\epsilon|$ yields a sequence diverging in n, with very little hope of proving bounds such as (3.73)—see (3.92). However, the support of γ_k is highly concentrated on the interval $[\aleph^{k+1} b, \aleph^k b]$. Iterating for $n \ge k$ steps, with κ on the latter interval,

$$\bar{p}_{n+1}(\kappa) \approx (1 - \bar{p}_1(\kappa)\gamma_k(\kappa))^{-1} \bar{p}_1(\kappa) = (1 + \mathcal{O}(\epsilon))\, \bar{p}_1(\kappa).$$

That is, $\bar{p}_n$ remains close to $\bar{p}_1$, which enables proving (3.73) through (3.92).

In fact, our argument is different still: since $\chi_n(\aleph\,\kappa) = \chi_{n-1}(\kappa)$ and $G(\aleph\,\kappa;0) \approx \aleph^{-1} G(\kappa;0)$, we have $\gamma_n(\aleph\,\kappa) \approx \aleph^{-1}\gamma_{n-1}(\kappa)$ for $n \ge$

2. Inserting this into (3.100), we notice that the *approximate scaling invariance*

$$\bar{p}_{n+1}(\aleph\,\kappa) \approx \aleph\,\bar{p}_n(\kappa)$$

is consistent with the flow. This is what the bounds (3.81)–(3.82) reflect.

(ii c) The constant part δ_{n+1} of the diagonal. Recall that γ may be viewed as a function of δ_n by the induction hypotheses; the identity $\delta_n = \Delta_n(\gamma)$ is bijective on J_n. The flow produces a *near-identity analytic* function $\delta_{n+1} = \delta_n + d_n(\delta_n)$ of δ_n on the disk I_n, such that, for ϵ small enough,

$$\delta_{n+1}(I_n) \supset I_{n+1}. \tag{3.101}$$

The analyticity of the map $\delta_n \mapsto d_n$ can be read off (3.78) and the expression of r_n. As far as estimates are concerned,

$$|d_n| \le \|r_n(0)\tilde{\pi}_n(0)\|_{n;-n} \le CC_n^2 C_\Gamma\,|\epsilon|^2 g \begin{cases} g^2 & \text{if } n = 0, \\ \aleph^n e^{-2c\mu^n} & \text{if } n \ge 1, \end{cases}$$

in the complex neighbourhood $2I_n$ of I_n of radius $\frac{1}{2}|I_n|$, where $|I_n|$ is the diameter of the disk I_n. Consequently, a Cauchy estimate yields the bound

$$\sup_{\delta_n \in I_n} |\partial d_n/\partial \delta_n| \le \frac{\sup_{\delta_n \in 2I_n} |d_n|}{\frac{1}{2}|I_n|} \le \tfrac{1}{2} \tag{3.102}$$

on the Lipschitz constant of d_n on I_n, provided $|\epsilon| \le \epsilon_{n+1}$ with

$$\epsilon_{n+1}^{-1} \ge 2C_0^{-1} CC_n^2 C_\Gamma \begin{cases} g & \text{if } n = 0, \\ \aleph^{-n} e^{-2c\mu^n} & \text{if } n \ge 1. \end{cases} \tag{3.103}$$

In this case also

$$|d_n| \le \tfrac{1}{2}|I_n| - \tfrac{1}{2}|I_{n+1}| \tag{3.104}$$

holds, which validates (3.101), considering how the boundary of I_n is transformed under δ_{n+1}.

Notice that (3.102) implies

$$\bigl|\delta_{n+1}(x) - \delta_{n+1}(y)\bigr| \ge \tfrac{1}{2}|x - y| \qquad (x, y \in I_n), \tag{3.105}$$

meaning that $\delta_n \mapsto \delta_{n+1}$ is *one-to-one*. By continuity and (3.101), there exists a closed set $\tilde{J}_{n+1} \subset I_n$ that is bijectively and analytically mapped onto I_{n+1}: $\tilde{J}_{n+1} := \delta_{n+1}^{-1}(I_{n+1})$. We can backtrack with the aid of the map Δ_n, obtaining a closed subset $J_{n+1} \subset I_\gamma$ (see (3.80)) that is bijectively and analytically mapped onto I_{n+1} by the map $\Delta_{n+1} := \delta_{n+1} \circ \Delta_n$:

$$J_{n+1} := \Delta_{n+1}^{-1}(I_{n+1}).$$

It follows immediately that

$$J_{n+1} \subset J_n.$$

(iii) Large values of n and the limit $g \to 0$. Suppose C_n is independent of g, which is the case for C_0. The recursive conditions (3.89), (3.91), (3.98), and (3.99) can be summarized in bounds of the form

$$C_{n+1} \geq K_n(g)\, C_n \quad \text{and} \quad C_{n+1} \geq L_n(g)|\epsilon|^{1/4}\, C_n^2.$$

Choosing $b := g/3$ (due to (3.98)) and $\aleph := \min\left(\frac{1}{8}, b^2\right)$ (due to (3.99); see also (3.90)), which is allowed, we may bound $K_n(g)$ and $L_n(g)$ *uniformly in g*: $\sup_{0<g<g_0} K_n(g) \leq K_n$ and $\sup_{0<g<g_0} L_n(g) \leq L_n$. This follows from the fact that $\aleph^{-1}e^{-c\mu} \to 0$ as $\aleph \to 0$. Moreover, $L_n \leq L$ for each n, such that we may choose

$$C_{n+1} := \max\left(K_n,\ L|\epsilon|^{1/4}C_n\right) C_n.$$

The numbers $K_n > 1$ converge to unity so fast that the number

$$K := \prod_{n=0}^{\infty} K_n > 1$$

is finite. Now choose ϵ so small that

$$L|\epsilon|^{1/4}KC_0 \leq 1.$$

In particular, $C_1 = K_0C_0$, and inductively $C_n = K_0 \cdots K_{n-1}C_0 \leq KC_0$. We conclude that the sequences (C_n) and (ϵ_n) (see (3.87) and (3.103)) converge to positive numbers.

(iv) Fine-tuning the Lyapunov exponent γ. The maps δ_n are relatively expansive; (3.105) holds, while the target I_n contracts by a factor of $\aleph^2 < \frac{1}{2}$ at each step. Thus, demanding $\Delta_n(J_n) = I_n$ at each step for the map $\Delta_n = \delta_n \circ \cdots \circ \delta_0$ amounts to

$$|x-y| \leq 2^n\big|\Delta_n(x) - \Delta_n(y)\big| \leq Cg\,(2\aleph^2)^n \qquad (x, y \in J_n),$$

or $\lim_{n\to\infty}|J_n| = 0$. Because the J_n's form an ever decreasing chain of closed disks, their intersection consists of precisely one point:

$$\{\gamma\} \equiv \bigcap_{n=0}^{\infty} J_n \subset I_\gamma.$$

The value of γ is an analytic function of ϵ, because the sequence $\Delta_n^{-1}(0)$ converges uniformly to γ with respect to ϵ. For real values of ϵ, Δ_n sends reals to reals, making γ real. □

Proof of Theorem 3.4. With x_n as in (3.49), the task is to show that the limiting function ξ—see (3.50)—is an analytic solution to (3.29).

Given the formal definition $y_\beta := \tau_\beta y$, (3.49) implies

$$x_{n\beta} = f_{n\beta}(0).$$

Recalling (3.46) and (3.47), one clearly has

$$f_{n+1,\beta} = f_{n\beta} \circ \Xi_{n\beta} \quad \text{and} \quad f_{n\beta} = \Xi_{0\beta} \circ \Xi_{1\beta} \circ \cdots \circ \Xi_{n-1,\beta}.$$

Hence, the recursion relation

$$x_{n+1,\beta} = x_{n\beta} + \big(f_{n\beta}(\Xi_{n\beta}(0)) - f_{n\beta}(0)\big)$$

follows. Here (3.43) extends to

$$\Xi_{n\beta}(y) \equiv (\mathbb{1} - \Gamma_n \pi_{n\beta})^{-1}(y + \Gamma_n \rho_{n\beta}). \tag{3.106}$$

Notice that the flows of $\rho_{n\beta}$ and $4\pi_{n\beta}(\,\cdot\,, 0)$ [4] are identical. Furthermore, the initial conditions agree according to (3.27), such that

$$\hat{\rho}_{n\beta}(q) \equiv 4\pi_{n\beta}(q, 0).$$

Due to the absence of second and higher order terms in $y \mapsto \Xi_{n\beta}(y)$,

$$D\Xi_{n\beta}(y) \equiv (\mathbb{1} - \Gamma_n \pi_{n\beta})^{-1},$$

such that the chain rule reveals

$$Df_{n\beta}(y) \equiv (\mathbb{1} - \Gamma_0 \pi_{0\beta})^{-1}(\mathbb{1} - \Gamma_1 \pi_{1\beta})^{-1} \cdots (\mathbb{1} - \Gamma_{n-1}\pi_{n-1,\beta})^{-1}.$$

Recursive implementation of Corollary 3.16 in the form

$$\|(\mathbb{1} - \Gamma_n \pi_{n\beta})^{-1}\|_{n;n-1} \le \|(\mathbb{1} - \Gamma_n \pi_{n\beta})^{-1}\|_{n;n} \le 2$$

implies that $Df_{n\beta}(y) \in \mathcal{L}(h_{n-1}; h_0)$ with

$$\sup_{y \in h_n} \|Df_{n\beta}(y)\|_{n-1;0} \le 2^n.$$

The Mean-Value Theorem ensures the existence of an element $y_0 \in h_n$ for which

$$\|f_{n\beta}(\Xi_{n\beta}(0)) - f_{n\beta}(0)\|_0 \le \|Df_{n\beta}(y_0)\|_{n-1;0}\, \|\Xi_{n\beta}(0)\|_{n-1}$$

and we go on to estimate

$$\|x_{n+1,\beta} - x_{n\beta}\|_0 \le 2^n \|\Xi_{n\beta}(0)\|_n \tag{3.107}$$

with the aid of the inequality $\|\cdot\|_{n-1} \le \|\cdot\|_n$.

[4] $\pi_{n\beta}(\,\cdot\,, 0)$ is shorthand for the function $\theta \mapsto \sum_q e^{iq\cdot\theta}\pi_{n\beta}(q, 0)$ through the identification of a function with its Fourier transform.

Lemma 3.17. *For parameters as in Lemma 3.13 and ϵ_0 small, we may perceive $\Xi_{n\beta}$ as an analytic map from h_n to $h_n \subset h_{n-1}$ with*

$$\|\Xi_{n\beta}(0)\|_n \le C|\epsilon| \begin{cases} g & \text{if } n = 0, \\ e^{-c\mu^n} & \text{if } n \ge 1. \end{cases}$$

Proof. Since Γ_n annihilates the zero mode ($\Gamma_n(0) = 0$), $\Gamma_n\rho_{n\beta} = 4(\Gamma_n\tilde{\pi}_{n\beta})(\,\cdot\,,0)$, which is super-exponentially small in the norm $\|\cdot\|_n$ by $\|\tilde{\pi}_{n\beta}(\,\cdot\,,0)\|_{-n} \le \|\tilde{\pi}_{n\beta}\|_{n;-n}$ and Lemma 3.13. According to (3.106), Lemma 3.17 clearly holds if we take ϵ small enough so as to validate $\|\Gamma_n\|_{-n;n}\|\pi_{n\beta}\|_{n;-n} \le \frac{1}{2}$, say, for each n. For the bounds on $\pi_{n\beta}$ and Γ_n we refer the reader to Corollary 3.16 and (3.65), respectively. □

By Lemma 3.17, $f_{n\beta}$ maps h_{n-1} to h_0, confirming that $x_{n\beta} \in h_0$ for each n. Coming back to (3.107), we can now write down

$$\|x_{n+1,\beta} - x_{n\beta}\|_0 \le C|\epsilon|g\,2^n e^{-c\mu^n} \qquad (|\Im\mathfrak{m}\,\beta| < \alpha_n).$$

Thus, taking $|\Im\mathfrak{m}\,\beta| < \alpha^* := \alpha_0/6$ (see (3.68)), the sequence $(x_{n\beta})_{n\in\mathbb{N}}$ is Cauchy in the Banach space h_0. Moreover, $x_{0\beta} = 0$ gives us

$$\|\xi_\beta\|_0 \le \sum_{n=0}^{\infty} \|x_{n+1,\beta} - x_{n\beta}\|_0 \le C|\epsilon|g,$$

which implies the bound

$$|\hat{\xi}(q)| \le C|\epsilon|g\,e^{-\alpha^*|q|} \qquad (q \in \mathbb{Z}^d).$$

We infer that ξ is real-analytic on $\mathbb{T}^d$.

Recalling that $\lim_{n\to\infty}\Gamma_{<n}(q) = G(q)$ for each $q \in \mathbb{Z}^d$, let us take the *pointwise* limit $n \to \infty$ of (3.29) in the Fourier representation:

$$\begin{aligned}\hat{\xi}(q) &= \lim_{n\to\infty}\Gamma_{<n}(q)\left(\widehat{\pi_0 x_n}(q) + \hat{\rho}_0(q)\right) = G(q)\lim_{n\to\infty}\left(\widehat{\pi_0 x_n}(q) + \hat{\rho}_0(q)\right) \\ &= G(q)\big(\widehat{\pi_0\xi} + \hat{\rho}_0\big)(q),\end{aligned}$$

because π_0 is a continuous operator on h_0. Indeed, ξ solves (3.29)!

Quite similarly—and merely out of curiosity—we conclude by the day-saving recursion invariance (3.41) that

$$\xi_n = G_n(\pi_n\xi_n + \rho_n) = G_n(\pi_0\xi + \rho_0) \tag{3.108}$$

converges to $\xi^* = 0$, pointwise in terms of the Fourier representation. Hence, equation (3.36) really trivializes in the large-n-limit. Another way of seeing this is the pointwise bound $|\widehat{G_n\rho_n}(q)| \le C|G(q)|\|\tilde{\pi}_n\|_{n;-n}$, which tends to zero and paraphrases $G^*\rho^* = 0$ below (3.51).

We still need to demonstrate that the solution ξ of (3.25) also solves (3.26), *i.e.*, that

$$\big(\widehat{\pi_0\xi} + \hat\rho_0\big)(0) = \pi_0(0,\,\cdot\,)\hat\xi + \hat\rho_0(0) = \sum_{p\in\mathbb{Z}^d} \pi_0(0,p)\hat\xi(p) + \hat\rho_0(0) = 0.$$

From (3.108), $G_n(q) := \chi_n(\omega\cdot q)G(q)$, $G(0)=0$, and (3.29),

$$\begin{aligned}
\big|\widehat{\pi_n\xi_n}(0)\big| &\le \sum_{q\in\mathbb{Z}^d} |\tilde\pi_n(0,q)|\chi_n(\omega\cdot q)\,\big|G(q)\big(\widehat{\pi_0\xi}+\hat\rho_0\big)(q)\big| \\
&\le \|\xi\|_0 \sup_{q\in\mathbb{Z}^d} |\tilde\pi_n(0,q)|\chi_n(\omega\cdot q) \\
&\le \|\xi\|_0\,\|\tilde\pi_n\|_{n;-n} \sup_{q\in\mathbb{Z}^d\setminus\{0\}} w_n(q)\chi_n(\omega\cdot q) \\
&\le C\|\xi\|_0\,\|\tilde\pi_n\|_{n;-n} \quad\longrightarrow\quad 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

Thus,

$$\big(\widehat{\pi_0\xi} + \hat\rho_0\big)(0) = \lim_{n\to\infty}\big(\widehat{\pi_n\xi_n} + \hat\rho_n\big)(0) = \lim_{n\to\infty}\hat\rho_n(0).$$

But

$$\lim_{n\to\infty}\hat\rho_n(0) = 4\lim_{n\to\infty}\pi_n(0,0) = 4\lim_{n\to\infty}\delta_n = 0,$$

and we are done.

If X_1 solves (3.1), it is a matter of applying the translation τ_β on both sides of the equation to get $(\mathcal{D}+\gamma)^2X_1' = D\Omega(X_0')X_1'$, where $X_0' = \tau_\beta X_0 + (0,\beta)$ and $X_1' = \tau_\beta X_1$. In other words, the translation property in the formulation of the theorem holds, and the value of γ does not change under such translations. □

Chapter 4

Proof of Theorem 1

LET us summarize what we have learned thus far. The general solution $X^0(z,\theta) \equiv X^0(z)$ to the equations of motion in the uncoupled case was found. In the coupled case we resolved KAM-type small denominator issues, which contributed the $t \to -\infty$ $(z = 0)$ asymptotic $X_0(\theta)$ of the general solution $X(z,\theta)$, as well as the linearization $X_1(\theta) \equiv \partial_z X(0,\theta)$.

We can now solve (1.11), and thus find the unstable manifold $\mathcal{W}^u_\lambda$ also "far away" from the torus $\mathcal{T}_\lambda$.

To begin with, we single out the uncoupled part X^0 of the complete solution X;

$$X = X^0 + \widetilde{X} \quad \text{with} \quad \widetilde{X}|_{\epsilon=0} \equiv 0.$$

Equation (1.11) now becomes

$$\mathcal{L}^2 \widetilde{X} = -(\gamma^2 \sin \Phi^0, 0) + \Omega(X^0 + \widetilde{X}),$$

as $\mathcal{L}^2 X^0 = (\gamma^2 \sin \Phi^0, 0)$ follows from (1.7) for $\mathcal{L}^2 X$ and the fact that $\phi(t) = \Phi^0(e^{\mu t})$ solves the equations $\dot{\phi} = I$ and $\dot{I} = \mu^2 \sin \phi$—see (1.2)—for any nonnegative μ, especially for $\mu = \gamma$. In other words, the map $\widetilde{X}$ has to satisfy

$$\mathcal{K}\widetilde{X} = \widetilde{W}(\widetilde{X}), \tag{4.1}$$

defining the linear operator

$$\mathcal{K} := \begin{pmatrix} L & 0 \\ 0 & \mathcal{L}^2 \end{pmatrix} \quad \text{with} \quad L := \mathcal{L}^2 - \gamma^2 \cos \Phi^0 \tag{4.2}$$

and the nonlinear operator $\widetilde{W}$ through the expression

$$\widetilde{W}(\widetilde{X}) := (-\gamma^2 \sin \Phi^0 - \gamma^2 (\cos \Phi^0)\widetilde{\Phi}, 0) + \Omega(X^0 + \widetilde{X}). \tag{4.3}$$

Throughout the rest of the work, we shall refer to different parts of the Taylor expansion of a suitable function $h(z,\theta)$ around $z = 0$ using the notation

$$h_k(\theta) := \frac{\partial_z^k h(0,\theta)}{k!}, \quad h_{\le k}(z,\theta) := \sum_{j=0}^{k} z^k h_k(\theta) \quad \text{and} \quad \delta_k h := h - h_{\le k-1}.$$

Observe that $X_0 = \widetilde{X}_0$ and $X_1 = (4,0) + \widetilde{X}_1$. Consequently, if $\widetilde{X}$ exists, there exists a map $Z : [0,1] \times \mathbb{T}^d \to \mathbb{R}^{d+1}$ (for real ϵ) satisfying

$$\widetilde{X}(z,\theta) \equiv X_{\le 1}(z,\theta) - (4,0)z + Z(z,\theta) \tag{4.4}$$

or, equivalently,

$$Z = \delta_2 \widetilde{X}.$$

We may transform equation (4.1) into the equation

$$\mathcal{K}Z = W(Z) \tag{4.5}$$

for Z, where we define W through

$$W(Z) := \delta_2 \left[\widetilde{W}(\widetilde{X}) + \begin{pmatrix} \gamma^2 (\cos \Phi^0) \widetilde{\Phi}_{\le 1} \\ 0 \end{pmatrix} \right], \tag{4.6}$$

taking now (4.4) as the *definition of* $\widetilde{X}$.

Let us consider the complex Banach space $\mathcal{A}$ of (bounded) analytic functions Z on the compact set

$$\Pi_\tau := \left\{ (z,\theta) \;\middle|\; \mathfrak{Re}\,(z,\theta) \in [-\tau, 1+\tau] \times \mathbb{T}^d,\ \mathfrak{Im}\,(z,\theta) \in [-\tau,\tau]^{d+1} \right\},$$

$\tau \ge 0$, equipped with the supremum norm, and its closed subspace

$$\mathcal{A}_1 := \{ Z \in \mathcal{A} \mid Z_{\le 1} = 0 \}. \tag{4.7}$$

For future use, let us also define the closed origin-centered balls

$$B(R) := \{ Z \in \mathcal{A} \mid \|Z\|_\infty \le R \} \quad \text{and} \quad B_1(R) := B(R) \cap \mathcal{A}_1.$$

Any element of $\mathcal{A}$ extends analytically to $\Pi_{\tau'}$ for some $\tau' > \tau$, allowing uniform estimates on its derivatives *on* Π_τ.

Remark 4.1. Whereas equation (4.1) is plagued by small denominators, equation (4.5) is not. This is so due to the decomposition (4.4) which separates the previously solved "KAM-asymptotics" $X_{\le 1}$ from $\widetilde{X}$ and enables reducing (4.1) to (4.5) on the space $\mathcal{A}_1$, which one could well call the small-denominator-free subspace of $\mathcal{A}$.

4.1. Existence and uniqueness of Z

Postponing the proofs until the end of this chapter, we make two observations, important in demonstrating that (4.5) is solvable.

LEMMA 4.2. *With sufficiently small R, τ, and ϵ (depending on the analyticity region of f), the operator $W : \mathcal{A} \to \mathcal{A}_1$ maps the ball $B(R)$ in $\mathcal{A}$ into a ball $B_1(R')$ in $\mathcal{A}_1$ with $R' = Cg^2(R^2 + |\epsilon|)$, and $W|_{\mathcal{A}_1}$ is Lipschitz continuous on $B_1(R)$ with a Lipschitz constant proportional to $g^2(R + |\epsilon|)$. If the restriction of $Z \in \mathcal{A}$ to a* real *neighbourhood of $[0,1] \times \mathbb{T}^d$ has the* real *range $\mathbb{R} \times \mathbb{R}^d$ and ϵ is real, then the same is true of $W(Z)$.*

LEMMA 4.3. *If $0 < \tau < 1$, the linear operator $\mathcal{K} : \mathcal{A}_1 \to \mathcal{A}_1$ has a bounded inverse $\mathcal{K}^{-1} \in \mathcal{L}(\mathcal{A}_1)$ obeying $\|\mathcal{K}^{-1}\|_{\mathcal{L}(\mathcal{A}_1)} \leq C\gamma^{-2}\tau^{-1}(1 - \tau^2)^{-2}$. It preserves analyticity in ϵ. If the restriction of $Z \in \mathcal{A}$ to a* real *neighbourhood of $[0,1] \times \mathbb{T}^d$ has the* real *range $\mathbb{R} \times \mathbb{R}^d$, the same is true of $\mathcal{K}^{-1}Z$.*

We have developed enough machinery to extract a solution from (4.5):

THEOREM 4.4. *For sufficiently small R, $\epsilon_0 < R/2$, and τ (depending on the analyticity regions of f and $X_{\leq 1}$), equation* (4.5) *has a unique solution $Z \in B_1(R)$. It is continuous on D, analytic in ϵ, and bounded uniformly by $C|\epsilon|$. The restriction $Z|_{[0,1]\times\mathbb{T}^d}$ takes values in $\mathbb{R} \times \mathbb{R}^d$, provided ϵ is real.*

Proof. We know by Lemmata 4.2 and 4.3, that $\mathcal{K}^{-1}W$ maps $B_1(R)$ into itself:

$$\mathcal{K}^{-1}W(B_1(R)) \subset B_1(C(R^2 + \epsilon_0)) \subset B_1(R). \tag{4.8}$$

We may furthermore choose ϵ_0 and R such that the operator $\mathcal{K}^{-1}W$ becomes contractive on $B_1(R)$ (*i.e.*, Lipschitz with constant strictly less than 1). The assumptions of the Banach Fixed Point Theorem are now satisfied, implying that $\mathcal{K}^{-1}W$ has a unique fixed point Z in the ball $B_1(R)$.

