Bargaining without Disagreement*

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Abstract

We identify weakenings of the independence axiom of Nash (1950) to derive a solution without disagreement point. The idea is to determine disagreement points simultaneously with the solutions, and as a function of the utility space only. Our version of the Nash solution maximizes the Nash product w.r.t. *both* the solution *and* the disagreement point. We show that that no other restriction on independence axiom can be reconciled with continuity axiom while still obtaining a uniquely defined solution. A similar alteration of the monotonicity axiom of Kalai-Smorodinsky (1975) is introduced.

JEL: C71, C78.
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1 Introduction

Since Nash (1950), a bargaining problem is characterized by two principal geometric properties: a utility set on the domain of utility vectors, and a *disagreement* point in the utility set. A bargaining solution is rule that assigns a unique solution to each bargaining problem. The solution is typically sensitive to both of its arguments; change in the disagreement point's position typically affects the solution.

The existence of a disagreement point can be motivated on many grounds. The most popular is the impassé-interpretation: disagreement point reflects what players get if they do not reach an agreement. The relative magnitudes of losses are then thought to affect players' negotiation power. Moreover, since any player can always guarantee his disagreement level of utility by refusing to negotiate, the diagreement point defines a lower bound of the solution. From the modeling angle, the disagreement outcome helps one to anchor the analysis. Without disagreement outcome or equivalent predetermined and fixed device,

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multi valuedness of a solution would become an immediate problem of most of the solutions. Finally, existence of a disagreement point is useful since it endows one with a natural reference point from where utility comparisons can be made. Many if not most prominent solutions are based on utility comparisons.

However, in many interesting scenarios assuming the existence of a disagreement point is disturbing. In fact, the impassé-interpretation stands in a sharp contradiction with Pareto optimality, the axiom that probably enjoys most general acceptance. How can players be threatened by the disagreement outcome if they are, at the outset, assumed to reach the Pareto frontier? The Coase Theorem argues that a Pareto-dominated disagreement outcome can never prevail if rational players cannot commit not to continue negotiations. Thus, if one assumes that disagreement outcome can effectively be reached, then one also needs to assume that players can commit not to continue negotiations. What happens if it is common knowledge that players lack any such commitment power?

In practice, the more serious problem with disagreement point is that there is no universally accepted criterion how to pick one. As the solution is sensitive to the position of the disagreement outcome, it follows, paradoxically, that once the solution has been identified one still needs to cope with the disagreement point selection problem (see Kalai and Rosenthal, 1978, for an elegant noncooperative solution to this problem). This problem manifests itself in noncooperative games. For example, in repeated games and situations where correlated strategies are feasible (e.g. Aumann 1974), equilibrium selection is a fundamental problem. As the set of equilibrium utilities is a convex set, a bargaining model could ideally be used as an equilibrium selection device (this idea is advanced e.g. by Luce and Raiffa, 1957). Applicability of the bargaining model is crucially hampered by the fact that there is no criterion to choose which equilibrium should play the role of the disagreement outcome. In fact, choosing the "disagreement equilibrium" should be as difficult as choosing the desirable equilibrium itself.

Defining the bargaining solution without a reference to a disagreement point, or an axiomatization which *generates* such point simultaneously with the actual solution, would constitute a remedy to the problem. This is the aim of this paper. We construct an axiomatization that allows one to remove the disagreement point altogether from the definition of the problem. Moreover, when comes to characterizing our *extended Nash* solution, we show that our axiomatization is essentially the only way to derive the solution. We believe that our results could be very useful in applied literature wherever disagreement point's position is difficult to determine.

The central property of our axiomatization is that we simply remove the disagreement point from the definition of the *symmetry* axiom (which is the only axiom where the disagreement point enters in Nash's system). Given this strenghtening of the symmetry axiom, both Nash and KS axiomatizations become too tight: no solution satisfies them. To solve this problem, we first restrict the domain where Nash's *independence* (IIA) axiom binds. Thus we introduce a *restricted* IIA (RIIA).

We first show that the unique solution that satisfies Nash's axioms with altered symmetry and IIA axioms is the utility vector (u_1^*, u_s^*) that maximizes together with some (d_1^*, d_2^*) the Nash product $(u_1 - d_1)(u_2 - d_2)$ with respect to *both* (u_1, u_2) *and* (d_1, d_2) in utility set U. Outcome (u_1^*, u_s^*) is called the extended Nash solution of U. Extended Nash solution is single valued almost everywhere. Thus the disagreement point (d_1^*, d_2^*) is in our model is determined "endogenously" and simultaneously with the solution (u_1^*, u_2^*).