The theorem also implies that Z is analytic in ϵ. Namely, Lemma 4.3 says that $\mathcal{K}^{-1}$ preserves such a property. Furthermore, the ϵ-dependence of W comes solely from γ, X_0, X_1, and Ω, making it analytic. Hence, the uniformly convergent sequence 0, $\mathcal{K}^{-1}W(0)$, $(\mathcal{K}^{-1}W)^2(0)$, ... of successive substitutions reveals the analyticity of the limit—the fixed point Z:

$$\|Z - (\mathcal{K}^{-1}W)^n(0)\|_\infty \leq \|(\mathcal{K}^{-1}W)(0)\|_\infty \frac{L^n}{1-L},$$

where $L \leq C(R+\epsilon_0) < 1$ is the Lipschitz constant of $\mathcal{K}^{-1}W$. If ϵ is real, this sequence consists of functions $\mathbb{R} \times \mathbb{R}^d$-valued on $[0,1] \times \mathbb{T}^d$, and the same goes for the limit Z. Finally,

$$\|Z\|_\infty \leq \|(\mathcal{K}^{-1}W)(Z) - (\mathcal{K}^{-1}W)(0)\|_\infty + \|(\mathcal{K}^{-1}W)(0)\|_\infty \leq L\|Z\|_\infty + C|\epsilon|$$

yields $\|Z\|_\infty \leq C|\epsilon|/(1-L)$. Here $(\mathcal{K}^{-1}W)(0)$ was bounded using R' of Lemma 4.2 at $R=0$. □

4.2. Putting it all together

To reach the statement of Theorem 1, we glue together the pieces provided by Theorems 2.2, 3.4, and 4.4.

Assuming $\langle \Psi_0 \rangle = 0$, we have constructed analytic maps γ and

$$X(z,\theta) = X_0(\theta) + zX_1(\theta) + \delta_2 X(z,\theta) \quad \text{with} \quad \delta_2 X = Z + \delta_2 X^0$$

that solve (1.11) in a complex neighbourhood of $[0,1] \times \mathbb{T}^d$ and satisfy the *physical constraint* $\Phi_1|_{\epsilon=0} = 4$. Recall now (1.16). Since (1.18) is not automatically satisfied, we are required to pinpoint specific values of α and β so as to fulfill $X_{\alpha,\beta}(1,0) = (\pi, 0)$. To this end, we utilize the Implicit Function Theorem.

First, set $\mathfrak{X}(\epsilon, g; \alpha, \beta) := X(\alpha,\beta) + (0,\beta) - (\pi, 0)$, which leaves us with the implicit equation $\mathfrak{X}(\epsilon, g; \alpha, \beta) = 0$. Both $\mathfrak{X}$ and $\frac{\partial \mathfrak{X}}{\partial(\alpha,\beta)}$ are continuous, and we get from $X = (\Phi^0, 0) + \mathcal{O}(\epsilon)$ that

$$\mathfrak{X}(0, g; 1, 0) = 0 \quad \text{and} \quad \det\left(\frac{\partial \mathfrak{X}(\epsilon, g; \alpha, \beta)}{\partial(\alpha,\beta)}\right) = \frac{4}{1+\alpha^2} + \mathcal{O}(\epsilon)$$

(*cf.* (1.23)) for $(\epsilon, g) \in D$ and for whichever values of α and β the map $\mathfrak{X}$ is well-defined. Hence, if we choose ϵ_0 small enough, there exist unique continuous functions α and β on D, analytic with respect to ϵ, such that $\alpha(0,g) = 1$, $\beta(0,g) = 0$, and

$$\mathfrak{X}(\epsilon, g; \alpha(\epsilon, g), \beta(\epsilon, g)) = 0.$$

Moreover, $\alpha(\epsilon, g) \in \mathbb{R}$ and $\beta(\epsilon, g) \in \mathbb{R}^d$ for ϵ real, as $\mathfrak{X}$ is then real-valued. A good reference here is [Chi96]. □

4.3. Proofs of Lemmata 4.2 and 4.3

We conclude the chapter by presenting the proofs of Lemmata 4.2 and 4.3 used in the proof of Theorem 4.4.

Proof of Lemma 4.2. Given $Z \in \mathcal{A}$ with $\|Z\|_\infty \leq R$, we study $W(Z)$—defined in (4.6), and clearly an element of $\mathcal{A}_1$. Notice that in the relation (4.4), expressing $\widetilde{X}$ in terms of Z, the maps X_0 and X_1 were previously determined and are independent of Z. Furthermore, taking advantage of (4.4) and Theorems 2.2 and 3.4, we deduce

$$\|\widetilde{X}\|_\infty \leq C(|\epsilon| + R). \tag{4.9}$$

With the aid of (1.10), cast equation (4.3) as

$$\widetilde{W}(\widetilde{X}) := (g^2 \sin(\Phi^0 + \widetilde{\Phi}) - \gamma^2 \sin \Phi^0 - \gamma^2 \cos(\Phi^0)\widetilde{\Phi}, 0) + \lambda\, \widetilde{\Omega}(X^0 + \widetilde{X}).$$

Recall that f is analytic on the strip $|\mathfrak{Im}\,\phi|, |\mathfrak{Im}\,\psi| \leq \eta$. Also, the imaginary part of $\Phi^0(z,\theta) \equiv 4 \arctan z = 4z + \mathcal{O}(z^3)$ is of order τ on Π_τ, when $\tau \ll 1$. Hence, owing to (4.9), our function $\widetilde{\Omega}(X^0 + \widetilde{X})$ is well-defined for λ and R sufficiently small and the strip Π_τ about $[0,1] \times \mathbb{T}^d$ narrow enough.

Since $\sin(\Phi^0 + \widetilde{\Phi}) = \sin \Phi^0 + \cos(\Phi^0)\widetilde{\Phi} + \mathcal{O}(\widetilde{\Phi}^2)$, in a neighbourhood of Π_τ

$$\|\widetilde{W}(\widetilde{X})\|_\infty \leq |g^2 - \gamma^2|\, \|\sin \Phi^0 + \cos(\Phi^0)\widetilde{\Phi}\|_\infty + Cg^2 \|\widetilde{\Phi}\|_\infty^2 + |\lambda|\, \|\widetilde{\Omega}(X^0 + \widetilde{X})\|_\infty.$$

The enticing factor $g^2 - \gamma^2$ is the reason we chose to subtract $\gamma^2 \cos(\Phi^0)\widetilde{\Phi}$ from both sides in equation (4.1), paying the price of making $\mathcal{K}$ and its inverse more complicated. Namely, $|g^2 - \gamma^2| = |2g + \gamma - g||g - \gamma| \leq (2g + \widetilde{C}g|\epsilon|)\, \widetilde{C}g|\epsilon| \leq Cg^2|\epsilon|$. Terms proportional to $\widetilde{\Phi}$ are dominated by (4.9). Thus, such estimates uniform in (z,θ) yield

$$\|W(Z)\|_\infty \leq Cg^2(R^2 + |\epsilon|)$$

for ϵ and R small—independently of each other and of g.

In order to obtain the Lipschitz continuity of $W|_{\mathcal{A}_1}$, it suffices to show that $Z \overset{(4.4)}{\mapsto} \widetilde{X} \mapsto \widetilde{W}(\widetilde{X})$ is Lipschitz, as neither $(\widetilde{W}(\widetilde{X}))_{\leq 1}$ nor $\widetilde{X}_{\leq 1}$ depend on $Z =: \delta_2 \widetilde{X}$. To that end, we use the Mean Value Theorem, see [Cha85], and conclude that for some $Z =: \delta_2 \widetilde{X}$ on the line segment between two points $Z' =: \delta_2 \widetilde{X}'$ and $Z'' =: \delta_2 \widetilde{X}''$ it holds that

$$\|\widetilde{W}(\widetilde{X}') - \widetilde{W}(\widetilde{X}'')\|_\infty \leq \|D\widetilde{W}(\widetilde{X})\|\, \|Z' - Z''\|_\infty.$$

Here the derivative is easily bounded by $Cg^2(R+|\epsilon|)$ when (4.9) holds and, therefore, when $\|Z\|_\infty \le R$.

From its explicit expression, one immediately recognizes that W preserves the class of functions whose restriction to $[0,1]\times\mathbb{T}^d$ has the *real* range $\mathbb{R}\times\mathbb{R}^d$, if ϵ is real. □

Proof of Lemma 4.3. $\mathcal{L}$ maps $\mathcal{A}_1$ into itself, and $\mathcal{K}$ in (4.2) inherits this feature.

Let us start with the "pendulum part" of $\mathcal{K}$, and solve

$$Lf = g$$

resorting to the method of characteristics; we first restrict ourselves to the case $(z,\theta) = (\zeta e^{\gamma t}, \vartheta + \omega t)$ (with fixed (ζ,ϑ)) in order to obtain an ordinary differential equation (ODE). Recalling the identity (1.7), we see that for an arbitrary ϑ

$$\left(\partial_t^2 - \gamma^2 \cos\Phi^0(\zeta e^{\gamma t})\right) f(\zeta e^{\gamma t}, \vartheta + \omega t) = g(\zeta e^{\gamma t}, \vartheta + \omega t), \tag{4.10}$$

and our task reduces to studying $L_t := \partial_t^2 - \gamma^2\cos\Phi^0(\zeta e^{\gamma t})$. Since a translation in t and ϑ eliminates ζ, we can just as well set $\zeta = 1$.

We proceed in the Fourier language. The function f solves equation (4.10) if and only if for an arbitrary $q\in\mathbb{Z}^d$ the functions $u(t) := e^{iq\cdot\omega t}\hat{f}(e^{\gamma t}, q)$ and $v(t) := e^{iq\cdot\omega t}\hat{g}(e^{\gamma t}, q)$ satisfy

$$L_t u = v.$$

Noticing that $\cos\Phi^0(e^{\gamma t}) = 2\tanh^2\gamma t - 1$, we see that L_t has got the zero mode

$$u_1(t) := (\cosh\gamma t)^{-1},$$

i.e., $L_t u_1 = 0$. Since $L_t u = 0$ is a linear second order ODE, there exists precisely one other zero mode u_2 of L_t that is linearly independent of u_1. Because $u_1(t)\neq 0$ for any $t\in\mathbb{R}$, u_2 may be found by a standard procedure:

$$u_2(t) := u_1(t)\int \frac{dt}{u_1^2(t)} = \frac{t}{2\cosh\gamma t} + \frac{\sinh\gamma t}{2\gamma},$$

omitting any additive constant emerging from the integral. Let us express the linear homogeneous equation $L_t u = 0$ as the first order system $\dot{U} = AU$ with $U := (u,\dot{u})^T$ and $A(t) := \left(\begin{smallmatrix} 0 & 1 \\ \gamma^2\cos\Phi^0(e^{\gamma t}) & 0\end{smallmatrix}\right)$. Then

$$w := \begin{pmatrix} u_1 & u_2 \\ \dot{u}_1 & \dot{u}_2 \end{pmatrix} \tag{4.11}$$

is a fundamental matrix solution of the system ($i.e.$, $\dot{w} = Aw$) with the Wronskian

$$\det w = u_1^2 \, \frac{d}{dt} \int \frac{1}{u_1^2(t)} \, dt \equiv 1,$$

and thus

$$w^{-1} = \begin{pmatrix} \dot{u}_2 & -u_2 \\ -\dot{u}_1 & u_1 \end{pmatrix} \quad \text{and} \quad w(t)w^{-1}(s) = \begin{pmatrix} * & u_2(t)u_1(s) - u_1(t)u_2(s) \\ * & * \end{pmatrix}.$$

In terms of a first order system, the complete equation $L_t u = v$ reads $\dot{U} = AU + V$, $V := (0, v)^T$. Varying constants,

$$U(t) = w(t) \left(w^{-1}(t_0)U(t_0) + \int_{t_0}^{t} w^{-1}(s)V(s) \, ds \right).$$

Next, we get rid of the dummy parameter t_0 by taking $t_0 \to -\infty$. In that limit $u(t_0) = \mathcal{O}(e^{2\gamma t_0})$, such that

$$w^{-1}(t_0)U(t_0) \sim \frac{1}{4e^{\gamma t_0}} \left(u(t_0) + \gamma^{-1}\dot{u}(t_0) \, , \, 0 \right)^T$$

tends rapidly to zero. Therefore, for $t \leq 0$,

$$u(t) = \int_{-\infty}^{t} \left[u_2(t)u_1(s) - u_1(t)u_2(s) \right] v(s) \, ds,$$

or equivalently

$$\hat{f}(e^{\gamma t}, q) = \int_{-\infty}^{0} \widetilde{K}_\Phi(s; e^{\gamma t}) \, \hat{g}(e^{\gamma t} e^{\gamma s}, q) \, e^{iq \cdot \omega s} \, ds \tag{4.12}$$

in terms of the kernel

$$\widetilde{K}_\Phi(s; z) := \mathcal{W}_{\Phi 2}(z)\mathcal{W}_{\Phi 1}(ze^{\gamma s}) - \mathcal{W}_{\Phi 1}(z)\mathcal{W}_{\Phi 2}(ze^{\gamma s}),$$

defined on $\{(s, z) \in \mathbb{R} \times \mathbb{C} \,|\, z \notin \{0, \pm i, \pm ie^{-\gamma s}\}\}$, where

$$\mathcal{W}_{\Phi 1} := 2P \quad \text{and} \quad \mathcal{W}_{\Phi 2} := \gamma^{-1} P \, \ln + \frac{1}{4}\gamma^{-1} Q$$

with the auxiliary functions

$$P(z) := (z^2 + 1)^{-1} z \quad \text{and} \quad Q(z) := z^{-1}(z^2 - 1). \tag{4.13}$$

This is so, because $\mathcal{W}_{\Phi j}(e^{\gamma t}) \equiv u_j(t)$, which also yields

$$L\mathcal{W}_{\Phi j} = L_t u_j = 0 \qquad (j = 1, 2). \tag{4.14}$$

For a fixed s, the function $\widetilde{K}_\Phi(s; z)$ has a removable singularity at $z = 0$. Accordingly, we define $\widetilde{K}_\Phi(s; 0) := \lim_{z \to 0} \widetilde{K}_\Phi(s; z)$ for each s.

In a complex strip $|\Im\mathfrak{m}\, z| \leq \tau < 1$, the inequality $|z^2+1| \geq 1-\tau^2$ yields

$$|\widetilde{K}_\Phi(s;z)| \leq C(1-\tau^2)^{-2}\gamma^{-1}e^{\gamma|s|}, \quad s \leq 0. \tag{4.15}$$

Since $\hat{f}(0,q) = \hat{g}(0,q) = 0$, we find that (4.12) remains true if 0 replaces $e^{\gamma t}$. Inserting all this into the Fourier series of $f(z,\theta)$ leads to

$$f(z,\theta) = \int_{-\infty}^0 \widetilde{K}_\Phi(s;z)\, g(ze^{\gamma s}, \theta + \omega s)\, ds, \quad (z,\theta) \in [0,1] \times \mathbb{T}^d. \tag{4.16}$$

Here, the compulsory change in the order of summation and integration was justified by virtue of Fubini's Theorem, taking advantage of the bound (4.15). Indeed, we may express $g(z,\theta)$ as the product of $z^2 h(z,\theta)$, where h is analytic in the same region as g. The analyticity in θ implies $|\hat{h}(z,q)| \leq \sup_{\theta\in\Pi_{\tau'}} |h(z,\theta)|e^{-\tau'|q|} \leq Ce^{-\tau'|q|}$ $(\tau < \tau' < 1)$—C being independent of z by compactness. Thus, if $\tau'' := \tau' - \tau$,

$$\sum_{q\in\mathbb{Z}^d} \int_{-\infty}^0 \left|\widetilde{K}_\Phi(s;z)\, \hat{g}(ze^{\gamma s}, q)\, e^{iq\cdot(\theta+\omega s)}\right| ds \leq C(1-\tau^2)^{-2}\gamma^{-2}|z|^2 \sum_{q\in\mathbb{Z}^d} e^{-\tau''|q|},$$

in which the right-hand side is finite.

The solution (4.16) is general, even though due to (4.14) the associated homogeneous equation admits—in a formal sense—a general solution of the form

$$f^0(z,\theta) := \sum_{j=1}^2 \mathcal{W}_{\Phi j}(z)\, c_j(\theta - \omega\gamma^{-1}\ln z),$$

where the maps $c_j : \mathbb{T}^d \to \mathbb{C}$ are arbitrary. But the condition $f^0 \in \mathcal{A}_1$ imposes $f^0(0,\theta) \equiv \partial_z f^0(0,\theta) \equiv 0$, which in turn constricts the c_j's to vanish identically.

Following the line of reasoning above, solving the "rotator part"

$$\mathcal{L}^2 f = g$$

amounts to integrating $\ddot{u} = v$ and results in an expression like (4.16) with the kernel

$$\widetilde{K}_\Psi(s;z) := \mathcal{W}_{\Psi 2}(z)\mathcal{W}_{\Psi 1}(ze^{\gamma s}) - \mathcal{W}_{\Psi 1}(z)\mathcal{W}_{\Psi 2}(ze^{\gamma s}) \equiv -s,$$

introducing

$$\mathcal{W}_{\Psi 1} := 1 \quad \text{and} \quad \mathcal{W}_{\Psi 2} := \gamma^{-1}\ln.$$

For each index $n \in \mathbb{N} \cup \{\infty\}$ define now

$$I_n(z,\theta) := \int_{-n}^0 \widetilde{K}(s;z)\, Z(ze^{\gamma s}, \theta + \omega s)\, ds \quad \text{with} \quad \widetilde{K} := \begin{pmatrix} \widetilde{K}_\Phi & 0 \\ 0 & \widetilde{K}_\Psi \end{pmatrix},$$

where $(z,\theta) \in \Pi_\tau$ and $Z \in \mathcal{A}_1$ are arbitrary. Also denote

$$\widetilde{\mathcal{K}} Z := I_\infty .$$

Since the integrand here is an analytic function of (z,θ) on the compact region Π_τ and continuous in $s \in [-n, 0]$, it follows from an exercise in function theory that I_n *with* $n < \infty$ is analytic on Π_τ; see p. 123 of [Ahl66]. As an element of $\mathcal{A}_1$, $Z(z,\theta)$ has the representation $z^2 \widetilde{Z}(z,\theta)$, where $\widetilde{Z}$ is analytic on Π_τ. Accordingly, (4.15) implies

$$\begin{aligned} \left| \widetilde{\mathcal{K}} Z(z,\theta) - I_n(z,\theta) \right| &\le C\gamma^{-1} \int_{-\infty}^{-n} e^{-\gamma s} \left| Z(ze^{\gamma s}, \theta + \omega s) \right| ds \\ &\le C|z|^2 \frac{\|\widetilde{Z}\|_\infty}{\gamma^2 \, e^{\gamma n}}, \end{aligned} \tag{4.17}$$

showing that $I_n \to \widetilde{\mathcal{K}} Z$ uniformly on Π_τ as $n \to \infty$. Hence, also $\widetilde{\mathcal{K}} Z$ is analytic on the latter region. Moreover, $I_n(z,\theta) = \mathcal{O}(z^2)$ as $z \to 0$, which by virtue of (4.17) yields $\widetilde{\mathcal{K}} : \mathcal{A}_1 \to \mathcal{A}_1$.

We showed above that if $Z \in \mathcal{A}_1$ and $\mathcal{K} Z = Z'$ (thus $Z' \in \mathcal{A}_1$), then $Z = \widetilde{\mathcal{K}} Z'$ holds on $[0,1] \times \mathbb{T}^d \subset \Pi_\tau$. But each side of the latter equation are analytic on Π_τ and hence agree there, meaning that $\widetilde{\mathcal{K}}$ is the left inverse of $\mathcal{K}$: $\widetilde{\mathcal{K}} \mathcal{K} = \mathbb{1}_{\mathcal{A}_1}$. A direct computation shows that it is also the right inverse. In other words,

$$\widetilde{\mathcal{K}} = \mathcal{K}^{-1} \text{ on } \mathcal{A}_1 .$$

$K(s;z) \in \mathbb{R}$, provided $z \in \mathbb{R}$. Thus, should the restriction $Z|_{[0,1]\times\mathbb{T}^d}$ be real-valued, so is $(\mathcal{K}^{-1} Z)|_{[0,1]\times\mathbb{T}^d}$.

The integrals I_n also depend analytically on γ. Thus, according to Theorem 3.4, they are analytic functions on the domain $|\epsilon| < \epsilon_0$. Since $|\gamma - g| < Cg|\epsilon|$, the trivia $\gamma > cg > 0$ and the final inequality in (4.17) guarantee that the convergence $I_n \to \mathcal{K}^{-1} Z$ takes place uniformly on compact subsets of D defined in (1.15) (g bounded away from zero).

It remains to be checked that $\mathcal{K}^{-1}$ is bounded. For any $Z \in \mathcal{A}_1$ and fixed θ

$$Z(z,\theta) = \sum_{k=2}^{\infty} \frac{1}{k!} \, Z_k(\theta) \, z^k$$

converges in the disk $\bar{\mathbb{D}}(0,\tau) := \left\{ z \in \mathbb{C} \;\middle|\; |z| \le \tau \right\}$. Using the Cauchy inequalities

$$|Z_k(\theta)| \le k! \, \tau^{-k} \, \|Z\|_\infty$$

we deduce the bound

$$|Z(z,\theta)| \leq 2(|z|/\tau)^2 \|Z\|_\infty$$

for $z \in \bar{\mathbb{D}}(0,\tau/2)$. In Π_τ, $|z| \leq R$ for a certain $R = 1 + \mathcal{O}(\tau)$, such that $ze^{\gamma s} \in \bar{\mathbb{D}}(0,\tau/2)$ whenever $s \leq S := -\gamma^{-1}\ln(2R/\tau))$. The bound in (4.15) for $\widetilde{K}_\Phi$ clearly applies to $\widetilde{K}_\Psi$ as well. Summarizing,

$$\begin{aligned}
\|\mathcal{K}^{-1}Z\|_\infty &\leq \frac{C\|Z\|_\infty}{\gamma(1-\tau^2)^2}\left(\int_S^0 e^{-\gamma s}\,ds + \int_{-\infty}^S \frac{e^{\gamma s}R^2}{\tau^2(1+\tau)^2}\,ds\right) \\
&\leq \frac{C\|Z\|_\infty}{\gamma^2\tau(1-\tau^2)^2},
\end{aligned}$$

which finishes the proof. □

Part 3

Measuring the Homoclinic Splitting

Chapter 5

Analytic Continuation of the Solution

HERE we assume that the reader is familiar with the notation introduced in the beginning of Chapter 4, as well as the results stated thereafter. The *existing* map $Z = \delta_2 \widetilde{X}$ solves (4.5) and, by virtue of $\widetilde{W}$'s analyticity, admits the representation

$$\begin{aligned}\delta_2 \widetilde{X} = \mathcal{K}^{-1} \delta_2 \Bigg[\begin{pmatrix} \gamma^2 \cos \Phi^0 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{X}_{\le 1} + \sum_{k=0}^{\infty} w^{(k)} \big(\widetilde{X}_{\le 1} \big)^{\otimes k} \Bigg] + \\ + \mathcal{K}^{-1} \sum_{k=1}^{\infty} \Big[w^{(k)} \big(\widetilde{X}_{\le 1} + \delta_2 \widetilde{X} \big)^{\otimes k} - w^{(k)} \big(\widetilde{X}_{\le 1} \big)^{\otimes k} \Big]\end{aligned} \tag{5.1}$$

on the set Π_τ, taking ϵ small enough, and denoting

$$w^{(k)} := \frac{1}{k!} D^k \widetilde{W}(0) \tag{5.2}$$

and a repeated argument of such a symmetric k-linear operator by

$$(x)^{\otimes k} := \underbrace{(x, \ldots, x)}_{k \text{ times}},$$

for the sake of brevity. Observe that we have omitted a δ_2 in front of the square brackets on the second line of (5.1) as redundant.

Equation (5.1) may be viewed as a recursion relation for $\delta_2 \widetilde{X}$. It is crucial that

$$w^{(0)},\, w^{(1)} = \mathcal{O}(\epsilon g^2), \tag{5.3}$$

when $(\epsilon, g) \in D$; see (1.15). Namely, any given order $\delta_2 \widetilde{X}^\ell$ in the convergent expansion

$$\delta_2 \widetilde{X} = \sum_{\ell=1}^{\infty} \epsilon^\ell \, \delta_2 \widetilde{X}^\ell$$

is then completely determined by $\widetilde{X}_{\leq 1}$ and the *lower orders* $\delta_2 \widetilde{X}^l$ $(1 \leq l \leq \ell - 1)$ through the right-hand side of (5.1). Moreover, since $\widetilde{X}_{\leq 1} = \mathcal{O}(\epsilon)$, only *finitely many* terms in the sum over the index k are involved. Together these facts imply that only *finitely many* recursive steps using (5.1) are needed to completely describe any given order $\delta_2 \widetilde{X}^\ell$ in terms of $\widetilde{X}_{\leq 1}$ alone and that, at each such step, only *finitely many* terms from the k-sum contribute. We next give a pictorial illustration of the above reasoning.

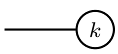
5.1. Tree expansion

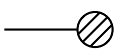
Despite the title of the thesis, we shall use plenty of tree diagrams. However, this is done for bookkeeping and pedagogical purposes, and our trees interact in no way; we simply choose to draw a tree instead of spelling out a formula.

Let us first define the auxiliary functions

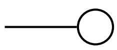
$$h^{(k)} := \begin{cases} w^{(0)} + \left[\begin{pmatrix} \gamma^2 \cos \Phi^0 & 0 \\ 0 & 0 \end{pmatrix} + w^{(1)} \right] \widetilde{X}_{\leq 1} & \text{if } k = 1, \\ w^{(k)} \big(\widetilde{X}_{\leq 1} \big)^{\otimes k} & \text{if } k = 2, 3, \ldots, \end{cases}$$

and make the identifications

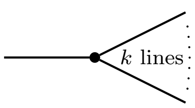


$:= \mathcal{K}^{-1} \delta_2 h^{(k)}$ and  $:= \mathcal{K}^{-1} \delta_2 \sum_{k=0}^{\infty} h^{(k)}.$ (5.4)

Furthermore, let

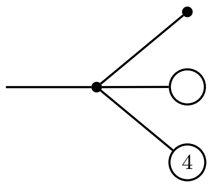
 $:= \delta_2 \widetilde{X},$ $:= \widetilde{X}_{\leq 1},$

and, for $k \geq 1$,



$:= \mathcal{K}^{-1} w^{(k)}.$

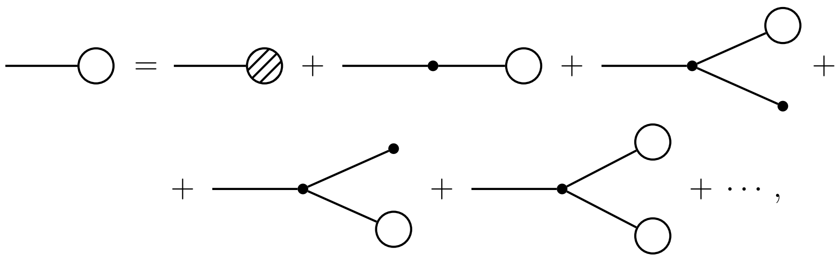
In the diagram representing the k-linear $w^{(k)}$, the k "free" lines to the right of the node stand for the arguments. We say that these lines *enter* the *internal* node, whereas the single line to the left of the node *leaves* it. For instance,



$$= \mathcal{K}^{-1} w^{(3)}\big(\widetilde{X}_{\leq 1},\, \delta_2 \widetilde{X},\, \mathcal{K}^{-1}\delta_2 h^{(4)}\big).$$

Notice that, as $w^{(k)}$ is symmetric, permuting the lines entering a node does not change the resulting function. We emphasize that all of the functions introduced above are analytic on Π_τ and $|\epsilon| < \epsilon_0$.

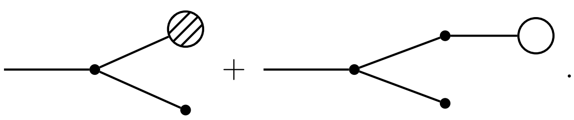
In terms of such *tree diagrams*, or simply *trees*, equation (5.1) reads



(5.5)

using multilinearity to split the sums $\widetilde{X}_{\leq 1} + \delta_2 \widetilde{X}$ into pieces. Above, the sum after the first tree consists of *all* trees having one internal node and an arbitrary number of *end nodes, at least one of which, however, is a white circle*. This rule encodes the fact that on the second line of (5.1) the summation starts from $k = 1$ and that the contributions with only $\widetilde{X}_{\leq 1}$ in the argument (*i.e.*, trees with only black dots as end nodes) are cancelled.

Using (5.1) recursively now amounts to replacing each of the lines with a white-circled end node by the complete expansion of such a tree above. This is to be understood additively, so that replacing one end node, together with the line leaving it, by a sum of two trees results in a sum of two new trees. For example, such a replacement in the third tree on the right-hand side of (5.5) by the first two trees gives the sum



Before proceeding, we introduce a little bit of terminology. The leftmost line in a tree is called the *root line*, whereas the node it leaves (*i.e.*, the uniquely defined leftmost node) is called the *root*. Having assigned such a special role to one of the nodes, our trees are in fact what are called rooted trees in the literature. A line leaving a node v and entering a node v' can always be interpreted as the root line of a *subtree*, the maximal tree consisting of lines and nodes in the original tree with v as its root. We call v a (not necessarily unique) *successor* of v', whereas v' is the unique *predecessor* of v.

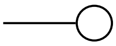
The recursion (5.5) can be repeated on a given tree if it has at least one white circle left. Otherwise, the tree in question must satisfy

(R1′) The tree has only filled circles and black dots (•) as its end nodes,

together with

(R2′) Any internal node has an entering (line that is the root of a) subtree containing at least one filled circle as an end node.

After all, the recursion can only stop by replacing an existing white circle with a filled one. Continuing *ad infinitum* yields the expansion

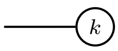
 $$= \sum \big(\text{Trees satisfying (R1}') \text{ and (R2}')\big) = \sum_{\text{trees } T}{}' \, T, \quad (5.6)$$

where the prime restricts the summation to trees T satisfying (R1′) and (R2′). We point out that each admissible tree appears precisely once in this sum, considering different two trees that can be superposed by permuting subtrees that enter the same node.

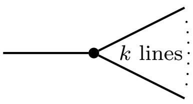
How does the earlier discussion concerning the description of $\delta_2\widetilde{X}^\ell$ in terms of a finite sum involving only $\widetilde{X}_{\leq 1}$ translate to the language of trees? In a straightforward fashion. First, the second part of (5.3) and $\widetilde{X}_{\leq 1} = \mathcal{O}(\epsilon)$ amount pictorially to

 $= \mathcal{O}(\epsilon)$ and $= \mathcal{O}(\epsilon)$.

Second, $w^{(k)} = \mathcal{O}(g^2)$ and the first part of (5.3) yield

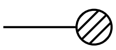
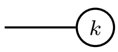
 $= \mathcal{O}(\epsilon^k) \qquad (k = 1, 2, \dots)$

and



$= \mathcal{O}(1) \qquad (k = 2, 3, \dots)$.

Expanding the filled end nodes

 $$= \sum_{k=1}^{\infty}$$  , (5.7)

according to (5.4), on the right-hand side of (5.6), we get a new version of the latter by replacing the rules (R1′) and (R2′), respectively, with

(R1) The tree has only numbered circles ($\textcircled{k}$ with arbitrary values of k) and black dots ($\bullet$) as its end nodes,

and

(R2) Any internal node has an entering (line that is the root of a) subtree containing at least one numbered circle as an end node.

Let us define the *degree* of a tree as the positive integer

$$\deg T := \#(\text{———}\bullet\text{———}) + \#(\text{———}\bullet) + \sum_{k=1}^{\infty} k\,\#(\text{———}\textcircled{k}) \qquad (5.8)$$

for any tree T satisfying (R1) and (R2). By $\#(G)$ we mean the number of occurrences of the graph G in the tree T. That is, the degree of a tree is the number of its end nodes with suitable weights plus the number of nodes with precisely one entering line. Since a tree has finitely many nodes, its degree is well-defined. Then a rearrangement of the sum arising from (5.7) being inserted into (5.6) yields formally

$$\text{———}\bigcirc = \sum_{l=1}^{\infty} \sum_{\substack{\text{trees } T \\ \deg T = l}}^{*} T, \qquad (5.9)$$

where the asterisk reminds us that the rules (R1) and (R2) are being respected by the sum.

According to the analysis above, the particular graphs appearing in the definition of $\deg T$ are the only possible single-node subgraphs of T proportional to a positive power of ϵ. Since each tree is an analytic function of ϵ, writing again $(\,\cdot\,)^k$ for the kth Taylor coefficient, we have

$$T = \sum_{k=\deg T}^{\infty} \epsilon^k\, T^k = \epsilon^{\deg T} \sum_{k=0}^{\infty} \epsilon^k\, T^{k+\deg T}.$$

Hence, only trees with degree *at most* equal to ℓ can contribute to $\delta_2 \widetilde{X}^\ell$:

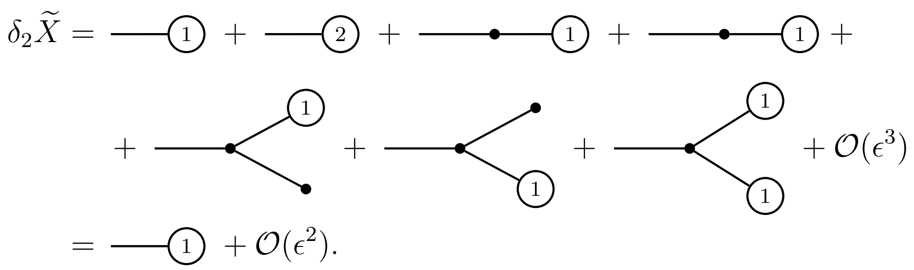
$$\delta_2 \widetilde{X}^\ell = \sum_{l=1}^{\ell} \sum_{\deg T = l}^{*} T^\ell = \Bigg(\sum_{\deg T \le \ell}^{*} T \Bigg)^{\ell} \qquad (5.10)$$

or, alternatively,

$$\delta_2 \widetilde{X} = \sum_{\deg T \le \ell}^{*} T + \mathcal{O}(\epsilon^{\ell+1}) \qquad (\epsilon \to 0) \tag{5.11}$$

for *each and every* $\ell = 1, 2, \dots$. The expansion in (5.9) is in fact just a compact way of writing (5.11). We emphasize that the latter can be derived completely rigorously, for each value of ℓ separately, but resorting to the use of formal series allowed us to treat all orders of $\delta_2 \widetilde{X}$ at once. We call the series (5.9) an *asymptotic expansion* of $\delta_2 \widetilde{X}$; the partial sums $\sum_{\deg T \le \ell}^{*} T$ need not converge to $\delta_2 \widetilde{X}$ for any fixed ϵ as $\ell \to \infty$, but for a fixed ℓ the error is bounded by an ℓ-dependent constant times $|\epsilon|^{\ell+1}$ on the mutual domain of analyticity, $|\epsilon| < \epsilon_0$.

Example 5.1. The beginning of the asymptotic expansion (5.11) is



5.2. Analyticity domain of trees

As already pointed out, all trees T above are analytic functions of (z, θ, ϵ) on $\Pi_\tau \times \{|\epsilon| < \epsilon_0\}$. Due to the projections δ_2 appearing in (5.4), they also satisfy $T|_{z=0} = \partial_z T|_{z=0} = 0$, *i.e.*, are elements of the space $\mathcal{A}_1$ defined in (4.7). On this space, the inverse of $\mathcal{K} = \begin{pmatrix} L & 0 \\ 0 & \mathcal{L}^2 \end{pmatrix}$ (see (4.2)) constructed in the proof of Lemma 4.3 satisfies

$$\mathcal{K}^{-1} h(z, \theta) = \int_{-\infty}^{0} \widetilde{K}(s; z)\, h(z e^{\gamma s}, \theta + \omega s)\, ds. \tag{5.12}$$

Consequently, we will now show that the analyticity domain of a tree in the z-variable is in fact much larger than the neighbourhood of $[0, 1]$ that is included in Π_τ; namely the wedgelike region

$$\mathbb{U}_{\tau,\vartheta} := \big\{|z| \le \tau\big\} \bigcup \big\{\mathfrak{Re}\, z > 0,\ |\arg z| \le \vartheta\big\} \subset \mathbb{C}$$

(with a new τ and "small" ϑ) in the right half-plane.

Lemma 5.2 (Analytic continuation of trees). *Without affecting the analyticity domain with respect to ϵ, there exist numbers $0 < \tau < 1$,*

$0 < \vartheta < \pi/2$, and $0 < \sigma < \eta$ such that each tree in the sums (5.9) *and* (5.11) *extends to an analytic function of (z, θ) on $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$.*

Proof. Observe that, as a polynomial, $\widetilde{X}_{\le 1}$ is an entire function of z. On the other hand, $\Phi^0(z) = 4 \arctan z = 2i\big(\log(1-iz) - \log(1+iz)\big)$, implying that $|\Im\mathfrak{m}\,\Phi^0(z)| \le \eta$ in $\mathbb{U}_{\tau,\vartheta}$ with τ and ϑ sufficiently small. By Example 2.1, $f(\Phi^0(z), \theta)$ is analytic, making the maps $h^{(k)}$ and $\widetilde{X}_{\le 1}$ analytic on $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$ for some $0 < \sigma < \eta$, where η is determined by f and σ by $\widetilde{X}_{\le 1}$ (ultimately by f and ω).

Suppose $h = \delta_2 h$ is a map analytic on $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$. Then the integrand in (5.12) is analytic in a neighbourhood of the latter set. By virtue of Fubini's theorem,

$$\oint_\Gamma \mathcal{K}^{-1} h(\zeta, \theta)\, d\zeta = \int_{-\infty}^0 \oint_\Gamma \widetilde{K}(s;\zeta)\, h(\zeta e^{\gamma s}, \theta + \omega s)\, d\zeta\, ds = 0$$

for any closed contour Γ inside a sufficiently small neighbourhood of $\mathbb{U}_{\tau,\vartheta}$ and enclosing z. Hence, Morera's theorem yields analyticity of $\mathcal{K}^{-1}h$ with respect to z. As always, analyticity with respect to θ follows from an exponentially decaying bound on the Fourier coefficients. Applying this argument at each node of a tree proves the claim. □

Since the number of terms in the sum in (5.10) is finite and the functions $\widetilde{X}_{\le 1}$ and X^0 in $X = X^0 + \widetilde{X}_{\le 1} + \delta_2 \widetilde{X}^\ell$ are analytic on $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$, we get

Proposition 5.3 (Analytic continuation). *Each order X^ℓ of the solution extends analytically to the region $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$. Moreover, if $\psi \mapsto f(\,\cdot\,, \psi)$ is a trigonometric polynomial of degree N, i.e., N is the minimal nonnegative integer such that $\hat{f}(\,\cdot\,, q) = 0$ if $|q| > N$, then $\theta \mapsto X^\ell(\,\cdot\,, \theta)$ is a trigonometric polynomial of degree ℓN, at most.*

However, (a straightforward upper bound on) X^ℓ grows without a limit as $\Re\mathfrak{e}\, z \to +\infty$, such that there is no reason whatsoever to expect absolute convergence of the series $\sum_{\ell=0}^\infty \epsilon^\ell X^\ell$ in an unbounded z-domain. In fact, it is known that the behavior of the unstable manifold ($X = X^u$) gets extremely complicated for large values of z even with relatively simple Hamiltonian systems.

Proof of the trigonometric polynomial part of Proposition 5.3. From the equations of motion, (1.11), a Taylor expansion yields

$$\mathcal{L}^2 \widetilde{X} = -\mathcal{L}^2 X^0 + \Omega(X^0) + \sum_{m=1}^{\infty} \frac{1}{m!} D^m \Omega(X^0)(\widetilde{X})^{\otimes m},$$

where the trigonometric degree of $D^m\Omega(X^0)$ is N for $\epsilon \neq 0$ but vanishes at $\epsilon = 0$ because X^0 does not depend on θ. For each $k \geq 1$, let n_k stand for the trigonometric degree of $\widetilde{X}^k$. Equating like powers of ϵ in the expansion above, we infer two things. First, $n_1 = N$. Second, we must have, for each $\ell \geq 2$,

$$n_\ell \leq \begin{cases} n_{k_1} + \cdots + n_{k_m} & \text{where } k_1 + \cdots + k_m = \ell, \\ N + n_{k_1} + \cdots + n_{k_m} & \text{where } k_1 + \cdots + k_m = \ell - 1, \end{cases}$$

because the trigonometric degree of a product is at most the sum of the trigonometric degrees of the factors; $e^{iq\cdot\theta}e^{iq\cdot\theta} = e^{i2q\cdot\theta}$ and $e^{iq\cdot\theta}e^{-iq\cdot\theta} = 1$.

Next, assume that $n_k \leq kN$ holds for each $1 \leq k \leq \ell - 1$, recalling that this is the case if $k = 1$. Subsequently, the estimate for n_ℓ above becomes $n_\ell \leq \ell N$. □

Chapter 6

Size of the Homoclinic Splitting

THE Lyapunov exponent $\gamma = \gamma(\epsilon, g)$, whose existence we have studied earlier, turned out to be an analytic function of ϵ in a neighbourhood of the origin ($|\epsilon| < \epsilon_0$). Hence, may be expand

$$\gamma = g + \sum_{\ell=1}^{\infty} \epsilon^\ell \gamma_\ell.$$

Further, since $\mathcal{L}$ depends on γ, we write its perturbation expansion as

$$\mathcal{L} = \mathcal{L}_0 + \sum_{\ell=1}^{\infty} \epsilon^\ell \mathcal{L}_\ell.$$

The operator-valued coefficients are given by

$$\mathcal{L}_0 = \omega \cdot \partial_\theta + gz\partial_z \quad \text{and} \quad \mathcal{L}_\ell = \gamma_\ell z \partial_z \quad (\ell \geq 1) \tag{6.1}$$

above. In particular, as $X^0(z,\theta) \equiv (4 \arctan z, 0)^T$ by (1.17),

$$Y^0(z,\theta) \equiv \begin{pmatrix} 4gz/(1+z^2) \\ 0 \end{pmatrix}.$$

The expansion of the splitting matrix Υ—see (1.25)—therefore begins linearly with respect to ϵ:

$$\Upsilon = \sum_{\ell=1}^{\infty} \epsilon^\ell \Upsilon^\ell.$$

It is customary to call the leading coefficient, Υ^1, the *Melnikov term* or the *Melnikov matrix.*

Recall now (1.10). For convenience, we also remind the reader of equations (2.1)–(2.2) and (3.1), *i.e.*,

$$\mathcal{D}^2 X_0 = \Omega(X_0) = (g^2 \sin\Phi_0, 0)^T + \lambda\,\widetilde{\Omega}(X_0) \tag{6.2}$$

and

$$(\mathcal{D}+\gamma)^2 X_1 = D\Omega(X_0)X_1 = \begin{pmatrix} g^2\cos\Phi_0 & 0 \\ 0 & 0 \end{pmatrix} X_1 + \lambda\, D\widetilde{\Omega}(X_0)X_1. \tag{6.3}$$

Curiously, a simple parity argument concerning (6.2) yields the following results.

Proposition 6.1. *The map $\theta \mapsto X_0(\theta-\beta) - (0,\beta)^T$ is odd for a unique phase shift $\beta \in \mathbb{T}^d$. The latter depends analytically on the parameters ϵ and g in the same domains as X_0, and $\beta|_{\epsilon=0} = 0$.*

Proof. Suppose $\beta := \langle\Psi_0\rangle$, such that the average of the Ψ-coordinate of the shifted solution $X_0'(\theta) \equiv X_0(\theta-\beta) - (0,\beta)^T$ vanishes. Let us set $X_0''(\theta) \equiv -X_0'(-\theta)$ and keep the even parity of f in mind. Surely $\mathcal{D}^2 X_0'' = \Omega(X_0'')$, $\|\Psi_0''\|_{\ell^1} = \|\Psi_0'\|_{\ell^1}$, and $\langle\Psi_0''\rangle = -\langle\Psi_0'\rangle$. By the uniqueness part of Theorem 2.2, we must have $X_0'' = X_0'$. In other words, the map $\theta \mapsto X_0'(\theta)$ is odd. The uniqueness of β follows readily, whereas its analyticity can be inferred directly from that of Ψ_0. □

As $\beta|_{\epsilon=0} = 0$ and $X_0^0 = 0$, we get

Corollary 6.2. *At the homoclinic point, the function $Y_0 = \mathcal{D}X_0$ does not contribute to the Melnikov term of the splitting matrix:*

$$\partial_\theta Y_0^1(0) = 0.$$

Remark 6.3. Similar considerations about X_1 are doomed to fail because of the parity breaking operator $2\gamma\mathcal{D}$ appearing on the left-hand side of (6.3).