We then show that IIA *cannot* be weakened in *any* other way if one wants the solution to be unique almost always, and derivable from somehow restricted IIA and other standard axioms. More precisely, if one wants to drop the disagreement point from the description of the model, and finds uniqueness of the solution along with *continuity,* Pareto optimality, symmetry, scale invariance and somehow restricted version of IIA appealing, then RIIA is the *only* acceptable axiom. Consequently, the extended Nash solution is the only feasible solution with somehow restricted IIA.

The extended *Kalai-Smorodinsky* (KS) (1977) solution is based on a similar logic. We remove the disagreement point from the symmetry axiom and weaken the *monotonicity* axiom by restricting the domain where it binds. Where the standard KS solution is characterized by the intersection of the Pareto-boundary and the line segment joining disagreement point to the "utopia" point, defined by the jointly maximal utilities of both players, our version of the solution picks the point in the intersection of the Pareto-boundary and the line segment joining the "utopia" point to the "anti-utopia" point, defined by the jointly minimal utilities of both players. Again, the disagreement point is determined "endogenously" by restricting the monotonicity axiom appropriately.

The driving force of the result is the extended use of symmetry properties. That is, we do not only appeal to the symmetry axiom that is defined w.r.t. positively sloped diagonal, but also to the *inverse* problem which is symmetric w.r.t. negatively sloped diagonal. This allows us demand that under inversely symmetric conditions a solution to the problem and its inverse must coincide.

This in turn gives us enough powetr to pin donwn the solution. Inverse symmetry is not needed whenever the disagreement point is fixed.

We begin with specifying the model. Then we introduce the axiomatic system, and characterize the extended Nash solution. In the third section, we establish that our restriction is the only feasible one. The final section introduces an extended Kalai-Smorodinsky solution.

2 The Model

Let $\mathcal{U} = \{U \subset \mathbb{R}^2 : U$ compact and convex$\}$. In particular, $\mathcal{U}$ contains all problems of zero measure. Note that almost every convex problem is also strictly convex. A *bargaining solution* is a non-empty, *convex* valued *mapping* $F : \mathcal{U} \to \mathbb{R}^2$ such that $F(U) \subseteq U$ for all $U \in \mathcal{U}$. The properties, or axioms, we are interested the solution to satisfy include the following. Denote the Pareto frontier by $P(U) = \{u \in U : v > u \text{ implies } v \notin U\}$.[1]

PAR (Pareto-efficiency) $F(U) \subset P(U)$, for all $U \in \mathcal{U}$.

Use notation $aU+b = \{(a_1u_1+b_1, a_2u_2+b_2) : (u_1, u_2) \in U, (a_1, a_2), (b_1, b_2) \in \mathbb{R}^2\}$. In particular, write $(-1, -1)U = -U$.

INV (Scale Invariance) $F(aU + b) = aF(U) + b$, for $a \in \mathbb{R}^2_{++}, b \in \mathbb{R}^2$, for all $U \in \mathcal{U}$.

To describe symmetric conditions, denote $U' := \{(u_2, u_1) \in \mathbb{R}^2 : (u_1, u_2) \in U\}$, for any $U \in \mathcal{U}$.

ESYM (Extended symmetry) $U = U'$ implies $F_1(U) = F_2(U)$, for all $U \in \mathcal{U}$.

Thus, in the extended symmetric case F contains an element in the diagonal. If F is single valued, ESYM requires that $F_1(U) = F_2(U)$ in symmetric problems. Observe that because of the absence of the disagreement point, this version of ESYM is *stronger* than the usual one which restricts the solution only when the disagreement point is equal to $(0, 0)$.

2.1 The Extended Nash

RIIA (Restricted Independence of Irrelevant Alternatives) $F(V) \cup F(-V) \subseteq U \cap (-U)$ and $U \subseteq V$ implies $F(V) = F(U)$, for all $U, V \in \mathcal{U}$.

[1] Vector inequalities: for $u, v \in \mathbb{R}^2$, $u \geqq v$ means $u - v \in \mathbb{R}^2_+$; $u \geq v$ means $u \geqq v \neq u$; $u > v$ means $u - v \in \mathbb{R}^2_{++}$.

Whenever RIIA binds, it necessarily also means that $F(-U) = F(-V)$, and $-F(U) = -F(V)$. Note well that RIIA is *weaker* than the usual IIA condition which in the current context can be stated as follows: let $F(V) \subseteq U$ and $U \subseteq V$, then $F(V) = F(U)$. Thus, if one views Nash's IIA axiom acceptable, then RIIA should be acceptable as well.