Extracting from (6.3) the terms that are linear in ϵ (which equals λg^{-2} by definition) and recalling $X_0^0 = 0$, we get

$$\left[(\mathcal{D}+g)^2 - \begin{pmatrix} g^2 & 0 \\ 0 & 0 \end{pmatrix}\right] X_1^1 = \left[g^2 D\widetilde{\Omega}(0) - \begin{pmatrix} 2g\gamma_1 & 0 \\ 0 & 0 \end{pmatrix}\right] X_1^0 \tag{6.4}$$

for X_1^1. As the average of the upper row on the left-hand side of (6.4) is zero and $X_1^0 = (4,0)^T$, we see that $\gamma_1 = \frac{1}{2} g \langle D_\Phi \widetilde{\Omega}_\Phi(0)\rangle$. This suggests that γ depends analytically on g. It also serves as motivation for (3.71).

6.1. Some tools

We define the operator $\mathcal{D}^{-1}$, acting on functions on $\mathbb{T}^d$ with *vanishing average*, by giving the expression of its kernel:

$$\mathcal{D}^{-1}(p,q) = \begin{cases} (i\omega\cdot q)^{-1} & \text{if } p = q \in \mathbb{Z}^d\setminus\{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{D}^{-1}$ is the right inverse of $\mathcal{D}$ and its range consists of functions on $\mathbb{T}^d$ with vanishing average.

Let us also define the integral operators

$$\mathcal{I}[t]h(z,\theta) := \int_{-\infty}^{t} h(ze^{\gamma\tau}, \theta+\omega\tau)\,d\tau \quad \text{with} \quad \mathcal{I} := \mathcal{I}[0].$$

The symbol $\mathcal{I}_0$ will stand for the "lowest order" of the operator $\mathcal{I}$—having γ replaced by g in the definition. Obviously,

$$\mathcal{I}h(ze^{\gamma t}, \theta+\omega t) \equiv \mathcal{I}[t]h(z,\theta) \equiv \mathcal{I}h(z,\theta) + \int_{0}^{t} h(ze^{\gamma\tau}, \theta+\omega\tau)\,d\tau.$$

Applying (1.7) at $t=0$, the formal identity

$$\mathcal{L}\mathcal{I} = \mathbb{1} \tag{6.5}$$

follows. Complementing (6.5), one also gets

$$\mathcal{I}\mathcal{L}\delta_1 = \delta_1 \tag{6.6}$$

by a direct computation.

We also point out that the inverse of $\mathcal{K} = \left(\begin{smallmatrix} L & 0 \\ 0 & \mathcal{L}^2 \end{smallmatrix}\right)$ on $\mathcal{A}_1$ constructed in the proof of Lemma 4.3 satisfies

$$\mathcal{K}^{-1}h(ze^{\gamma t}, \theta+\omega t) = \int_{-\infty}^{t} \widetilde{K}(\tau - t; ze^{\gamma t})\, h(ze^{\gamma\tau}, \theta+\omega\tau)\,d\tau, \tag{6.7}$$

where the kernel can conveniently be expressed in terms of the matrices

$$\mathcal{W}_j := \begin{pmatrix} \mathcal{W}_{\Phi j} & 0 \\ 0 & \mathcal{W}_{\Psi j} \end{pmatrix} \qquad (j=1,2),$$

namely,

$$\widetilde{K}(\tau - t, ze^{\gamma t}) \equiv \mathcal{W}_2(ze^{\gamma t})\mathcal{W}_1(ze^{\gamma\tau}) - \mathcal{W}_1(ze^{\gamma t})\mathcal{W}_2(ze^{\gamma\tau}). \tag{6.8}$$

6.2. The homoclinic trajectory

Consider the homoclinic point at $(z,\theta) = (1,0)$. If we let the system evolve with time, the flow of the coordinates (ϕ,ψ) is given by $t \mapsto (0,\omega t) + X(e^{\gamma t},\omega t)$, according to (1.6). As the initial point lies on both $\mathcal{W}^u_\lambda$ and $\mathcal{W}^s_\lambda$, so does the whole trajectory, which means that $X^u(e^{\gamma t},\omega t) = X^s(e^{\gamma t},\omega t)$ must hold true at all times (after analytic continuation to large $|t|$).

On the other hand, defining the time-reversal operator (recall (1.8))

$$\widehat{T}h(z,\theta) := (h\circ T)(z,\theta) = h(z^{-1},-\theta),$$

(1.20) states that $X^s = (2\pi,0) - \widehat{T}X^u$. Thus, we have

$$X(e^{\gamma t},\omega t) = (2\pi,0) - \widehat{T}X(e^{\gamma t},\omega t) \qquad \forall\, t\in\mathbb{R}. \tag{6.9}$$

Moreover, differentiation with respect to t yields

$$Y(e^{\gamma t},\omega t) = \widehat{T}Y(e^{\gamma t},\omega t) \qquad \forall\, t\in\mathbb{R}.$$

Denoting the ℓth Taylor coefficient of a function at the origin by a superscript ℓ and recalling $\Psi^0 = \beta^0 = 0$, Proposition 6.1 implies—in full agreement with (6.9), we stress—that

$$X_0^1 = \tfrac{1}{2}\big(\mathbb{1}-\widehat{T}\big)X_0^1 + (0,\beta^1)^T.$$

This is the unique decomposition of X_0^1 into its odd and even parts. Notice now that (6.6) and $\mathcal{L}_0^2\Psi^1 = g^2\widetilde{\Omega}_\Psi(X^0)$ imply

$$\delta_1\Psi^1 = g^2\mathcal{I}_0^2\delta_1\widetilde{\Omega}_\Psi(X^0). \tag{6.10}$$

The imposed condition $X(1,0) = (\pi,0)^T$ contains the relations

$$X_0^\ell(0) = -\delta_1 X^\ell(1,0) \qquad (\ell\geq 1),$$

such that, in fact,

$$\beta^1 = \Psi_0^1(0) = -\delta_1\Psi^1(1,0).$$

6.3. The Melnikov term

Let us study the lowest nonvanishing contribution to the splitting in detail. The symmetry (1.3) comes in very handy, since it implies, together with the even parity of f, that

$$f(\Phi^0(z^{-1}),-\theta) \equiv f(\Phi^0(z),\theta). \tag{6.11}$$

By Corollary 6.2 and (1.21), the Melnikov matrix reads

$$\Upsilon^1 = 2\partial_\theta\delta_1 Y_\Psi^1(1,0) = 2g^2\mathcal{I}_0\delta_1 D_\Psi\widetilde{\Omega}_\Psi(X^0)(1,0),$$

the last equality being obtained from (6.10) by applying $\mathcal{L}_0$. By equation (6.11), $\delta_1 D_\Psi \widetilde{\Omega}_\Psi(X^0)(z,\theta) = \partial_\psi^2 f(\Phi^0(z),\theta) - \partial_\psi^2 f(0,\theta)$ is invariant under $(z,\theta) \mapsto (z^{-1}, -\theta)$, such that setting

$$F(s,\theta) \equiv \int_{-\infty}^{\infty} \partial_\psi^2 f(\Phi^0(e^{gt+s}), \theta + \omega t) - \partial_\psi^2 f(0, \theta + \omega t)\, dt \qquad (6.12)$$

yields

$$\Upsilon^1 = g^2 F(0,0).$$

Here the second term in the integrand removes the quasiperiodic limit of the first one, making the integral absolutely convergent. In fact, the integrand tends to zero at an exponential rate as $|t| \to \infty$.

We now give a result, due to Lazutkin in it's original form, [Laz03], and extended by [DGJS97], [Sau01], and [LMS03] to the quasiperiodic setting.

LEMMA 6.4 (Lazutkin). *Suppose a function F is analytic on the set $S := [-i\vartheta, i\vartheta] \times \{|\Im\mathfrak{m}\, \theta| < \eta\}$, continuous on its closure $\bar{S}$, and satisfies the identity*

$$F(s,\theta) \equiv F(0, \theta - g^{-1}\omega s). \qquad (6.13)$$

Then F extends analytically to $\{|\Im\mathfrak{m}\, s| \leq \vartheta\} \times \{|\Im\mathfrak{m}\, \theta| < \eta\}$, with (6.13) still holding. On $\mathbb{R} \times \mathbb{T}^d$ it obeys a bound of the form

$$|F(s,\theta) - \widetilde{F}| \leq C B(g) e^{-c\, g^{-1/(\nu+1)}}, \quad B(g) := \sup_{(s,\theta)\in\bar{S}} |F(s,\theta)|,$$

where $\widetilde{F}$ stands for the θ-average $\langle F(s,\,\cdot\,)\rangle$ and is independent of s.

The lemma applies to the function defined in (6.12), due to Example 2.1 and Lemma 5.2. Because the ψ-derivatives in the integrand are interchangeable with total θ-derivatives, $\widetilde{F} = 0$, and we get

PROPOSITION 6.5. *There exist positive constants c and C such that the Melnikov term satisfies*

$$|\Upsilon^1| \leq C g^2 e^{-c\, g^{-1/(\nu+1)}}.$$

Proof of Lemma 6.4. (Adapted from [Sau01]). The Fourier transform of (6.13) yields

$$\hat{F}(s,q) = \hat{F}(0,q)\, e^{-i(g^{-1}\omega\cdot q)s}, \qquad (6.14)$$

which extends to an entire function of s. For $s \in [-i\vartheta, i\vartheta]$,

$$|\hat{F}(0,q)|\, e^{(g^{-1}\omega\cdot q)\, \Im\mathfrak{m}\, s} = |\hat{F}(s,q)| \leq B(g) e^{-\eta|q|},$$

and therefore,

$$|\hat{F}(0,q)| \le B(g)e^{-\vartheta g^{-1}|\omega\cdot q|-\eta|q|}.$$

Finally, plugging this into (6.14), we get

$$|\hat{F}(s,q)| \le B(g)e^{-(\vartheta-|\Im\mathfrak{m}\, s|)g^{-1}|\omega\cdot q|-\eta|q|} \qquad (s\in\mathbb{C}).$$

For $|\Im\mathfrak{m}\, s| \le \vartheta$ and $|\Im\mathfrak{m}\,\theta| \le \eta' < \eta$, the series $\sum_{q\in\mathbb{Z}^d}\hat{F}(s,q)e^{iq\cdot\theta}$ is uniformly convergent and, as such, provides the analytic extension.

Since $\alpha x^{-\nu}+\beta x \ge \alpha(\nu+1)\left(\frac{\alpha\nu}{\beta}\right)^{-\nu/(\nu+1)}$ for positive α, β, ν, and x, we get from the Diophantine condition (1.4) that

$$|\hat{F}(s,q)| \le B(g)e^{-\vartheta g^{-1}a|q|^{-\nu}-\eta|q|} \le B(g)e^{-\delta|q|}e^{-w(\vartheta,\eta-\delta)g^{-1/(\nu+1)}}$$

holds if $s\in\mathbb{R}$, $q\in\mathbb{Z}^d\backslash\{0\}$, $0<\delta<\eta$, and $w(\vartheta,\eta-\delta) := (\vartheta a)^{1/(\nu+1)}(\eta-\delta)^{\nu/(\nu+1)}(\nu+1)\nu^{-\nu/(\nu+1)}$. Moreover,

$$\sum_{q\in\mathbb{Z}^d\backslash\{0\}} e^{-\delta|q|} \le C\delta^{-d},$$

where C only depends on the dimension d. The bound of the lemma now follows.

By (6.14), we compute $\widetilde{F} := \langle F(s,\,\cdot\,)\rangle = \hat{F}(s,0) = \hat{F}(0,0)$, such that $\widetilde{F}$ does not depend on (s,θ). □

Remark 6.6. Notice that, for small values of g, the analyticity domain of F in Lemma 6.4 is much larger than what the right-hand side of (6.13) suggests.

Remark 6.7. Identity (6.13) in Lazutkin's lemma is equivalent to

$$(\partial_s + g^{-1}\omega\cdot\partial_\theta)F = 0, \qquad (6.15)$$

or, calling $\bar{F}(z,\theta) \equiv F(g^{-1}\ln z,\theta)$, to $\mathcal{L}_0\bar{F} = 0$. There exists a whole industry trying to push through the argument given above replacing Υ^1 with a "better measure" of the splitting, such as Υ, by searching for a coordinate system in which the "measure" satisfies (6.15). The state of this quest is best described in [LMS03].

Neither Lemma 6.4 nor Proposition 6.5 is optimal. There is a refinement in [Sau01] which, however, is not optimal. In many special cases one can derive a much stronger bound.

Example 6.8. (Gallavotti *et al.*) A good concrete perturbation to dwell on is $f(\phi,\psi) = \sum_{k=1}^{d}\cos(\phi+\psi_k)$. The Melnikov matrix is now diagonal. In fact,

$$e^{in\Phi^0(e^{-gt})} = \frac{(i+e^{gt})^{4n}}{(1+e^{2gt})^{2n}} = \frac{1}{\cosh^{2|n|}(gt)} \begin{cases} [\sinh(gt)+i]^{2|n|}, & n \in \mathbb{N}, \\ [\sinh(gt)-i]^{2|n|}, & n \in \mathbb{Z} \setminus \mathbb{N}, \end{cases}$$

which implies

$$\begin{aligned} \Upsilon^1_{kk} &= 2g^2 \int_{-\infty}^{\infty} \frac{\cos(\omega_k t)}{\cosh^2(gt)} - \frac{\sin(\omega_k t)\sinh(gt)}{\cosh^2(gt)}\, dt \\ &= 2\pi\omega_k \left[\sinh\left(\frac{\pi\omega_k}{2g}\right)\right]^{-1} - 2\pi\omega_k \left[\cosh\left(\frac{\pi\omega_k}{2g}\right)\right]^{-1} \\ &\sim 8\pi|\omega_k| \cdot \begin{cases} \exp\left(-\frac{\pi|\omega_k|}{2g}\right) & \text{if } \omega_k < 0 \\ \exp\left(-\frac{3\pi|\omega_k|}{2g}\right) & \text{if } \omega_k > 0 \end{cases}, \quad \text{as} \quad g \to 0. \end{aligned}$$

The integrals can be found using the Residue Theorem after lifting the contour of integration to $\mathfrak{Im}\, t = \pi/g$ or, alternatively, by looking them up in [EMOT54]. We emphasize that the domain of integration being the *whole of* $\mathbb{R}$ and the analyticity of the oscillatory integrand above are absolutely crucial for obtaining exponentially small asymptotics.

6.4. Hiding γ

We adhere to the notation

$$X(t; z, \theta) := X(ze^{\gamma t}, \theta + \omega t), \tag{6.16}$$

which makes the dependence on the Lyapunov exponent γ *implicit*, such that the perturbation expansion reads compactly

$$X(t; z, \theta) = \sum_{\ell=0}^{\infty} \epsilon^\ell X^\ell(t; z, \theta)$$

instead of the more verbose form [1]

$$X(ze^{\gamma t}, \theta + \omega t) = \sum_{\ell=0}^{\infty} \epsilon^\ell \left[\frac{1}{\ell!}\frac{d^\ell}{d\epsilon^\ell} X(ze^{\gamma t}, \theta + \omega t)\right]\Bigg|_{\epsilon=0}.$$

With little trickery, the equations of motion retain their appearance: *defining*

$$\Omega(X)(t; z, \theta) \equiv [(g^2 \sin\Phi, 0) + \lambda\, \widetilde{\Omega}(X)](t; z, \theta)$$

together with

$$\widetilde{\Omega}(X)(t; z, \theta) \equiv \partial f(X(t; z, \theta) + (0, \theta + \omega t)),$$

[1] Recall that γ is a function of ϵ.

and implementing (1.7), we obtain the analogue of (1.11):

$$\partial_t Y(t;z,\theta) := \partial_t^2 X(t;z,\theta) = \Omega(X)(t;z,\theta). \tag{6.17}$$

The explicit dependence of a function on z and θ will often be suppressed. Notice, however, that

$$X(t) := X(t;z,\theta) \equiv X(0;ze^{\gamma t},\theta+\omega t). \tag{6.18}$$

But due caution should be exercised when dealing with derived quantities; for instance, the *partial* derivative with respect to z obeys

$$\partial_z X(t;z,\theta) = e^{\gamma t}\partial_z X(0;ze^{\gamma t},\theta+\omega t) \not\equiv \partial_z X(0;ze^{\gamma t},\theta+\omega t).$$

On the other hand, $\partial_\theta X(t)$ and $z\partial_z X(t)$, as well as $\partial_t X(t)$, inherit the functional quality (6.18), and $\Omega(X)(t)$ was constructed so as to have it, too. Furthermore, smoothness guarantees that the partial derivatives ∂_t, ∂_z, and ∂_θ commute.

6.5. Regularized integrals

This section borrows heavily from Gallavotti, *et al.*; see for instance [CG94] and [GGM99].

We consider functions $h(t;z,\theta)$ that can be expanded as finite sums

$$\sum_{p=0}^{P} t^p h^p(ze^{\gamma t},\theta+\omega t), \tag{6.19}$$

where h^p is analytic on $\{0<|z|<\tau\}\times\mathbb{T}^d$—a punctured neighbourhood of the origin times the d-torus—with at worst a *finite* pole at $z=0$.

Let us then denote by $T_N h(t;z,\theta)$ the truncated Fourier series $\sum_{|q|\le N}\hat h(t;z,q)e^{iq\cdot\theta}$ and define a *regularized integral*

$$⨍ h\,(t;z,\theta) := \lim_{N\to\infty}\operatorname*{res}_{R=0}\frac{1}{R}\int_{-\infty}^{t} e^{-R|\tau|}T_N h(\tau;z,\theta)\,d\tau,$$

the residue at $R=0$ meaning that of the analytic extension from the complex half-plane with $\mathfrak{Re}\,R$ sufficiently large. This is well-known from the theory of Laplace transforms. The truncation, T_N, is needed to insure that the origin of the R-plane is not an accumulation point for poles, since factors $(R+i\omega\cdot q)^{-1}$ appear from the integral.

Proposition 6.9. *The regularized integral is linear. Moreover,*

(1) $⨍ h\,(t;z,\theta) = ⨍ h\,(t_0;z,\theta) + \int_{t_0}^{t} h(\tau;z,\theta)\,d\tau$ *for any real* t_0,

(2) *if h and h' are trigonometric polynomials (in θ) and $h(t;z,\theta) = h'(t;z,\theta)$ for all t, then $⨍ h\,(t;z,\theta) = ⨍ h'\,(t;z,\theta)$ for all t, and*

(3) $T_N ⨍ h = ⨍ T_N h$.

Proof. For a finite t_0, $\operatorname{res}_{R=0}\frac{1}{R}\int_{t_0}^t e^{-R|\tau|}u(\tau)\,d\tau$ is equal to the evaluation $\int_{t_0}^t e^{-R|\tau|}u(\tau)\,d\tau\big|_{R=0} = \int_{t_0}^t u(\tau)\,d\tau$, since the integral is entire in R, from which (1) follows. If h is a trigonometric polynomial, then $⨍ h\,(t;z,\theta) = \operatorname*{res}_{R=0}\frac{1}{R}\int_{-\infty}^t e^{-R|\tau|}h(\tau;z,\theta)\,d\tau$, yielding (2). Claim (3) is trivial. □

It is thus natural to employ the compelling notation

$$⨍_{-\infty}^t h(\tau;z,\theta)\,d\tau \equiv ⨍ h\,(t;z,\theta),$$

and to view z and θ as mere parameters. We also define

$$⨍_t^\infty h(\tau;z,\theta)\,d\tau := \lim_{N\to\infty}\operatorname*{res}_{R=0}\frac{1}{R}\int_t^\infty e^{-R|\tau|}T_N h(\tau;z,\theta)\,d\tau$$

when it makes sense (for instance, if $h(t;z,\theta) \equiv \bar{h}(-t;z^{-1},-\theta)$ with $\bar{h}$ as in (6.19) and $\bar{h}^p$ analytic on $\{0<|z|<\tau\}\times\mathbb{T}^d$), and write

$$⨍_{\pm\infty}^t = -⨍_t^{\pm\infty} \quad \text{and} \quad ⨍_{-\infty}^t + ⨍_t^\infty = ⨍_{-\infty}^\infty .$$

Finally, $⨍_\infty$ stands for the operator that is defined by

$$⨍_\infty h\,(t;z,\theta) \equiv ⨍_\infty^t h(\tau;z,\theta)\,d\tau,$$

together with the previous line. Property (1) in Proposition 6.9 yields

Corollary 6.10. *The (regularized) integral satisfies*

$$\partial_t ⨍ h\,(t;z,\theta) = \frac{d}{dt}⨍_{-\infty}^t h(\tau;z,\theta)\,d\tau = h(t;z,\theta).$$

Proposition 6.11. *Suppose h is analytic on $\{|z|<\tau\}\times\mathbb{T}^d$, and denote $h(t;z,\theta) \equiv h(ze^{\gamma t},\theta+\omega t)$. Then*

$$⨍_{-\infty}^t h(\tau;z,\theta)\,d\tau = \langle h_0\rangle t + \mathcal{D}^{-1}h_0(\theta+\omega t) + \mathcal{I}\delta_1 h(ze^{\gamma t},\theta+\omega t).$$

Proof. Decompose $h = \langle h_0\rangle + [h_0 - \langle h_0\rangle] + \delta_1 h$ and use the linearity of $⨍$. □

Remark 6.12. Observe that, unless the integrals converge in the traditional sense, $⨍_{-\infty}^{t}$ does generically *not* tend to $⨍_{-\infty}^{\infty}$ or 0 as t tends to ∞ or $-\infty$, respectively.

COROLLARY 6.13. *Under the conditions of Proposition 6.11*

$$⨍_{-\infty}^{t} \partial_\tau h(\tau; z, \theta)\, d\tau = h(t; z, \theta) - \langle h_0 \rangle.$$

Proof. Notice $\partial_\tau h(\tau; z, \theta) = \mathcal{D} h_0(\theta + \omega\tau) + \mathcal{L}\delta_1 h(ze^{\gamma\tau}, \theta + \omega\tau)$, and recall the identities $\mathcal{D}^{-1}\mathcal{D} h_0 = h_0 - \langle h_0 \rangle$ and $\mathcal{I}\mathcal{L}\delta_1 h = \delta_1 h$. □

PROPOSITION 6.14. *Analogously to the ordinary iterated integral,*

$$⨍⨍ h\,(t; z, \theta) = ⨍_{-\infty}^{t} ⨍_{-\infty}^{\tau} h(s; z, \theta)\, ds\, d\tau = ⨍_{-\infty}^{t} (t - \tau)\, h(\tau; z, \theta)\, d\tau.$$

Proof. The first identity is a definition. In order to confirm the second one, we first assume that it holds for the trigonometric polynomials $T_N h$, and compute by item (3) of Proposition 6.9 that

$$\begin{aligned} ⨍_{-\infty}^{t} ⨍_{-\infty}^{\tau} h(s)\, ds\, d\tau &= \lim_{N\to\infty} ⨍_{-\infty}^{t} T_N ⨍_{-\infty}^{\tau} h(s)\, ds\, d\tau \\ &= \lim_{N\to\infty} ⨍_{-\infty}^{t} ⨍_{-\infty}^{\tau} T_N h(s)\, ds\, d\tau = \lim_{N\to\infty} ⨍_{-\infty}^{t} (t-\tau) T_N h(\tau)\, d\tau \\ &= \lim_{N\to\infty} ⨍_{-\infty}^{t} T_N\big[(t-\tau)h(\tau)\big]\, d\tau \equiv ⨍_{-\infty}^{t} (t-\tau) h(\tau)\, d\tau \end{aligned}$$

keeping z and θ implicit. Notice that the last identity is a definition.

Suppose now that h is a trigonometric polynomial in θ. If $t \le 0$, an integration by parts (with sufficiently large $\mathfrak{Re}\, R > 0$ followed by analytic continuation) yields

$$\begin{aligned} ⨍_{-\infty}^{t} ⨍_{-\infty}^{\tau} h(s; z, \theta)\, ds\, d\tau &= \operatorname*{res}_{R=0} \frac{1}{R} \int_{-\infty}^{t} e^{R\tau} ⨍_{-\infty}^{\tau} h(s; z, \theta)\, ds\, d\tau \\ &= t ⨍_{-\infty}^{t} h(s; z, \theta)\, ds - \operatorname*{res}_{R=0} \frac{1}{R^2} H(R, t), \end{aligned}$$

where

$$H(R, t) := \int_{-\infty}^{t} e^{R\tau} h(\tau; z, \theta)\, d\tau.$$

But one easily checks that $\operatorname{res}_{R=0} \frac{1}{R^2} H \equiv \operatorname{res}_{R=0} \frac{1}{R}\frac{d}{dR} H$ from the Laurent series, and that the latter equals $⨍_{-\infty}^{t} \tau\, h(\tau; z, \theta)\, d\tau$. The case $t > 0$ follows either by analytic continuation or a direct computation. □

We need to extend the definition of the matrices $\mathcal{W}_j$, given above (6.8), by setting

$$\mathcal{W}_j(t;z) \equiv \mathcal{W}_j(ze^{\gamma t}) \qquad (j=1,2).$$

Then, motivated by (6.8) and its usage in the expression (6.7) of $\mathcal{K}^{-1}$ on the class of functions $h(z,\theta)$ that are analytic at $z=0$ and satisfy $h=\delta_2 h$, we also define $K=\operatorname{diag}(K_\Phi,K_\Psi)$ such that $K(t,\tau;z)\equiv \widetilde{K}(\tau-t,ze^{\gamma t})$, *i.e.*,

$$K(t,\tau;z) := \mathcal{W}_2(t;z)\mathcal{W}_1(\tau;z) - \mathcal{W}_1(t;z)\mathcal{W}_2(\tau;z).$$

For some functions $\mathcal{W}'$ and $\mathcal{W}''$, and an operator O, the shorthand notation

$$[\mathcal{W}',\mathcal{W}'']_O h := \mathcal{W}'\,O\big(\mathcal{W}''h\big) - \mathcal{W}''\,O\big(\mathcal{W}'h\big)$$

turns out to be practical. For instance,

$$⨍_{-\infty}^{t} K(t,\tau;z)\,h(\tau;z,\theta)\,d\tau = [\mathcal{W}_2,\mathcal{W}_1]_{⨍} h\,(t;z,\theta), \tag{6.20}$$

whereas, the operator in (6.7) reads $[\mathcal{W}_2,\mathcal{W}_1]_{\mathcal{I}}$. Thanks to the regularization in the integral, $[\mathcal{W}_2,\mathcal{W}_1]_{⨍}$ can be applied to a much wider class of functions than $[\mathcal{W}_2,\mathcal{W}_1]_{\mathcal{I}}$—the latter being restricted to $h(t;z,\theta)\equiv h(ze^{\gamma t},\theta+\omega t)$ with fast decay as $t\to-\infty$. But how is this operator related to (the inverse of) $\mathcal{K}=\left(\begin{smallmatrix} L & 0\\ 0 & \mathcal{L}^2\end{smallmatrix}\right)$? We now dwell on this question.

LEMMA 6.15. *Let $\mathcal{S}_1$ denote the space of functions h analytic on $\{|z|<\tau\}\times\mathbb{T}^d$ and satisfying $\langle h_1\rangle=0$. Setting*

$$Uh := (\mathcal{D}^2-\gamma^2)^{-1}h_0 + z\,\mathcal{D}^{-1}(\mathcal{D}+2\gamma)^{-1}h_1,$$

the operator $\widetilde{L}:\mathcal{S}_1\to\mathcal{S}_1$ defined by

$$\widetilde{L}h := Uh + [\mathcal{W}_{\Phi 2},\mathcal{W}_{\Phi 1}]_{\mathcal{I}}\big\{\delta_2 h + \gamma^2(\delta_2\cos\Phi^0)Uh\big\}$$

inverts L:

$$\widetilde{L} = L^{-1} \ \text{on}\ \mathcal{S}_1.$$

Remark 6.16. The fact $(LUh)_{\le 1} = h_{\le 1}$ makes the definition of $\widetilde{L}$ somewhat more understandable, since then

$$\gamma^2(\delta_2\cos\Phi^0)Uh = \delta_2(\gamma^2\cos\Phi^0\,Uh) = -\delta_2(LUh) = -LUh + h_{\le 1}.$$

Proof. First, by virtue of $L[\mathcal{W}_{\Phi 2},\mathcal{W}_{\Phi 1}]_{\mathcal{I}}\delta_2=\delta_2$, compute

$$\begin{aligned} L\widetilde{L}h &= LUh + \delta_2 h + \gamma^2(\cos\Phi^0-1)Uh = (\mathcal{L}^2-\gamma^2)Uh + \delta_2 h\\ &= (\mathcal{D}^2-\gamma^2)(Uh)_0 + z\,\mathcal{D}(\mathcal{D}+2\gamma)(Uh)_1 + \delta_2 h\\ &= h_0 + z(h_1-\langle h_1\rangle) + \delta_2 h = h - z\langle h_1\rangle. \end{aligned}$$

So, if $\langle h_1\rangle = 0$, $\widetilde{L}$ is the right inverse of L. Next, notice that $h = Lf$ is equivalent to the system

$$\begin{cases} h_0 = (\mathcal{D}^2 - \gamma^2)f_0, \quad h_1 = (\mathcal{D} + 2\gamma)\mathcal{D}f_1, \\ \delta_2 h = L(\delta_2 f) - \gamma^2(\delta_2 \cos\Phi^0)f_{\le 1}, \end{cases}$$

or, using $[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{\mathcal{I}} L\delta_2 = \delta_2$, to

$$\begin{cases} f_{\le 1} - z\langle f_1\rangle = Uh_{\le 1}, \\ \delta_2 f = [\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{\mathcal{I}}\big\{\delta_2 h + \gamma^2(\delta_2\cos\Phi^0)f_{\le 1}\big\}. \end{cases}$$

Finally, since $Uh_{\le 1} = Uh$,

$$\widetilde{L}Lf = \widetilde{L}h = f - \langle f_1\rangle\big[z + [\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{\mathcal{I}}\big\{\gamma^2(\delta_2\cos\Phi^0)z\big\}\big].$$

Demanding $\langle f_1\rangle = 0$ proves that $\widetilde{L}$ is also the left inverse of L. □

PROPOSITION 6.17. *Denote* $h(t; z, \theta) \equiv h(ze^{\gamma t}, \theta + \omega t)$ *for a function* $h \in \mathcal{S}_1$*; see Lemma 6.15. Then* $L_t = \partial_t^2 - \gamma^2\cos\Phi^0(ze^{\gamma t})$ *and* $L = \mathcal{L}^2 - \gamma^2\cos\Phi^0$ *satisfy*

$$L_t h(t; z, \theta) \equiv Lh(ze^{\gamma t}, \theta + \omega t).$$

The operator $[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍}$ *maps* $\mathcal{S}_1$ *into itself and, in fact,*

$$[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍} = L_t^{-1} \text{ on } \mathcal{S}_1.$$

Proof. First, identifying $h(t; z, \theta) \equiv h(ze^{\gamma t}, \theta + \omega t)$ for $h \in \mathcal{S}_1$, notice that

$$[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍}\delta_2 h = [\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{\mathcal{I}}\delta_2 h,$$

because the integrals converge even without regularization. Recalling (4.13) and setting $P(t; z) \equiv P(ze^{\gamma t})$ and $Q(t; z) \equiv Q(ze^{\gamma t})$, we have

$$[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍} h_{\le 1} = \frac{1}{2\gamma}[Q, P]_{⨍} h_{\le 1} + 2P ⨍⨍ Ph_{\le 1},$$

where Proposition 6.14 was used for the double integral. Splitting $Q = Q_1 - Q_2$, where $Q_2(z) \equiv z^{-1}$, all regularized integrals above—except for $⨍ Q_2 h_{\le 1}$—can be replaced by the convergent $\int_{-\infty}^{\cdot}$, since $P(z)$ and $Q_1(z)$ are both $\mathcal{O}(z)$. The exceptional one can be explicitly computed:

$$\Big(⨍ Q_2 h_{\le 1}\Big)^{\widehat{}}(t; z, q) := \operatorname*{res}_{R=0} \frac{1}{R}\int_{-\infty}^{t} \frac{e^{(R-\gamma+iq\cdot\omega)\tau}}{z}\big(\hat{h}_0(q) + ze^{\gamma\tau}\hat{h}_1(q)\big)\,d\tau$$

$$= e^{iq\cdot\omega t}\begin{cases} \hat{h}_1(0)t - z^{-1}e^{-\gamma t}\gamma^{-1}\hat{h}_0(0) & \text{if } q = 0, \\ (i\omega\cdot q)^{-1}\hat{h}_1(q) + z^{-1}e^{-\gamma t}(i\omega\cdot q - \gamma)^{-1}\hat{h}_0(q) & \text{if } q \in \mathbb{Z}^d \setminus \{0\}, \end{cases}$$

by analytic continuation from $\Re\mathfrak{e}\, R > \gamma$ to a neighbourhood of $R = 0$. On $\mathcal{S}_1$, the coefficient $\hat{h}_1(0)$ vanishes, such that $[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍} h(t; z, \theta) \equiv H(ze^{\gamma t}, \theta + \omega t)$, where H is an analytic function on $\{|z| < \tau\} \times \mathbb{T}^d$.

Leaving the details to the reader, we point out that $K_\Phi(t, \tau; z) = \gamma^{-1} \sinh\big(\gamma(t - \tau)\big) + \mathcal{O}(z^2)$ yields

$$\begin{aligned} H_1(\theta) &= -⨍_{-\infty}^{0} \gamma^{-1} \sinh(\gamma\tau)\, e^{\gamma\tau} h_1(\theta + \omega\tau)\, d\tau \\ &= -\frac{\hat{h}_1(0)}{4\gamma^2} + \mathcal{D}^{-1}(\mathcal{D} + 2\gamma)^{-1} h_1(\theta). \end{aligned}$$

In particular, $\langle H_1 \rangle = -\langle h_1 \rangle / 4\gamma^2$, meaning that $[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍}$ maps $\mathcal{S}_1$ into itself.

Since L is invertible on $\mathcal{S}_1$, by Lemma 6.15, we are left with checking that $L_t[\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍} = \mathbb{1}$ on $\mathcal{S}_1$. This boils down to the identities

$$L_t(uv) = (L_t u)v + 2(\partial_t u)(\partial_t v) + u\, \partial_t^2 v,$$

$$L_t \mathcal{W}_{\Phi 1} = L_t \mathcal{W}_{\Phi 2} = 0 \quad \text{and} \quad \mathcal{W}_{\Phi 1} \partial_t \mathcal{W}_{\Phi 2} - \mathcal{W}_{\Phi 2}\, \partial_t \mathcal{W}_{\Phi 1} = 1,$$

together with Corollary 6.10. □

Proposition 6.18. *Let $\mathcal{S}_0$ be the space of analytic functions h on $\{|z| < \tau\} \times \mathbb{T}^d$ and denote $h(t; z, \theta) \equiv h(ze^{\gamma t}, \theta + \omega t)$. Then $\mathcal{L} = \mathcal{D} + \gamma z \partial_z$ satisfies*

$$\partial_t h(t; z, \theta) \equiv \mathcal{L} h(ze^{\gamma t}, \theta + \omega t).$$

The operator $⨍$ maps $\mathcal{S}_0$ into itself and, in fact,

$$⨍ = \partial_t^{-1} \text{ on } \mathcal{S}_0.$$

Proof. This follows from Corollaries 6.10 and 6.13, and Proposition 6.11. □

Corollary 6.19. *With the interpretation $h(t; z, \theta) \equiv h(ze^{\gamma t}, \theta + \omega t)$, the operator $\mathcal{K} = \left(\begin{smallmatrix} L & 0 \\ 0 & \mathcal{L}^2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} L_t & 0 \\ 0 & \partial_t^2 \end{smallmatrix}\right)$ is invertible on the space $\mathcal{S}_1 \times \mathcal{S}_0$, where*

$$\mathcal{K}^{-1} = [\mathcal{W}_2, \mathcal{W}_1]_{⨍} = \begin{pmatrix} [\mathcal{W}_{\Phi 2}, \mathcal{W}_{\Phi 1}]_{⨍} & 0 \\ 0 & ⨍⨍ \end{pmatrix}.$$

In terms of coordinates, by virtue of (6.20),

$$\mathcal{K}^{-1} h(t; z, \theta) \equiv ⨍_{-\infty}^{t} K(t, \tau; z) h(\tau; z, \theta)\, d\tau.$$

Proof. Invertibility follows from Propositions 6.17 and 6.18. Recall that $K = \left(\begin{smallmatrix} K_\Phi & 0 \\ 0 & K_\Psi \end{smallmatrix}\right)$, where $K_\Psi(t,\tau;z) \equiv t-\tau$. Hence, Proposition 6.14 finishes the proof. □

6.6. Asymptotic expansion for the splitting matrix

Starting from (6.17) and using Proposition 6.18 with $\langle Y_0^u \rangle = \langle \mathcal{D} X_0^u \rangle = 0$, we have

$$Y^u(t;z,\theta) = ⨍_{-\infty}^{t} \Omega(X^u)(\tau;z,\theta)\,d\tau$$

The identity $X^s(t;z,\theta) \equiv (2\pi,0) - X^u(-t;z^{-1},-\theta)$ yields

$$Y^s(t;z,\theta) \equiv Y^s(ze^{\gamma t},\theta{+}\omega t) \equiv Y^u(z^{-1}e^{-\gamma t},-\theta{-}\omega t) \equiv Y^u(-t;z^{-1},-\theta),$$

such that defining the d component column vector

$$\Delta(t;z,\theta) := (Y_\Psi^u - Y_\Psi^s)(t;z,\theta)$$

and the functions

$$f^{s,u}(t;z,\theta) := f(X^{s,u}(t;z,\theta) + (0,\theta+\omega t))$$

thus results in an expression for the splitting matrix Υ of (1.25) in terms of regularized integrals: $\Upsilon = \partial_\theta \Delta(0;1,0)$ with

$$\partial_\theta \Delta(t;z,\theta) = \lambda ⨍_{-\infty}^{t} \partial_\theta f_\psi^u(\tau;z,\theta)\,d\tau + \lambda ⨍_{t}^{\infty} \partial_\theta f_\psi^s(\tau;z,\theta)\,d\tau,$$

where the subindices attached to $f^{s,u}$ stand for partial derivatives (our convention is $\bar\psi = (\phi,\psi)$ and $\partial = \partial_{\bar\psi} = (\partial_\phi,\partial_\psi)$).

Remark 6.20. Extracting here the first order in $\epsilon = g^{-2}\lambda$ casts the Melnikov matrix in a compact form in terms of the regularized integrals:

$$\Upsilon^1 = \partial_\theta \Delta^1(0;1,0) = g^2 ⨍_{-\infty}^{\infty} \partial_\psi^2 f(\Phi^0(e^{g\tau}),\omega\tau)\,d\tau.$$

The reader is invited to compare this with (6.12) and the equation following it. Without going into the details, we point out that a similar procedure, using Lemma 6.4, that resulted in the exponentially small bound on Υ^1 can be applied here, despite the "unusual" integrals.

Suppose that, even with the superscript u, the integral

$$⨍_{t}^{\infty} \partial_\theta f_\psi^u(\tau;z,\theta)\,d\tau \tag{6.21}$$

exists. If the integrand is a trigonometric polynomial in θ, then Proposition 6.9 implies that the value of the integral only depends on the integrand's restrictions to $\{(t;z,\theta)\,|\,t\in\mathbb{R}\}$, as the notation suggests [2]. Now, *fixing* $(z,\theta)=(1,0)$ and dropping it from the notation,

$$\partial_\theta\Delta(t)=\lambda\fint_{-\infty}^{\infty}\partial_\theta f_\psi^u(\tau)\,d\tau+\lambda\fint_t^{\infty}\big[f_{\psi\bar\psi}^s\,\partial_\theta(X^s-X^u)\big](\tau)\,d\tau, \quad (6.22)$$

since $X^u(\tau;1,0)\equiv X^s(\tau;1,0)$.

Let us make the assumption that $\psi\mapsto f(\,\cdot\,,\psi)$ is a trigonometric polynomial, such that, by Proposition 5.3, each order of X in ϵ is a trigonometric polynomial in θ. Even so, equation (6.22) is *formal* in the sense that the integrands are defined for $\tau\gg 0$ only up to an arbitrarily high order in ϵ. However, it is *asymptotic*, which is to say that at each order the identity is *exact*. Put differently, (6.22) is a collection of exact identities, one for each order in ϵ, written in closed form. Moreover, at each order, the integrands are *analytic functions* by virtue of the extension result in Proposition 5.3. Of course, we need to check that, *each order of* (6.21), *i.e.*, the integral $\fint_t^\infty\big(\partial_\theta f_\psi^u\big)^\ell(\tau;z,\theta)\,d\tau$ with arbitrary $\ell\in\mathbb{N}$, exists [3].

The point of the formula above is twofold. First, the integral $\fint_{-\infty}^{\infty}\partial_\theta f_\psi^u(\tau)\,d\tau$ (at each order in ϵ) turns out to be exponentially small in the limit $g\to 0$. Second, we can actually construct a (formal) recursion relation for the function $\partial_\theta(X^s-X^u)(\tau)$, taking us to ever-increasing orders in ϵ, as follows. Differentiate both sides of (6.17) with respect to θ and obtain

$$\begin{pmatrix}L_t & 0\\ 0 & \partial_t^2\end{pmatrix}\partial_\theta X^{s,u}=M^{s,u}\partial_\theta X^{s,u}+\lambda f_{\bar\psi\psi}^{s,u}$$

with $L_t=\partial_t^2-\gamma^2\cos\Phi^0(ze^{\gamma t})$, as in Proposition 6.17, and

$$M^{s,u}(t;z,\theta):=\begin{pmatrix}g^2\cos\Phi^{s,u}(t;z,\theta)-\gamma^2\cos\Phi^0(ze^{\gamma t}) & 0\\ 0 & 0\end{pmatrix}+\lambda f_{\bar\psi\bar\psi}^{s,u}(t;z,\theta).$$

Owing to the θ-derivative acting on $X^{s,u}$ and Corollary 6.19, we get

$$\partial_\theta X^{s,u}(t;z,\theta)=\fint_{\pm\infty}^{t}K(t,t';z)\,\big[M^{s,u}\partial_\theta X^{s,u}+\lambda f_{\bar\psi\psi}^{s,u}\big](t';z,\theta)\,dt',$$

upon choosing ∞ in conjunction with X^s and $-\infty$ with X^u. The most convenient way to see this is to infer the identity for $\partial_\theta X^u$ and

[2]This is the first of the two places where we need to restrict ourselves to the use of trigonometric polynomials...

[3]... and this is the second.

employing $K(-t,-\tau;z^{-1}) \equiv -K(t,\tau;z)$ and $X^s(t;z,\theta) \equiv (2\pi,0) - X^u(-t;z^{-1},-\theta)$.

At $(z,\theta) = (1,0)$, $M^s(t) \equiv M^u(t)$ and $f^s_{\bar\psi\psi}(t) \equiv f^u_{\bar\psi\psi}(t)$, such that (2) of Proposition 6.9 validates

$$\begin{aligned}\partial_\theta(X^s - X^u)(\tau) = \int_\infty^{-\infty} K(\tau,\tau')\left[M^u\partial_\theta X^u + \lambda f^u_{\bar\psi\psi}\right](\tau')\,d\tau' + \\ + \int_\infty^{\tau} K(\tau,\tau')\left[M^s\partial_\theta(X^s - X^u)\right](\tau')\,d\tau'.\end{aligned} \tag{6.23}$$

By construction, the quantities appearing in square brackets above are θ-gradients:

$$M^{s,u}\partial_\theta X^{s,u} + \lambda f^{s,u}_{\bar\psi\psi} = \partial_\theta \mathfrak{A}^{s,u}$$

with

$$\begin{aligned}\mathfrak{A}^{s,u}(t;z,\theta) &:= \begin{pmatrix} L_t & 0 \\ 0 & \partial_t^2 \end{pmatrix} X^{s,u}(t;z,\theta) \\ &= \left[\Omega(X^{s,u}) - \begin{pmatrix} \gamma^2\cos\Phi^0(ze^{\gamma t}) & 0 \\ 0 & 0 \end{pmatrix} X^{s,u}\right](t;z,\theta)\end{aligned}$$

regardless of the value of (z,θ) and, but only at $(z,\theta) = (1,0)$,

$$\left[M^s\partial_\theta(X^s - X^u)\right](t) = \left[\partial_\theta(\mathfrak{A}^s - \mathfrak{A}^u)\right](t).$$

Notice the structure of (6.23); it is an affine fixed point equation of the form $\zeta = \mathcal{E} + BM^s\zeta$ solved formally by $\zeta(t) = \partial_\theta(X^s - X^u)(t)$. This is highly interesting from the point of view of asymptotic analysis, since $M^s = \mathcal{O}(\epsilon)$ multiplies ζ on the right-hand side. Indeed, truncating the Taylor series in ϵ, *e.g.*,

$$\zeta^{\le k} = \sum_{i=0}^{k} \epsilon^i \zeta^i,$$

the identities $\zeta^k = (\mathcal{E} + BM^s\zeta)^k = (\mathcal{E} + BM^s\zeta^{\le k-1})^k$ become exact. Hence, we can recursively construct arbitrarily high orders of $\partial_\theta(X^s - X^u)(t)$ from its lower orders, without the need of diverting from $\{(t;z,\theta) = (t;1,0)\,|\,t \in \mathbb{R}\}$ in order to compute the θ-derivative. Cumulatively, we have $\zeta^k = \left[\sum_{j=0}^{k-1}(BM^s)^j\mathcal{E}\right]^k$, where the index j is a power.