Construct a new solution as follows: define a correspondence $(F^N, D) : \mathcal{U} \to \mathbb{R}^2 \times \mathbb{R}^2$ where

$$(F^N, D)(U) = \underset{(u,v)\in U\times U}{\arg\max}\,(u_1 - v_1)(u_2 - v_2), \quad \text{for all } U \in \mathcal{U}. \tag{1}$$

The geometry of $(F^N, D)(U) \subseteq U \times U$ meeting (1) will play influential role in the analysis. W.l.o.g., let $f^N \geq d$ for all $f^N \in F^N$, $d \in D$. The difference between F^N and the standard Nash solution is that the latter is a solution to program (1) with some *fixed* $D = d$. In contrast, in our model D is determined endogenously, and simultaneously with F^N. However, F^N and D have interpretation of being one anothers' "dual" solutions, as the following parity manifests:

$$F^N(U) = -D(-U),\ F^N(-U) = -D(U). \tag{2}$$

Moreover, it is clear that $(F^N, D)(U) = (F^N, D)(V)$ if $U \subseteq V$ and $(F^N, D)(V) \subseteq U \times U$. In particular, note that if $U = -U$, then

$$F^N(U) = -D(-U) = F^N(-U) = -D(U).$$

In such case, $F^N(U)$ and $D(U)$ draw a line segment of equal length and slope, at the opposite sides of the origin. Thus, the convex hull of points in $F^N(U)$ and $D(U)$, denoted by $\mathrm{co}(F^N(U), D(U))$, is a parallelogram whose diagonals run through the origin. Finally, note that $(F^N, D)(U)$ contains a *unique* element almost everywhere: for example, $F(U)$ is single valued if U is strictly convex, has continuously differentiable Pareto-frontier, or if U does not contain any two parallel line segments in its exterior surface. I.e. the solution is generically unique for bargaining problems arising from correlated equilibria of a finite normal form game, or equilibria of a repeated finite normal form game.

To characterize geometrically the solution, note that if $F(U) = F(-U) = \{(1,1)\}$ and there is a tangent of U and $-U$ with slope -1, running through $(1,1)$, then $F(U) = F^N(U)$. To see why, fix $D(U) = \{(-1,-1)\}$ and derive the standard Nash solution $(1,1)$. The tangency of U at $(1,1)$ has the slope -1. By (2), the inverse problem has the same property.

Theorem 1 *F satisfies PAR, INV, ESYM and RIIA if and only if $F = F^N$.*

Proof. First we argue that F^N satisfies PAR, INV, ESYM and RIIA. Seeing that F^N meets PAR, INV and ESYM is obvious. We check RIIA. Suppose that $U \subset V$ and $F^N(V) \cup F^N(-V) \subseteq U \cap (-U)$, for some $U, V \in \mathcal{U}$. By (2), $(-D(V)) \subseteq U \cap (-U)$. Multiplying both sides by -1 gives $D(V) \subseteq (-U) \cap U$. Thus $(F^N, D)(V) \subseteq U \times U$. Therefore, $(F^N, D)(U) = (F^N, D)(V)$, and RIIA holds.

Now we argue that if F satisfies PAR, INV, ESYM and RIIA, then $F(T) = F^N(T)$, for all $T \in \mathcal{U}$. Identify the element $(f^N, d)(U)$ of $(F^N, D)(U)$ such that, for $i = 1, 2$,

$$\begin{aligned} f_i^N(U) &= \frac{\sup F_i^N(U) + \inf F_i^N(U)}{2}, \\ d_i(U) &= \frac{\sup D_i(U) + \inf D_i(U)}{2}. \end{aligned}$$

Take T and let $U = aT + b$ where

$$a_i := \frac{2}{f_i^N(T) - d_i(T)},\ b_i := -\frac{f_i^N(T) + d_i(T)}{f_i^N(T) - d_i(T)},\ i = 1, 2.$$

Then, $f^N(U) = -d(U) = (1, 1)$. By INV, it now suffices to show that $F^N(U)$ is the only solution for U meeting PAR, ESYM, and RIIA. With this normalization, $(1, 1) = f^N(U) = f^N(-U) \in F^N(U) = F^N(-U)$ Identify[2]

$$V = \text{co}(U \cup (-U) \cup U' \cup (-U')).$$

Clearly such V belongs to $\mathcal{U}$, satisfies $V = -V = V' = -V'$, and runs through points $(1, 1)$ and $(-1, -1)$. Moreover $U \subseteq V$, and $f^N(U) = f^N(V) = f^N(-V)$. By ESYM and PAR, $f^N(U) \in F(V) = F(-V)$. This and convexity of F imply that $F(V) \subseteq F^N(V) = F^N(U)$. Thus $F(V) = F(-V) \subset U \cap (-U)$ and, by RIIA, $F(U) = F(V)$. From these we conclude that $F(U) \subseteq F^N(U)$. We still need to establish the other direction.