PROPOSITION 6.21. *In brief, the (presumably) divergent Neumann series* $\sum_{j=0}^{\infty}(BM^s)^j\mathcal{E}$ *is an asymptotic expansion of* $\partial_\theta(X^s - X^u)(t)$ *in the sense that*

(1) *At each order in ϵ it terminates after finitely many well-defined terms, and*
(2) $\Big[\partial_\theta(X^s - X^u) - \big[\sum_{j=0}^{\infty}(BM^s)^j\mathcal{E}\big]^{\le k}\Big](t) = \mathcal{O}(\epsilon^{k+1})$ *for each* $k \in \mathbb{N}$.

Once inserted into (6.22), the latter series provides us with an *asymptotic expansion for the splitting matrix $\partial_\theta\Delta(t)$ on the homoclinic trajectory* $(t; z, \theta) \equiv (t; 1, 0)$, and in particular for $\Upsilon = \partial_\theta\Delta(0; 1, 0)$:

Corollary 6.22. *In the asymptotic sense of Proposition 6.21,*

$$\partial_\theta\Delta(t) = \lambda \fint_{-\infty}^{\infty} \partial_\theta f^u_\psi(\tau)\, d\tau + \lambda \fint_t^{\infty} \Big[f^s_{\psi\bar\psi} \sum_{j=0}^{\infty}(BM^s)^j\mathcal{E}\Big](\tau)\, d\tau$$

The operator B above has the expression

$$Bh(t; z, \theta) = \fint_{\infty}^{t} K(t, \tau; z) h(\tau; z, \theta)\, d\tau, \tag{6.24}$$

whereas $\mathcal{E}$ is the restriction of

$$\mathcal{E}(t; z, \theta) := \fint_{\infty}^{-\infty} K(t, \tau; z)\, \partial_\theta \mathfrak{A}^u(\tau; z, \theta)\, d\tau$$

to $(z, \theta) = (1, 0)$. We may split the kernel K into the useful sum

$$K(t, \tau; z) = \sum_{i=0}^{1}\sum_{j=1}^{2}(t-\tau)^i K_{ij}(t; z)\bar K_{ij}(\tau; z), \tag{6.25}$$

setting

$$K_{01} = \frac{1}{2\gamma}\begin{pmatrix} Q & 0 \\ 0 & 0\end{pmatrix}, \quad \bar K_{01} = \begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix}, \quad K_{02} = -\frac{1}{2\gamma}\begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix},$$
$$\bar K_{02} = \begin{pmatrix} Q & 0 \\ 0 & 0\end{pmatrix}, \quad K_{11} = 2\begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix}, \quad \bar K_{11} = \begin{pmatrix} P & 0 \\ 0 & 0\end{pmatrix},$$

with $P(t; z) \equiv P(ze^{\gamma t})$ and $Q(t; z) \equiv Q(ze^{\gamma t})$ defined in (4.13), and

$$K_{12} = \bar K_{12} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}\end{pmatrix}.$$

In this way, also $\mathcal{E}$ splits into pieces:

$$\mathcal{E}(t; z, \theta) = \sum_{i=0}^{1}\sum_{p=0}^{i}\sum_{j=1}^{2} t^p K_{ij}(t; z)\, \mathcal{E}^p_{ij}(z, \theta),$$

where, in the Kronecker delta notation,

$$\mathcal{E}_{ij}^{p}(z,\theta) := -\!\!\!\!\!\!\int_{-\infty}^{\infty} (-\tau)^{\delta_{i1}-p} \bar{K}_{ij}(\tau;z)\, \partial_\theta \mathfrak{A}^u(\tau;z,\theta)\, d\tau.$$

We shall see that, for each fixed pair (z,θ), these *t-independent factors* are *exponentially small* with respect to g, as the latter tends to zero. Furthermore, if

$$K_{ij}^{p}(t;z) \equiv t^p K_{ij}(t;z),$$

then

$$(BM^s)^l \mathcal{E} = \sum_{i=0}^{1} \sum_{p=0}^{i} \sum_{j=1}^{2} \big[(BM^s)^l K_{ij}^p\big]\, \mathcal{E}_{ij}^p.$$

This is so because, by Proposition 6.9, all integrals due to (6.24) *factorize* formally: if h is a trigonometric polynomial at each order, then

$$-\!\!\!\!\!\!\int_{t}^{\infty} (h\mathcal{E})(\tau;z,\theta)\, d\tau = \sum_{i=0}^{1} \sum_{p=0}^{i} \sum_{j=1}^{2} \left[-\!\!\!\!\!\!\int_{t}^{\infty} h(\tau;z,\theta)\tau^p K_{ij}(\tau;z)\, d\tau \right] \mathcal{E}_{ij}^p(z,\theta).$$

By virtue of Corollary 6.22, we infer

PROPOSITION 6.23. *On the homoclinic trajectory, the following asymptotic expansion of the splitting matrix holds:*

$$\partial_\theta \Delta(t) = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \lambda\, \partial_\theta f_\psi^u(\tau)\, d\tau + c_{ij}^p(t)\mathcal{E}_{ij}^p, \tag{6.26}$$

where repeated indices are contracted ($i \in \{0,1\}$, $p \in \{0,i\}$, $j \in \{1,2\}$*),* $(z,\theta) = (1,0)$*, and*

$$c_{ij}^p(t) := -\!\!\!\!\!\!\int_{t}^{\infty} \left[\lambda\, f_{\psi\bar{\psi}}^s \sum_{l=0}^{\infty} (BM^s)^l K_{ij}^p\right](\tau)\, d\tau.$$

As we already pointed out, but have not yet proved, the terms appearing in the asymptotic expansion of Proposition 6.23 are exponentially small with respect to g. The factors c_{ij}^p shall pose no problems, the functions M^s and K_{ij}^p being explicit and simple. In brief, Theorem 2 begins to emerge!

For the record (at $(z,\theta) = (1,0)$),

$$c_{ij}^p(t) = (-1)^{p+j\delta_{i0}} -\!\!\!\!\!\!\int_{-\infty}^{-t} \left[\lambda\, f_{\psi\bar{\psi}}^u \sum_{l=0}^{\infty} (B^u M^u)^l K_{ij}^p\right](\tau)\, d\tau, \tag{6.27}$$

where B_u is shorthand for the operator $[\mathcal{W}_2, \mathcal{W}_1]_{\not f}$ appearing in (6.20).

6.7. Emergence of exponential smallness

The "asymptotic" integrals in (6.26), including $\mathcal{E}^p_{ij}$, are of the form

$$\fint_{-\infty}^{\infty} \partial_\theta F(X^u;t;z,\theta)\,dt \equiv \sum_{\ell=0}^{\infty}\epsilon^\ell \sum_{0<|q|\le \ell N} iq\,e^{iq\cdot\theta}\fint_{-\infty}^{\infty}\bigl[\hat F(\hat X^u;t;z,q)\bigr]^\ell\,dt,$$

evaluated at $(z,\theta)=(1,0)$, where the integrand on the right-hand side is a trigonometric polynomial of degree $\le \ell N$ by Proposition 5.3. We also point out that, by construction, the latter are θ-gradients, which allows us to omit the harmful $q=0$ terms. Above, F depends on $X=X^u$ locally, *i.e.*, only through $X(t;z,\theta)$, as well as analytically near X^0, *i.e.*, the series

$$F(X)=F(X^0+\widetilde X)=\sum_{k=0}^{\infty}F^{(k)}\bigl(\widetilde X\bigr)^{\otimes k} \tag{6.28}$$

converges. In fact, since F is one of the functions in

$$\bigl\{\lambda f^u_\psi\bigr\}\bigcup\bigl\{-(-t)^{\delta_{i1}-p}\bar K_{ij}\,\mathfrak{A}^u \bigm| 0\le p\le i\le 1 \text{ and } 1\le j\le 2\bigr\}, \tag{6.29}$$

we observe that

$$F(X^u;t;z,\theta)\equiv t^pF(X^u;0;ze^{\gamma t},\theta+\omega t)\qquad (p\in\{0,1\}), \tag{6.30}$$

with $(z,\theta)\mapsto[F(X^u;0;z,\theta)]^\ell$ analytic on $(\mathbb{U}_{\tau,\vartheta}\setminus\{0\})\times\{|\Im\mathfrak{m}\,\theta|\le\sigma\}$ for all ℓ by virtue of Proposition 5.3. At $z=0$ there is a simple pole, due to $\bar K_{02}$, at worst.

Next, we present a lemma that we use for analyzing such integrals. Its proof is given at the end of the section.

Lemma 6.24 (Shift of contour). *Suppose that the function $h(t;z,\theta)\equiv t^ph(ze^{gt},\theta+\omega t)$ is analytic with respect to $(z,\theta)\in(\mathbb{U}_{\tau,\vartheta}\setminus\{0\})\times\{|\Im\mathfrak{m}\,\theta|\le\sigma\}$ and $\hat h(\,\cdot\,,0)=0$. Moreover, set $t_q:=\operatorname{sgn}(\omega\cdot q)\vartheta g^{-1}$ and*

$$H_q(R):=\int_0^\infty e^{(iq\cdot\omega-R)t}(t+it_q)^p\hat h(e^{g(t+it_q)},q)\,dt$$

for each $q\in\mathbb{Z}^d\setminus\{0\}$. If

$$\left|\fint_{-\infty}^{0} e^{iq\cdot\omega t}(t+it_q)^p\hat h(e^{g(t+it_q)},q)\,dt\right|\le A(g)e^{-\sigma|q|}$$

and if $H_q(R)$ admits an analytic continuation to $\{0<|R|\le\rho\}$ with a pole of order k at $R=0$, respecting the bound

$$\sup_{|R|=\rho}|H_q(R)|\le B(g)e^{-\sigma|q|},$$

then we obtain the exponentially small ($c > 0$) bound

$$\left|\fint_{-\infty}^{\infty} h(t;1,0)\,dt\right| \le C\left[A(g) + B(g)\sum_{j=0}^{k}\frac{1}{j!}\Big(\frac{\rho\vartheta}{g}\Big)^{j}\right]e^{-cg^{-1/(\nu+1)}}.$$

With the aid of Lemma 6.24, we shall prove in Chapter 7 the following key result:

PROPOSITION 6.25 (Convergence vs. exponential smallness). *Fix a $t \in \mathbb{R}$. There exist positive constants c, ϵ_1, ϵ_2, and C, such that the estimates*

$$|\partial_\theta \Delta^\ell(t)| \le C\begin{cases}\epsilon_2^{-\ell}\ell!^{4(\nu+1)}e^{-cg^{-1/(\nu+1)}}\\ \epsilon_1^{-\ell}\end{cases}$$

both hold true for all $\ell = 1, 2, \ldots$.

This dichotomy, in which the exponential smallness competes with the usual bound due to convergence, is not new; see [Gal94, GGM99, Pro03]. It is remarkable that we get the same exponent of the factorial, $4(\nu + 1)$, as the latter articles—even though our method is quite different.

Theorem 2 follows immediately from Proposition 6.25 by an argument due to Gallavotti, *et al.*: For each g, let $n(g)$ be a positive integer. If $|\tilde\epsilon| < \frac{1}{2}$ and $\epsilon = \tilde\epsilon \min(\epsilon_2 n(g)^{-4(\nu+1)}, \epsilon_1)$,

$$\begin{aligned}|\partial_\theta\Delta(t)| &\le Ce^{-cg^{-1/(\nu+1)}}\sum_{\ell=1}^{n(g)}\Big(\frac{|\epsilon|}{\epsilon_2}\Big)^{\ell}\ell!^{4(\nu+1)} + C\sum_{\ell=n(g)+1}^{\infty}\Big(\frac{|\epsilon|}{\epsilon_1}\Big)^{\ell}\\ &\le Ce^{-cg^{-1/(\nu+1)}}\frac{|\epsilon|}{\epsilon_2}n(g)^{4(\nu+1)} + C\Big(\frac{|\epsilon|}{\epsilon_1}\Big)^{n(g)+1}\\ &\le C|\tilde\epsilon|e^{-cg^{-1/(\nu+1)}} + C|\tilde\epsilon|e^{n(g)\ln|\tilde\epsilon|},\end{aligned}$$

since $\ell! \le \ell^\ell \le n(g)^\ell$ in the first sum. Now, set $n(g) = cg^{-1/(\nu+1)}/\ln 2$, such that $n(g)\ln|\tilde\epsilon| \le -cg^{-1/(\nu+1)}$ and

$$\tilde\epsilon = (\epsilon/\epsilon_2)n(g)^{4(\nu+1)} = \epsilon_2^{-1}(c/\ln 2)^{4(\nu+1)}\,\epsilon g^{-4}$$

for sufficiently small g.

Remark 6.26. We used the exponentially small but diverging estimate to bound the partial sum $\sum_{\ell=1}^{n(g)}\epsilon^\ell\partial_\theta\Delta^\ell$, whereas the remainder of the series was easily controlled by convergence. As $n(g) \to \infty$ with $g \to 0$, the important thing here is to have the exponentially small bound on $\partial_\theta \Delta^\ell$ for *arbitrarily* large ℓ, in addition to the ϵ-analyticity of $\partial_\theta \Delta$.

Proof of Lemma 6.24. By shifting the contour of integration from $\mathbb{R}$ to the complex plane by $it_q := i\,\mathrm{sgn}(\omega\cdot q)\vartheta g^{-1}$ units, we compute

$$\begin{aligned}\fint_{-\infty}^{\infty} t^p \hat{h}(e^{gt},q)e^{iq\cdot\omega t}\,dt &= \operatorname*{res}_{R=0}\int_{-\infty}^{\infty} e^{-R|t|}t^p\hat{h}(e^{gt},q)e^{iq\cdot\omega t}\,dt \\ &= \operatorname*{res}_{R=0}\frac{1}{R}\Big\{e^{-iRt_q}H_q(R)+e^{iRt_q}I_q(R)\Big\}e^{-\vartheta g^{-1}|\omega\cdot q|},\end{aligned}$$

where H_q is the integral defined in the formulation of the lemma and

$$I_q(R) := \int_{-\infty}^{0} e^{(iq\cdot\omega+R)t}(t+it_q)^p\hat{h}(e^{g(t+it_q)},q)\,dt.$$

There were, *a priori*, two additional line integrals $\int_0^{it_q}$, but they cancel due to the residue at $R=0$, as is easily checked. Because $\hat{h}(\,\cdot\,,0)=0$, (the analytic continuation of) $I_q(R)$ cannot have a pole at $R=0$, which we infer from the Taylor expansion of $h(z,\theta)$ at $z=0$. Hence,

$$\operatorname*{res}_{R=0}\frac{e^{iRt_q}I_q(R)}{R} = \fint_{-\infty}^{0} e^{iq\cdot\omega t}(t+it_q)^p\hat{h}(e^{g(t+it_q)},q)\,dt.$$

If $H_q(R)$ has a pole of order k at $R=0$, then

$$\left|\operatorname*{res}_{R=0}\frac{e^{-iRt_q}H_q(R)}{R}\right| = \left|\sum_{j=0}^{k}\frac{(-it_q)^j H_{q,-j}}{j!}\right| \le \sum_{j=0}^{k}\frac{1}{j!}\Big(\frac{\rho\vartheta}{g}\Big)^j \sup_{|R|=\rho}|H_q(R)|,$$

because the Laurent coefficients

$$H_{q,-j} := \frac{1}{2\pi}\oint_{|R|=\rho}\frac{H_q(R)}{R^{-j+1}}\,dR$$

satisfy

$$|H_{q,-j}| \le \rho^j \sup_{|R|=\rho}|H_q(R)|,$$

whenever the circle $|R|=\rho$ is inside the domain of H_q.

Under our current assumptions, for any $0<\delta<\sigma$ and $q\in\mathbb{Z}^d\setminus\{0\}$,

$$\left|\fint_{-\infty}^{\infty} t^p\hat{h}(e^{gt},q)e^{iq\cdot\omega t}\,dt\right| \le D(g)e^{-\delta|q|}e^{-w(\vartheta,\sigma-\delta)g^{-1/(\nu+1)}},$$

where

$$D(g) := \left[A(g)+B(g)\sum_{j=0}^{k}\frac{1}{j!}\Big(\frac{\rho\vartheta}{g}\Big)^j\right],$$

since—mimicking the proof of Lemma 6.4—we are free to take $w(\vartheta, \sigma - \delta) := (\vartheta a)^{1/(\nu+1)} \left(\frac{\sigma-\delta}{\nu}\right)^{\nu/(\nu+1)} (\nu + 1)$. Summation over q produces a factor $C\delta^{-d}$. □

Chapter 7

Proof of Theorem 2

We are left with proving Proposition 6.25, since Theorem 2 was already shown to be its corollary. Here things are most conveniently explained using tree diagrams. Consequently, and contrary to what the title of the work suggests, there will be trees all over the following pages! However, each tree will be treated as an *individual*, solely for bookkeeping benefits, and no cancellations nor regroupings are forced upon them.

By Proposition 6.23, we need to consider the simple factors c_{ij}^{p}, as well as the more involved integrals

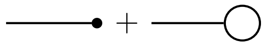
$$\fint_{-\infty}^{\infty} \hat{F}(\hat{X}^u; t, z, q)\, dt \equiv \sum_{\ell=0}^{\infty} \epsilon^{\ell} \fint_{-\infty}^{\infty} \big[\hat{F}(\hat{X}^u; t; z, q)\big]^{\ell}\, dt \qquad (7.1)$$

of Section 6.7, at each order $\ell \geq |q|/N > 0$. As F comes from the collection in (6.29), all integrals of the latter type shall be controlled with the aid of Lemma 6.24.

Due to the superscript u—referring to the unstable manifold—in the integrand above, the required bounds on the integrals over $\mathbb{R}_-$ in $\fint_{-\infty}^{\infty} = \fint_{-\infty}^{0} + \fint_{0}^{\infty}$ are straightforward, and are discussed later.

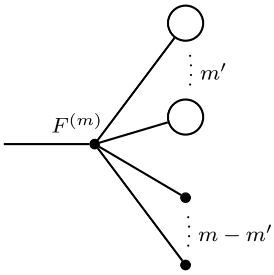
In order to deal with $\fint_{0}^{\infty}$, we present the procedure below, which amounts to *little more than integration by parts.* First, we expand F according to (6.28) and split

$$\widetilde{X} = \widetilde{X}_{\leq 1} + \delta_2 \widetilde{X} =$$



like in Chapter 5, at the same time dropping the superscript u from the notation. We can then express F pictorially as

$$\sum_{m=0}^{\infty} \sum_{0 \le m' \le m} \binom{m}{m'}$$



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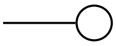
where the binomial coefficient in front of the tree comes from the combinatorics of shuffling the arguments of the symmetric $F^{(m)}$ [1], which is attached to the root (node). Next, we replace the subtrees  $= \delta_2 \widetilde{X}$ with the expansion (5.9) derived from (5.5).

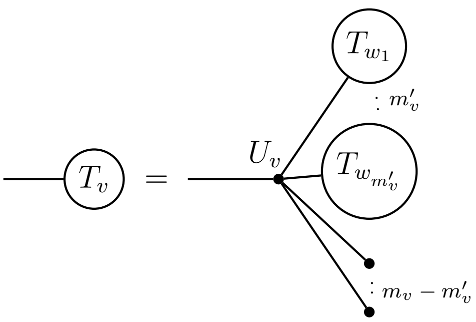
In a tree T_{v_0} with root v_0 obtained like this, a generic subtree T_v with root v has the expression

$$T_v = U_v\big(\mathcal{K}^{-1} T_{w_1}, \dots, \mathcal{K}^{-1} T_{w_{m'_v}}; \big(\widetilde{X}_{\le 1}\big)^{\otimes (m_v - m'_v)}\big), \tag{7.2}$$

where T_{w_j} is a subtree—with root w_j—entering v and the *node function*

$$U_v = \begin{cases} F^{(m_v)} \text{ coming from (6.29)}, & \text{if } v = v_0, \\ w^{(m_v)} \text{ as in (5.2)}, & \text{otherwise.} \end{cases}$$

In a pictorial representation, there are m_v lines entering the node v and precisely $m_v - m'_v$ of the latter are leaving a black dot (•) end node:



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The "whole" tree T_{v_0} contributes at orders ℓ satisfying

$$\ell \ge 1 + (m_{v_0} - m'_{v_0}) + \sum_{j=1}^{m'_v} \deg T_{w_j} =: \mathrm{d}(T_{v_0}), \tag{7.3}$$

[1] This convention merely facilitates drawing.

where $\deg T_w$—defined in (5.8)—counts end nodes with suitable weights as well as nodes with exactly one entering line in the subtree T_w. By this we mean that $\mathrm{d}(T_{v_0})$ is the largest integer such that

$$T_{v_0} = \mathcal{O}(\epsilon^{\mathrm{d}(T_{v_0})}) \quad \text{as} \quad \epsilon \to 0.$$

7.1. Some combinatorics

LEMMA 7.1. *The number of nodes,* $\mathrm{n}(T_{v_0})$*, of a tree* T_{v_0} *contributing at order* $\ell \geq 2$ *is bounded by*

$$\mathrm{n}(T_{v_0}) \leq 2(\ell - 1). \tag{7.4}$$

The number of end nodes is at most $\ell - 1$*.*

Proof. $\mathrm{n}(T_{v_0})$ attains its maximum with respect to the “degree” $\mathrm{d}(T_{v_0})$—which is a lower bound on ℓ—as follows:

(1) If $\mathrm{n}(T_{v_0})$ is even, there is only one line entering the root v_0 and the rest of the tree is binary, *i.e.*, contains only end nodes and nodes with exactly two entering lines.

(2) If $\mathrm{n}(T_{v_0})$ is odd, the tree is binary.

Moreover, each of the end nodes is either $\bullet$ or $\textcircled{1}$, which contribute the least to the degree; see (5.8). These choices minimize the number of end nodes when the $\mathcal{O}(\epsilon)$ nodes having exactly one entering line are excluded (except at v_0 which is always $\mathcal{O}(\epsilon)$). Therefore, $\mathrm{d}(T_{v_0})$ gets minimized with respect to the number of all nodes, $\mathrm{n}(T_{v_0})$. Since in a binary tree of j end nodes there are $2j - 1$ nodes, we infer (7.4).

If there were more than $\ell - 1$ end nodes, each of which is $\mathcal{O}(\epsilon)$, $\mathrm{d}(T_{v_0})$ would exceed ℓ. $\square$

COROLLARY 7.2. *At most* $2^{6\ell}$ *trees contribute at order* ℓ*.*

Proof. It is well-known that the number of (rooted) trees with k indistinguishable nodes is $N(k) := \frac{1}{k}\binom{2k-2}{k-1} \leq \frac{1}{k}4^{k-1}$, which follows from generating functions (see [Drm04]) and the bound $(2m)! \leq 4^m (m!)^2$. By Lemma 7.1, the number of end nodes is less than ℓ. We label the latter arbitrarily by the labels in $\{\bullet\} \bigcup \big\{\textcircled{m} \mid 1 \leq m \leq \ell - 1\big\}$ in order to form an upper bound on the number of our trees; these are the only possible labels, as otherwise $\mathrm{d}(T_{v_0})$ certainly exceeds the order ℓ—contradicting (7.3). The labeling of a tree with j end nodes can be carried out in at most $\binom{j+\ell-1}{j-1} \leq 2^{j+\ell-1}$ ways. The desired bound on the number of the resulting trees is thus obtained from

$$1+\sum_{k=2}^{\ell-1} N(k)\binom{\ell+k-2}{k-2}+\binom{2\ell-2}{\ell-2}\sum_{k=\ell}^{2(\ell-1)} N(k) \leq 2^{6\ell},$$

because, by Lemma 7.1 the number of the end nodes is bounded by $\ell-1$ even though the number of nodes can be as large as $2(\ell-1)$. The term 1 on the left-hand side counts the single node tree $F^{(0)}$. ☐

7.2. Simplification of integrals: scalar trees

We take a preliminary step towards bounding the values of the trees.

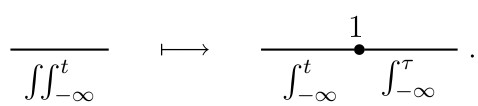
Let us split the kernels K of the operators $\mathcal{K}^{-1}$ appearing in (7.2) into four pieces according to (6.25). These operators are attached to the lines between the nodes of a tree. Each of the $2^{6\ell}$ trees counted in Corollary 7.2 thus breaks into at most $4^{2\ell}$ new trees, as there are no more than 2ℓ such lines by Lemma 7.1.

At the same time, we also expand the matrix products due to the coordinate representation of (7.2) at each node:

$$(U_v)^i(\widetilde{T}_1,\ldots,\widetilde{T}_m)=\sum_{j_1=1}^{d+1}\cdots\sum_{j_m=1}^{d+1}(U_v)^i_{j_1\ldots j_m}\widetilde{T}_1^{j_1}\ldots\widetilde{T}_m^{j_m},$$

where the $\widetilde{T}_k$'s represent all arguments of U_v (*i.e.*, lines entering the node v)—including $\widetilde{X}_{\leq 1}$—and the superindices specify vector components. We next separate each scalar term into its own tree, thus getting up to $(d+1)^{2\ell}$ of these *scalar trees* from each old tree.

There are lines carrying a factor $t-\tau$, coming from the $i=1$ terms of (6.25). They correspond to double integrals: $\iint_{-\infty}^{t}(d\tau)^2=\int_{-\infty}^{t}d\tau\,(t-\tau)$. We remove these factors, insert a new node $\tilde{v}$ on the line with a node function $U_{\tilde{v}}\equiv 1$ and an integral sign both on the line leaving and on the line entering $\tilde{v}$. This operation can be depicted as



(7.5)

It does not affect the number of trees, but slightly simplifies the discussion below, even though the number of lines in a single tree can be as much as doubled.

Altogether, we arrive at the following conclusion:

PROPOSITION 7.3. *There are less than C_d^ℓ scalar trees to be considered, with $C_d = 2^{10}(d+1)^2$, at each order ℓ. Each of these trees has at most $4(\ell-1)$ lines.*

Remark 7.4 (Some conventions). From now on, by a tree we will always refer to a scalar tree, where all the decompositions above have been carried out. In order to alleviate notation, we systematically omit all vector component indices.

Recall that the node w' is the unique predecessor of the node w. Let T_{v_0} be a generic (scalar) tree with root v_0. Denote by V the set of all nodes and by V_{int} the set of *integrated nodes*, *i.e.*, the nodes whose leaving line carry an integral. We consider the root v_0 an integrated node, such that V_{int} consists of all nodes of T_{v_0} except black dot ($\bullet$) end nodes. We can describe T_{v_0} by giving its node structure recursively: if T_v is the subtree of T with root $v \in V_{\text{int}}$,

$$T_v(t) = u_v(t) \prod_{\substack{w \in V_{\text{int}} \\ w'=v}} \int_{-\infty}^{t} T_w(\tau)\, d\tau. \tag{7.6}$$

We call u_v the *multiplier* of the node v. It comprises all factors in the expression of the tree carrying the same variable of integration ("time label"), constricted in-between integral signs. In particular, it contains all subtrees $\longrightarrow\!\!\bullet$ entering v, as these are the functions $\widetilde{X}_{\leq 1}$ involving no integrals.

To be completely explicit, the multiplier u_v of a generic node $v \in V_{\text{int}}$ is one of the functions in the list below:

(1) $\bar{K}_v \delta_2 h^{(k)}$, if v is the end node $\textcircled{k}$; see (5.4).

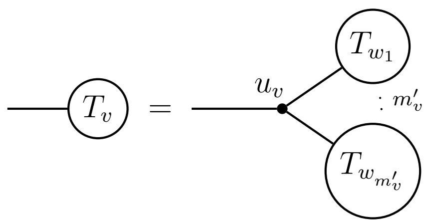
(2) $\bar{K}_v w^{(m_v)} \Big(\prod_{\substack{w \notin V_{\text{int}} \\ w'=v}} \widetilde{X}_{\leq 1}\Big)\Big(\prod_{\substack{w \in V_{\text{int}} \\ w'=v}} K_w\Big)$, if v is neither an end node nor the root v_0.

(3) 1, if v appeared from splitting a double integral; see (7.5).

(4) $F^{(m_v)} \Big(\prod_{\substack{w \notin V_{\text{int}} \\ w'=v}} \widetilde{X}_{\leq 1}\Big)\Big(\prod_{\substack{w \in V_{\text{int}} \\ w'=v}} K_w\Big)$, if $v = v_0$; see (6.29).

The functions K_v and $\bar{K}_v$ refer to diagonal elements of K_{ij} and $\bar{K}_{ij}$ in (6.25), respectively. m'_v is the cardinality of $\{w \in V_{\text{int}} \mid w' = v\}$ and $m_v - m'_v$ the cardinality of $\{w \in V \setminus V_{\text{int}} \mid w' = v\}$. We point out that each factor in u_v—and u_v itself—is a scalar, due to Remark 7.4.

We draw the scalar (sub)tree in (7.6)—originated from (7.2)—as



,

where m'_v is the number of factors in the product. This diagram is reminiscent of the one below (7.2), except that the operator U_v has changed into the multiplier u_v and the subtrees ——• have been absorbed into u_v. Moreover, recalling the decompositions above, the present diagram carries $\int_{-\infty}^{t}$ on its lines instead of $\mathcal{K}^{-1}$—compare (7.6) with (7.2)—and has possibly more nodes due to the diagram in (7.5).

In what follows, we do not consider the end nodes with a black dot (•) nodes anymore. Subsequently, we will not mention the set V_{int} below, and refer to "integrated nodes" as just "nodes".

7.3. Integration by parts

Reinserting the implicit arguments (z, θ), the multipliers u_v obey

$$u_v(t; z, \theta) \equiv t^p u_v(ze^{\gamma t}, \theta + \omega t) \tag{7.7}$$

for the obviously defined $u_v(z, \theta)$. The power p can be nonzero only at the root, $v = v_0$, where it possibly assumes the value 1. The latter is caused by case 4 in the list collecting all possible multipliers above.

We will now turn our attention to the integrals $⨍_0^\infty T_v(t; z, \theta)\, dt$, *i.e.*,

$$\sum_{q_v + \sum q_w = q} \operatorname*{res}_{R=0} \frac{1}{R} \int_0^\infty e^{-Rt}\, u_v(t; z, q_v) \prod_{w'=v} \int_{-\infty}^{t} T_w(\tau; z, q_w)\, d\tau\, dt, \tag{7.8}$$

at each order $\ell \geq |q|/N > 0$, which arise from (7.1) together with (7.6). In fact, at this stage $v = v_0$ (the original root), but we preferred writing down the more general form. We have omitted the usual $\widehat{\text{hat}}$ in the Fourier transforms, since there is no danger of confusion.

Remark 7.5. The task is to show that the t-integral in (7.8) extends analytically from large and positive values of $\mathfrak{Re}\, R$ to a (punctured) neighbourhood of $R = 0$, such that the residue can be computed. For this, we need the specific structure of the multipliers u_v.

Let us define

$$\xi_{01} = \bar{\xi}_{02} = 1, \quad \bar{\xi}_{01} = \xi_{02} = \xi_{11} = \bar{\xi}_{11} = -1, \quad \text{and} \quad \xi_{12} = \bar{\xi}_{12} = 0.$$

There exist positive, continuous, functions a_{ij} and $\bar{a}_{ij}$ such that

$$|K_{ij}(t;z)| \le a_{ij}(z)e^{\xi_{ij}\gamma|t|} \quad \text{and} \quad |\bar{K}_{ij}(t;z)| \le \bar{a}_{ij}(z)e^{\bar{\xi}_{ij}\gamma|t|} \tag{7.9}$$

hold on $\{(t,z) \in \mathbb{R} \times \mathbb{C} \mid |z| = 1 \text{ and } ze^{\gamma t} \ne \pm i\}$. Moreover,

$$a_{ij}\bar{a}_{ij} = \frac{A_{ij}}{\gamma^{1-i}}, \tag{7.10}$$

where A_{ij} is independent of γ. Notice that, in the Kronecker delta notation,

$$\xi_{ij} + \bar{\xi}_{ij} = -2\,\delta_{i1}\delta_{j1} \le 0. \tag{7.11}$$

Following the notation used with the multipliers, we drop the subindices ij and just write ξ_v and $\bar{\xi}_v$.

For each node v, define the numbers

$$r_v := \sup_{(z,\theta)\in B} \min\left\{k \in \mathbb{Z} \;\middle|\; \forall\,\delta > 0 : u_v(t;z,\theta)e^{-(k+\delta)\gamma t} \to 0 \text{ as } t \to \infty\right\}$$

with

$$B := \{|z| = 1, |\arg z| \le \vartheta\} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}. \tag{7.12}$$

Thus, recalling (7.7), we are inside the analyticity domain $\mathbb{U}_{\tau,\vartheta} \times \{|\Im\mathfrak{m}\,\theta| \le \sigma\}$ of Lemma 5.2. These numbers measure the divergence rate of the multipliers in the limit $t \to \infty$.

Lemma 7.6. *Ordered according to the list of possible u_v's on p. 107,*

$$r_v = \begin{cases} \bar{\xi}_v + k, & \textit{case (1)},\\ \bar{\xi}_v + (m_v - m'_v) + \sum_{w'=v} \xi_w, & \textit{case (2)},\\ 0, & \textit{case (3)},\\ \bar{\xi}_F + (m_v - m'_v) + \sum_{w'=v} \xi_w, & \textit{case (4)}, \end{cases}$$

where $\bar{\xi}_F \in \{0,1\}$ depends on the choice of F in (6.29). *Moreover $u_v(z,\theta)$, see* (7.7), *is analytic on $\{|\Im\mathfrak{m}\,\theta| \le \sigma\}$ with respect to θ and on $\mathbb{U}_{\tau,\vartheta} \setminus \{0\}$ with respect to z. It is also analytic in the punctured neighbourhood $\{|z| \ge \tau^{-1}\}$ of $z = \infty$, at which point there is a (possible) pole of order r_v (if $r_v > 0$).*

Proof. The maps $\Phi^0(z)$ and $f(\Phi^0(z),\theta)$ are analytic in these domains, without singularities at $z = 0, \infty$; see (1.3). The rest follows by staring at the expression of u_v in each particular case. ☐

Starting from the end nodes of a tree—setting $s_v := 0$ for them—we recursively define the integer numbers

$$s_v := \sum_{w'=v} \max(0, n_w) \quad \text{and} \quad n_v := r_v + s_v. \tag{7.13}$$

They measure the divergence rate of (sub)trees in the limit $t \to \infty$:

LEMMA 7.7. *For T_v as in* (7.6), *and any $\delta > 0$,*

$$T_v(t; z, \theta) e^{-(n_v+\delta)\gamma t} \to 0 \quad \text{as} \quad t \to \infty \tag{7.14}$$

and

$$\left(\prod_{w'=v} \int_{-\infty}^{t} T_w(\tau)\, d\tau \right) e^{-(s_v+\delta)\gamma t} \to 0 \quad \text{as} \quad t \to \infty \tag{7.15}$$

hold true in the region B of (7.12).

Proof. Assume that (7.14) is true for each successor w of v ($w' = v$). By (7.6), we only need to observe that, given $\delta > 0$,

$$\left| \int_{-\infty}^{t} T_w(\tau; z, \theta)\, d\tau \right| \le C_1 + C_2 \int_0^t e^{(n_w+\delta)\gamma\tau}\, d\tau = \mathcal{O}(e^{[\max(0,n_w)+\delta]\gamma t})$$

in the limit $t \to \infty$. For then (7.13) implies (7.14) for the node v.

Since (7.14) is clear in the case of end nodes ($n_v = r_v$), we have an induction proof for the claim. □

Let us now come back to the integral in (7.8), *i.e.*,

$$\int_0^\infty e^{-Rt}\, u_v(t; z, q_v) \prod_{w'=v} \int_{-\infty}^{t} T_w(\tau; z, q_w)\, d\tau\, dt, \tag{7.16}$$

recalling Remark 7.5. An obvious problem is the exponential divergence of the integrand, $T_v(t, z; q)$, in (7.14). Our cure is the following. Since u_v is meromorphic at $z = \infty$, by Lemma 7.6, we may expand

$$u_v(z, \theta) = \sum_{k=-s}^{r_v} z^k u_{v,k}(\theta) + u_{v,<-s}(z, \theta) \tag{7.17}$$

for any integer $s \ge r_v$, with

$$u_{v,<-s}(z, \theta) = \mathcal{O}(z^{-s-1}) \quad \text{as} \quad z \to \infty.$$

Extending (7.7), we write

$$\begin{cases} u_{v,k}(t; \theta) \equiv t^p\, e^{k\gamma t}\, u_{v,k}(\theta + \omega t), \\ u_{v,<-s}(t; z, \theta) \equiv t^p\, u_{v,<-s}(z e^{\gamma t}, \theta + \omega t). \end{cases}$$

In particular, the integral

$$\int_0^\infty e^{-Rt}\, u_{v,<-s_v}(t;z,q_v) \prod_{w'=v} \int_{-\infty}^t T_w(\tau;z,q_w)\,d\tau\,dt$$

is convergent for $\Re\mathfrak{e}\, R > -\gamma$, by virtue of (7.15), and can be estimated on a circle $|R| = \rho < \gamma$ for the purpose of Lemma 6.24.

The rest of the integral (7.16) is *integrated by parts*: for $-s_v \le k \le r_v$ and sufficiently large values of $\Re\mathfrak{e}\, R$,

$$\begin{aligned}
&\int_0^\infty e^{-Rt}\, z^k u_{v,k}(t;q_v) \prod_{w'=v} \int_{-\infty}^t T_w(\tau;z,q_w)\,d\tau\,dt \\
&= z^k u_{v,k}(q_v) \int_0^\infty e^{(k\gamma + iq_v\cdot\omega - R)t}\, t^p \prod_{w'=v} \int_{-\infty}^t T_w(\tau;z,q_w)\,d\tau\,dt \\
&= \frac{z^k u_{v,k}(q_v)}{R_v} \left\{ \frac{E_v(0)}{R_v^p} + \sum_{p'=0}^{p} \frac{1}{R_v^{p-p'}} \int_0^\infty e^{-R_v t}\, t^{p'} \frac{dE_v}{dt}\,dt \right\}, \qquad (7.18)
\end{aligned}$$

where $p \in \{0,1\}$ depends on the choice of F in case (4) on p. 107,

$$R_v := R - k\gamma - i\omega \cdot q_v$$

and

$$E_v := E_v(t;z,\{q_w\}) := \prod_{w'=v} \int_{-\infty}^t T_w(\tau;z,q_w)\,d\tau.$$

According to the Leibniz rule and the recursion relation in (7.6),

$$\begin{aligned}
\frac{dE_v}{dt} &= \sum_{w'=v} T_w(t) \prod_{\substack{\bar w'=v\\ \bar w\ne w}} \int_{-\infty}^t T_{\bar w}(\tau)\,d\tau \\
&= \sum_{w'=v} u_w(t) \Big(\sum_{\text{conv}} \prod_{\bar w'=w} \int_{-\infty}^t T_{\bar w}(\tau)\,d\tau \Big) \Big(\prod_{\substack{\bar w'=v\\ \bar w\ne w}} \int_{-\infty}^t T_{\bar w}(\tau)\,d\tau \Big).
\end{aligned}$$

leaving z and a bunch of Fourier indices implicit. Above, $\sum_{\text{conv}}$ stands for a convolution. Throwing the sums out of the integral, we observe that the remaining integral in (7.18) produces integrals similar to the original (7.16), except that

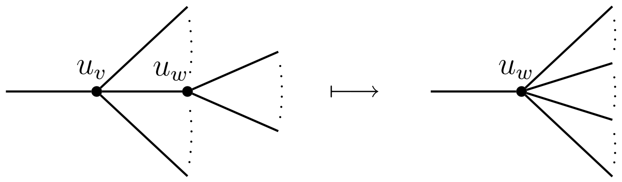
(i) u_v changes to u_w,

(ii) R changes to R_v,

(iii) p changes to p', and

(iv) the integral on the line *leaving* w, namely $\int_{-\infty}^{t} T_w(\tau)\,d\tau$, is replaced by the integrals on the lines *entering* w, namely the product $\prod_{\bar{w}'=w} \int_{-\infty}^{t} T_{\bar{w}}(\tau)\,d\tau$.

Thus, a single step in the integration-by-parts scheme can be described in terms of trees as follows: Given a tree, consider one of the successors, w, of the root, $v = w'$.



The line from w to v is "contracted" by erasing the node w, reattaching to v all subtrees originally entering w, and replacing the multiplier of the root, v, by u_w. Finally, we rename the root w.

We can proceed recursively, as follows. Let us number the nodes of the original tree, say T_0, with the aid of the figure above. First, set $v_0 := v$. Then choose a successor w and define $v_1 := w$, contracting the line from w to v. Next, in the *new tree* (the rightmost diagram) called T_1, choose a successor of w and call it v_2. Contract the line from v_2 to v_1. Repeat until the tree has been exhausted and all nodes have been numbered. We can express the sequence of trees formed as

$$T_i(t) = u_{v_i} \prod_{w \in T_i:\, w'=v_i} \int_{-\infty}^{t} T_w(\tau)\,d\tau, \tag{7.19}$$

where in the product we consider the tree T_i with the root v_i having the successors w. Define the numbers

$$r_i := r_{v_i} \quad (i = 0, 1, \dots).$$

Analogously to the numbers s_v in (7.13), we set

$$s_i := \sum_{w \in T_i:\, w'=v_i} \max(0, n_w),$$

where the numbers n_w are the ones defined in (7.13) for the original tree. Notice that, although $s_0 = s_{v_0}$, s_i is not simply equal to s_{v_i}, but is the analogue in the tree T_i of which v_i is the *root*. Similarly to (7.15), s_i bounds the divergence rate of the product in (7.19): if $\delta > 0$,

$$\left(\prod_{w \in T_i:\, w'=v_i} \int_{-\infty}^{t} T_w(\tau)\,d\tau \right) e^{-(s_i+\delta)\gamma t} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.20}$$

The following, elementary, property is useful.

Lemma 7.8. *For every i,*

$$s_{i+1} + r_{i+1} \le s_i.$$

Proof. We compute

$$s_{i+1} := \sum_{\substack{w \in T_{i+1} \\ w' = v_{i+1}}} \max(0, n_w) = \sum_{\substack{w \in T_i\colon w \neq v_{i+1} \\ w' = v_i}} \max(0, n_w) + \sum_{\substack{w \in T_i \\ w' = v_{i+1}}} \max(0, n_w).$$

Since $r_{i+1} := r_{v_{i+1}}$, $n_{v_{i+1}} := r_{v_{i+1}} + s_{v_{i+1}}$, and

$$\sum_{\substack{w \in T_i \\ w' = v_{i+1}}} \max(0, n_w) = s_{v_{i+1}},$$

we get

$$s_{i+1} + r_{i+1} = \sum_{\substack{w \in T_i\colon w \neq v_{i+1} \\ w' = v_i}} \max(0, n_w) + n_{v_{i+1}} \le s_i,$$

as claimed. □

Let $\tilde{k}_0 := 0$ and $Q_0 := 0$. Suppose that at the ith step we are considering the integral

$$\int_0^\infty e^{(iQ_i \cdot \omega + \tilde{k}_i \gamma - R)t}\, t^{p_i} u_{v_i}(ze^{\gamma t}, q_{v_i}) e^{iq_{v_i} \cdot \omega t} \prod_{w \in T_i\colon w' = v_i} \int_{-\infty}^t T_w(\tau)\, d\tau\, dt.$$

Then the integration-by-parts procedure described above takes us at the $(i+1)$st step to a similar integral with i replaced by $i+1$, defining

$$\tilde{k}_{i+1} := \tilde{k}_i + k_i \quad \text{and} \quad Q_{i+1} := Q_i + q_{v_i}, \tag{7.21}$$

multiplied by the factor

$$\frac{z^{k_i} u_{v_i,k_i}(q_{v_i})}{\big(R - \tilde{k}_{i+1}\gamma - i\omega \cdot Q_{i+1}\big)^{1+p_i-p_{i+1}}}. \tag{7.22}$$

The integer indices p_{i+1} and k_i can assume the values

$$0 \le p_{i+1} \le p_i \le 1 \quad \text{and} \quad k_i = -\tilde{k}_i - s_i, \dots, r_i. \tag{7.23}$$

With the aid of Lemma 7.8, it is straightforward to see that

$$0 \le \tilde{k}_i + r_i + s_i \le n_{v_0}, \tag{7.24}$$

where $n_{v_0} = s_{v_0} + r_{v_0} = s_0 + r_0$ is the number describing the divergence rate of the original tree, T_0, in the sense of (7.14). The number of possible values of k_i above is thus at most $1 + n_{v_0}$. This implies the two-sided (equivalent) bounds

$$r_i - n_{v_0} \le k_i \le r_i \quad \text{and} \quad 0 \le r_i - k_i \le n_{v_0}.$$

We continue recursively until there are no nodes—alternatively, integrals that require regularization—left. The result is a sum with terms of three different species. Setting

$$R_i := R - \tilde{k}_i \gamma - i\omega \cdot Q_i,$$

such that $R_0 = R$, the first ones are of the form

$$\prod_{i=0}^{|V_{\text{int}}|-1} \frac{z^{k_i} u_{v_i,k_i}(q_{v_i})}{R_{i+1}^{1+p_i-p_{i+1}}}, \tag{7.25}$$

where $|V_{\text{int}}|$ is the number of nodes in the original tree; see above (7.6). We need to define

$$E_j(t) := E_j(t; z, \{q_w | w \in T_j, w' = v_j\}) := \prod_{w \in T_j \colon w' = v_j} \int_{-\infty}^{t} T_w(\tau; z, q_w)\, d\tau,$$

which is the analogue of E_v in (7.18), in order to make the presentation more compact. There is the second class of terms

$$\left(\prod_{i=0}^{j-1} \frac{z^{k_i} u_{v_i,k_i}(q_{v_i})}{R_{i+1}^{1+p_i-p_{i+1}}} \right) \int_0^\infty e^{(iq_{v_j}\cdot\omega - R_j)t}\, t^{p_j}\, u_{v_j,<-\tilde{k}_j - s_j}(ze^{\gamma t}, q_{v_j})\, E_j(t)\, dt, \tag{7.26}$$

for $0 < j < |V_{\text{int}}|$. Last, again for $0 < j < |V_{\text{int}}|$,

$$\left(\prod_{i=0}^{j-1} \frac{z^{k_i} u_{v_i,k_i}(q_{v_i})}{R_{i+1}^{1+p_i-p_{i+1}}} \right) \frac{z^{k_j} u_{v_j,k_j}(q_{v_j})}{R_{j+1}^{1+p_j}}\, E_j(0). \tag{7.27}$$

7.4. Estimates

By the very definition of s_j, we can bound

$$|E_j(t)| \le C_{E_j} e^{s_j \gamma t} e^{-\sigma \sum |q_w|} \qquad (t \ge 0),$$

for $|z| = 1$ with $|\arg z| \le \vartheta$. Postponing the proof by a few paragraphs, we formulate

Lemma 7.9. *Suppose $u_{v_i} \neq 1$ (case (3) on p. 107). The coefficients $u_{v_i,k_i}(q)$ have the bounds*

$$|u_{v_i,k_i}(q_{v_i})| \le C_{v_i} \tau^{k_i - r_i} e^{-\sigma |q_{v_i}|}.$$

On $\{1 \le |z| \le \tau^{-1} \text{ with } |\arg z| \le \vartheta\} \bigcup \{|z| \ge \tau^{-1}\}$, *for a new* C_{v_i},

$$|u_{v_i,<-\tilde{k}_i-s_i}(z,q_{v_i})| \le C_{v_i}(\tau|z|)^{-\tilde{k}_i-s_i-1}(1+3^{n_{v_0}+1}\tau^{-r_i})e^{-\sigma|q_{v_i}|}.$$

Remark 7.10. It is worth pointing out that the second bound in Lemma 7.9 appears from only one node, such that the powers $c^{n_{v_0}}$ cannot accumulate. After its first appearance, there is no need to further decompose the multipliers.

For $|z| = 1$ with $|\arg z| \le \vartheta$, the integral in (7.26) is bounded by

$$(3\tau^{-1})^{n_{v_0}+1}\frac{C_{v_j}C_{E_j}}{\gamma^{1+p_j}}e^{-\sigma|q_{v_j}|-\sigma\sum|q_w|},$$

where (7.24) has been used. Thus, the bounds on (7.25), (7.26), and (7.27) read

$$\left(\prod_{i=0}^{|V_{\text{int}}|-1}\frac{C_{v_i}}{|R_{i+1}^{1+p_i-p_{i+1}}|}\right)\tau^{\sum_{i=0}^{|V_{\text{int}}|-1}(k_i-r_i)}e^{-\sigma\sum_{i=0}^{|V_{\text{int}}|-1}|q_{v_i}|},$$

$$\left(\prod_{i=0}^{j-1}\frac{C_{v_i}}{|R_{i+1}^{1+p_i-p_{i+1}}|}\right)\frac{C_{v_j}C_{E_j}}{\gamma^{1+p_j}}(3\tau^{-1})^{n_{v_0}+1}\tau^{\sum_{i=0}^{j-1}(k_i-r_i)}e^{-\sigma\sum_{i=0}^{j}|q_{v_i}|-\sigma\sum|q_w|}$$

and

$$\left(\prod_{i=0}^{j-1}\frac{C_{v_i}}{|R_{i+1}^{1+p_i-p_{i+1}}|}\right)\frac{C_{v_j}C_{E_j}}{|R_{j+1}^{1+p_j}|}\tau^{\sum_{i=0}^{j}(k_i-r_i)}e^{-\sigma\sum_{i=0}^{j}|q_{v_i}|-\sigma\sum|q_w|},$$

respectively. Here the products of the factors $e^{-\sigma|q_v|}$ are bounded by $e^{-\sigma|q|}$, because the tree is a Fourier transform with index q, and the q_v's come from convolutions. It is an exercise that $n_{v_0} \le \ell$ in a tree contributing at order ℓ.

LEMMA 7.11. *For all* $j = 0, 1, \dots,$

$$\sum_{i=0}^{j}(k_i - r_i) \ge -n_{v_0}.$$

Proof. From (7.21) and (7.23), we clearly have $\tilde{k}_{i+1} \ge -s_i$. Then,

$$\begin{aligned}\sum_{i=0}^{j}(k_i - r_i) &= \tilde{k}_{j+1} - \sum_{i=0}^{j} r_i \ge -(s_j + r_j) - \sum_{j=0}^{i-1} r_j \\ &\ge s_{i-1} - \sum_{j=0}^{i-1} r_j \ge \dots \ge -s_0 - r_0 = -n_{v_0},\end{aligned}$$

with the aid of Lemma 7.8. ∎

Proof of Lemma 7.9. Here we usually omit the subindices i, v, and v_0. According to Lemma 7.6, u_v is analytic on $\{|z| \geq \tau^{-1}\} \times \{|\Im\mathfrak{m}\,\theta| \leq \sigma\}$ with a pole of order r_v at $z = \infty$. A Cauchy estimate then reads

$$|u_{v,k}(\theta)| \leq \tau^k \sup_{|z|=\tau^{-1}} |u_v(z,\theta)| \leq C\tau^{k-r}.$$

Now $|u_{v,k}(q)| \leq e^{-\sigma|q|} \sup_{|\Im\mathfrak{m}\,\theta|\leq\sigma} |u_{v,k}(\theta)|$ yields the claimed bound.

The second inequality is a trivial consequence of the first one for $|z| > 2\tau^{-1}$. For other values, use (7.17) to bound $|u_{v,<-\tilde{k}-s}(z,q)|$ by

$$\begin{aligned}
&\leq |z|^{-\tilde{k}-s-1}\Big\{|z|^{\tilde{k}+s+1}|u_v(z,q)| + Ce^{-\sigma|q|}\tau^{-\tilde{k}-s-1}\sum_{l=0}^{r+\tilde{k}+s}|z\tau|^{l+1}\Big\}\\
&\leq Ce^{-\sigma|q|}|z|^{-\tilde{k}-s-1}\Big\{(2\tau^{-1})^{\tilde{k}+s+r+1} + \tau^{-\tilde{k}-s-r-1}3^{r+\tilde{k}+s}\Big\},
\end{aligned}$$

where $|z|^{-r}|u_v(z,q)| \leq Ce^{-\sigma|q|}$. Now, recall $\tau < 1$ and (7.24). □

Suppose that, in the domain B of (7.12),

$$|h(t;z,\theta)| \leq C_h \begin{cases} e^{2\gamma t}, & \text{if } t < 0,\\ x(t)e^{n\gamma t}, & \text{if } t \geq 0, \end{cases} \tag{7.28}$$

for a polynomial x. Then, by virtue of the bounds (7.9), equations (7.10) and (7.11), and elementary computations,

$$\left|K_{ij}(t;z)\int_{-\infty}^{t}(t-\tau)^i\bar{K}_{ij}(\tau;z)\,h(t;z,\theta)\,d\tau\right| \leq \frac{C_hA_{ij}}{\gamma^2}\begin{cases} e^{2\gamma t}, & t<0,\\ y(t)e^{n\gamma t}, & t\geq 0,\end{cases}$$

where y is the polynomial

$$y(t) := \begin{cases} 1 + i + x(t), & \text{if } n + \bar{\xi}_{ij} \geq 1,\\ 1 + \gamma t\, x(t), & \text{if } n + \bar{\xi}_{ij} = 0 \ (\textit{i.e.}, n = j = 1). \end{cases}$$

The functions $h^{(n)}$ in (5.4), representing end nodes, admits a bound like (7.28) with C_h proportional to g^2. For them, the factor γ^{-2} is essentially cancelled; C_h/γ^2 is bounded in the limit $g \to 0$. We can recursively bound an entire tree by beginning from the end nodes and proceeding towards the root. At each node whose multiplier is not identically 1 (case (3) on p. 107) there is a node function bounded by C_*g^2 with $C_* = \mathcal{O}(1)$ as $g \to 0$, such that the factors γ^{-2} above do not accumulate but are balanced;

$$\prod_{i=0}^{|V_{\text{int}}|-1} C_{v_i}, \ \left(\prod_{i=0}^{j-1} C_{v_i}\right)C_{E_j} \leq (C_*A)^{|V_{\text{int}}|}C_*^{|V\setminus V_{\text{int}}|} \leq C^{|V|}.$$

Here V, defined above (7.6), is the set of all nodes including the black dot ($\bullet$) end nodes. Moreover, A is the maximum of the four possible A_{ij}'s above, and there is one of the latter for each of the $|V_{\text{int}}|$ lines carrying an integral.

Notice that the fraction $1/R_{i+1}$ is analytic on the domain where

$$|R| \neq 2\rho_{i+1} := |\tilde{k}_{i+1}\gamma + i\omega \cdot Q_{i+1}|.$$

Thus, they are all analytic in a punctured neighbourhood of the origin, namely

$$0 < |R| \leq \rho := \min\{\rho_i \mid \rho_i > 0\}.$$

On the circle $|R| = \rho$ they satisfy the bounds

$$\left|\frac{1}{R_{i+1}^{1+p_i-p_{i+1}}}\right| \leq \big(\max(\rho_{i+1}, \rho)\big)^{-1-p_i+p_{i+1}} \leq \rho^{-1-p_i+p_{i+1}}.$$

Since $Q_{|V_{\text{int}}|} = \sum_{i=0}^{|V_{\text{int}}|-1} q_{v_i} = q \neq 0$ by assumption, $\rho > 0$ indeed. Further, for some j,

$$\rho = \rho_j = \tfrac{1}{2}|\tilde{k}_j\gamma + i\omega \cdot Q_j| \geq \tfrac{1}{2}\begin{cases}\gamma & \text{if } Q_j = 0,\\ a|Q_j|^{-\nu} & \text{if } Q_j \neq 0.\end{cases}$$

We can associate with v_i the orders ℓ_{v_i} with the property that

$$\sum_{i=0}^{|V_{\text{int}}|-1} \ell_{v_i} = \ell,$$

where ℓ is the order at which the tree contributes, and which bound the Fourier indices we have to consider: $|q_{v_i}| \leq \ell_{v_i} N$. This is a consequence of Proposition 5.3. In particular, for all j,

$$|Q_j| \leq \ell N.$$

Thus,

$$\prod_{i=0}^{j-1}\left|\frac{1}{R_{i+1}^{1+p_i-p_{i+1}}}\right| \leq \rho^{-j-1} \qquad (0 < j \leq |V_{\text{int}}|).$$

By Proposition 7.3, the order of the pole at $R = 0$ in our integrals does not exceed $K := 4\ell$. We insert this into Lemma 6.24 and compute, for $0 \leq J \leq K$,

$$\left(\frac{\rho}{g}\right)^J \rho^{-K} = \frac{1}{g^J \rho^{K-J}} \leq C^\ell \max\left(g^{-4\ell}, a^K (ga(\ell N)^\nu)^{-J} (\ell!)^{4\nu}\right)$$

There are at most

$$(4\ell)! \leq 4^{4\ell}(\ell!)^4$$

orders in which we can exhaust all lines of a tree contributing at order ℓ, since the latter has $|V| \leq 4(\ell-1)$ lines, according to Proposition 7.3.

7.5. Remaining integrals

The integral over $\mathbb{R}_-$ in (7.1) is simple, because the integrand satisfies the identity (6.30) and has the analyticity properties stated below that equation. In other words, we can separate the (possible) pole and the constant term from the rest: writing $F(X^u; 0, z, \theta) \equiv F(z, \theta)$,

$$F(X^u; t, z, \theta) = t^p z^{-1} e^{-\gamma t} F_{-1}(\theta+\omega t) + t^p F_0(\theta+\omega t) + t^p \delta_1 F(z e^{\gamma t}, \theta+\omega t).$$

Applying $⨍_{-\infty}^{0}$ on each term separately, we get

$$⨍_{-\infty}^{0} F(X^u; t, z, \theta)\, dt = (-1)^p \sum_{q'} \frac{z^{-1} e^{iq'\cdot\theta} \hat{F}_{-1}(q)}{(iq'\cdot\omega - \gamma)^{1+p}} + \\ + (-1)^p \sum_{q'\neq 0} \frac{e^{iq'\cdot\theta} \hat{F}_0(q)}{(iq'\cdot\omega)^{1+p}} + \int_{-\infty}^{0} t^p \delta_1 F(z e^{\gamma t}, \theta + \omega t)\, dt.$$

These terms are small compared to to the large bounds obtained for the $⨍_0^\infty$ part above.

Studying the coefficients c_{ij}^p appearing in (6.26) is most conveniently done in terms of the representation (6.27). Since $M^u = \mathcal{O}(\epsilon)$, only

$$⨍_{-\infty}^{-t} \Big[f^u_{\psi\bar{\psi}} (B^u M^u)^l K_{ij}^p \Big](\tau)\, d\tau$$

with $l < \ell$ can contribute to c_{ij}^p at order ℓ. Moreover, we only need to consider $t \geq 0$, because $\partial_\theta \Delta(t) = \partial_\theta \Delta(-t)$. The integral consists, through (6.25) and obvious change of notation, of 4^l terms of the form

$$⨍_{-\infty}^{-t} (f K_l)(\tau_l) ⨍_{-\infty}^{\tau_l} (\tau_l - \tau_{l-1})^{p_l} (\bar{K}_l M)(\tau_{l-1}) \cdots K_1(\tau_1) ⨍_{-\infty}^{\tau_1} (\tau_1 - \tau_0)^{p_1} (\bar{K}_l M)(\tau_0) \tau_0^{p_0} K_0(\tau_0).$$

If $p_0 = 1$, we use $\tau_0 = (\tau_0 - \tau_1) + (\tau_1 - \tau_2) + \cdots + (\tau_l - t) + t$, getting $l + 2$ terms of the original form except that $p_0 = 0$ and either it has a factor t or precisely one $p_i \mapsto p_i + 1$. We proceed by computing

$$⨍_{-\infty}^{\tau} (\tau - \tau')^m h(\omega \tau')\, d\tau' = \sum_{q \neq 0} \frac{\hat{h}(q) e^{iq\cdot\omega\tau}}{(i\omega\cdot q)^{m+1}} + \frac{\hat{h}(0)\tau^{m+1}}{(m+1)!} \qquad (m \in \mathbb{N}).$$

Suppose that, starting from the integral with respect to τ_0 and proceeding all the way to τ_l we always pick the "resonant part" of the integrand. This results in terms of the form

$$e^{iq\cdot\omega\tau}\frac{\hat{h}_l(q_l)}{(i\omega\cdot Q_l)^{p_l+1}}\cdots\frac{\hat{h}_0(q_0)}{(i\omega\cdot Q_0)^{p_1+1}} \tag{7.29}$$

Here

$$Q_i := \sum_{j=0}^{i} q_i \quad \text{and} \quad Q_l = q.$$

Due to $M = \mathcal{O}(\epsilon g^2)$, (7.9) and (7.10), each $K_i\bar{K}_iM$ produces a factor $CA|\epsilon|g^{p_i+1}$ to the upper bound, whereas analyticity produces $e^{-\sigma|q_i|}$ (which we prefer even though at each order we reduce to trigonometric polynomials). In particular,

$$\prod_{i=0}^{l} e^{-\sigma|q_i|} \le e^{-\sigma|q|}.$$

Since, for all i,

$$|i\omega\cdot Q_i|^{-1} \le a^{-1}|Q_i|^{\nu} \le a^{-1}(N\ell)^{\nu}$$

and $p_i \le 1$ (except for one index, for which it is ≤ 2), we obtain the upper bound

$$C^{\ell}|\epsilon|^l g^l e^{-\sigma|q|}(\ell!)^{2\nu}$$

for (7.29)

Skipping further details, this is the general upper bound on the integral. It is smaller than what was derived for the integral $\fint_{-\infty}^{\infty} F$—and thus for $\mathcal{E}_{ij}^p$ in (6.26)—above. In conclusion, no matter which term

$$\left(c_{ij}^p\right)^{\ell_1}\left(\mathcal{E}_{ij}^p\right)^{\ell_2} \quad \text{with} \quad \ell_1+\ell_2=\ell$$

contributing to $\partial_\theta\Delta^\ell$ we choose, it is always smaller than the upper bound on $\left(\mathcal{E}_{ij}^p\right)^{\ell}$. Obviously, ℓ_i and ℓ refer to coefficients in the power series.

Remark 7.12 (The sums over q_i). We should have multiplied the above estimates by the number of possible sequences (q_i), with $\sum_i q_i = q$, that arise from the convolutions. However, we took a shortcut. Since the analyticity domain with respect to θ is the compact $\{|\Im\mathfrak{m}\,\theta| \le \sigma\}$, it can be substituted by some $\{|\Im\mathfrak{m}\,\theta| \le \sigma'\}$ with $\sigma < \sigma'$. Then, for every i we actually have the factor $e^{-\sigma'|q_i|}$ in the estimates above. While $e^{-\sigma\sum_i|q_i|} \le e^{-\sigma|q|}$, we get rid of the sums over q_i:

$$\sum_{q_i} e^{-(\sigma'-\sigma)q_i} \le C_{\sigma'-\sigma}.$$

A similar remark applies to the Fourier indices in Section 7.4. □

Chapter 8

Discussion

THE main achievement in this work is the derivation and subsequent analysis of the asymptotic expansion (6.26) of the splitting matrix Υ. Each term in the expansion is proportional to an integral of the form $⨍_{-\infty}^{\infty} \partial_\theta F(X^u; \tau, z, \theta)\, d\tau$, which we showed to be exponentially small in the limit $g \to 0$ at all orders ℓ.

The latter involves extending the integrands analytically into a wedge $\{|\arg z| \leq \vartheta\}$ on the complex plane. Because of the identity $X^u(t; z, \theta) \equiv X^u(0; ze^{\gamma\tau}, \theta + \omega\tau)$, this implies that shifting the contour of integration becomes possible. One of the key points is that we need to do this *order by order*, because there is no reason why the series $X^u(z, \theta) = \sum_{\ell=0}^{\infty} \epsilon^\ell X^{u,\ell}(z, \theta)$ should converge for large values of $|z|$.

Our estimates are not optimized for trigonometric polynomials. The proof was cooked up bearing analytic perturbations in mind, although we had to abandon this case as too intricate. In our notation, Procesi, see [Pro03], obtains a bound (in the case of trigonometric polynomials) on the determinant of Υ proportional to $e^{-cg^{-1/d}}$. This is smaller than our $e^{-cg^{-1/(\nu+1)}}$, at least when the tori are abundant ($\nu > d - 1$).

In any case, the greatest interest lies in the *novel method*, which in our opinion underlines central features of the problem. First, the source of exponentially small contributions is completely obvious. Second, the need to use order-by-order analysis culminates in the success of extending $X^{u,\ell}$ to $\{|\arg z| \leq \vartheta\}$ analytically and the failure to do so for the "full" X^u. Third, the large powers of the factorial $\ell!$, associated with the regularized integrals, are produced by accumulation of poles at the origin in the R plane.

To some extent the factorials produced by poles in the R plane are artifact, as is shown by the following simple example. In order to study, say, the regularized integral

$$\fint_0^\infty u(\theta+\omega t)\int_{-\infty}^t v(\theta+\omega\tau)ze^{\gamma\tau}\,d\tau\,dt,$$

we have to show that the sums

$$\sum_p \hat{u}(p)\hat{v}(q-p)\int_0^\infty e^{(ip\cdot\omega-R)t}\int_{-\infty}^t e^{(i(q-p)\cdot\omega+\gamma)\tau}\,d\tau\,dt$$

extend analytically to a (punctured) neighbourhood of the origin, $R=0$. First, we integrate by parts, just as was done in (7.18) when constructing an analytic extension of the tree integral (7.8), and get

$$\sum_p \hat{u}(p)\hat{v}(q-p)\frac{1}{R-ip\cdot\omega}\left\{\int_{-\infty}^0 e^{(i(q-p)\cdot\omega+\gamma)\tau}\,d\tau+\int_0^\infty e^{(iq\cdot\omega+\gamma-R)\tau}\,d\tau\right\}$$

$$=\sum_p \hat{u}(p)\hat{v}(q-p)\frac{1}{R-ip\cdot\omega}\left\{\frac{1}{(i(q-p)\cdot\omega+\gamma)}+\frac{1}{R-(\gamma+iq\cdot\omega)}\right\}.$$

Here the pole at $ip\cdot\omega$ gets arbitrarily close to the origin, unless we restrict p somehow—for instance, by considering trigonometric polynomials. Either by simplification, or by computing the same expression directly by starting from the inner integral, we obtain

$$\sum_p \frac{1}{i(q-p)\cdot\omega+\gamma}\cdot\frac{1}{R-(\gamma+iq\cdot\omega)}.$$

In the latter form there is no problem; the pole has cancelled. Of course, this is a naive example and in general it is hard, if not impossible, to see whether a given pole popping out of the integration-by-parts procedure should really be there.

There was another place, namely the coefficients c_{ij}^p appearing in Proposition 6.23, that produced large powers of $\ell!$. Even though integration by parts was not exploited, the source of the factorials was again the accumulation of poles at the origin in the R plane. In both this case and the previous, the type of the "divergence" is very similar to what is encountered in KAM theory. There repeated resonances, or arbitrarily many occurrences of the operator $\mathcal{D}^{-1}$ in convolutions, ruin absolute convergence of the Fourier–Taylor expansion of a solution by producing high powers of the factorial $\ell!$. On the other hand, the state of affairs can be cured by well-known resummations, as in [Gal94]. Such resummations still escape us in the context of homoclinic splitting.

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