Suppose that $F(U) \neq F^N(U)$. Since $U \subseteq V$, and $F^N(V) = F^N(-V) \subset U \cap (-U)$, it must be that $F(V) \neq F^N(V)$. By SYM, $F^N(V) \backslash F(V) = (F^N(V) \backslash F(V))'$. Identify a rectangle $\bar{V} \subset V$ such that $\bar{V}$:=co$(F^N(V), D(V))$. As $\bar{V} = -\bar{V}$ and $F(V) = F(-V) \subset \bar{V}$, it follows by RIIA that $F(\bar{V}) = F(V)$. Now there is a rectangle $\bar{T} \neq V$ such that $\bar{T} \cup (-\bar{T}) = \bar{V}$ and $\text{co}(F(\bar{V}), -F(\bar{V})) \subset \bar{T}$. By multiplying both sides by -1, we have $\text{co}(F(\bar{V}), -F(\bar{V})) \subset -\bar{T}$. By RIIA this implies that $F(\bar{T}) = F(-\bar{T}) = F(\bar{V})$. However, $\bar{T} = -\bar{T} + c$ for some $c \neq 0$. Thus, by INV, $F(\bar{T}) = F(-\bar{T}) + c$, a contradiction. ■

[2] co$A$ = convex hull of A.

The necessity part of the proof runs as follows. Suppose U is smooth so that F^N is the single valued and denote $f^N = F^N$. Normalize the situation such that $(1,1) = f^N(U) = f^N(-U)$. Identify the convex hull V of the union of U, the inverse of U, the transpose of U, and the inverse transpose of U. Then V is symmetric and by construction $(1,1) = f(U)$. Thus $f = f^N$. A slightly more elaborate argument is needed in the non-generic case where U is non-smooth and F^N set valued.

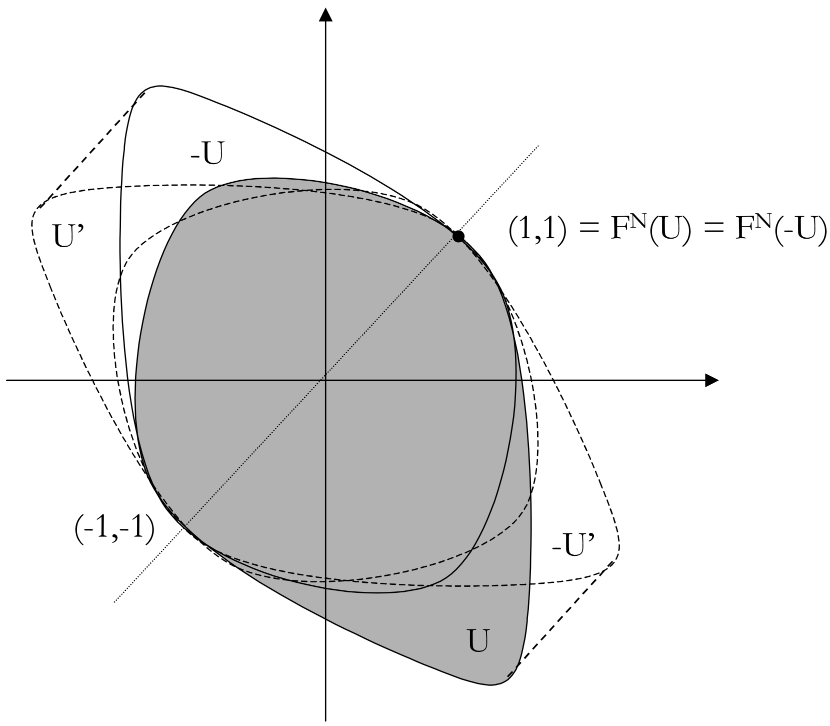

Figure 1

As discussed above, the only cases where F^N may obtain multiple values are the degenerate ones where U has parallel flat sides. Let $\mathcal{U}^* \subset \mathcal{U}$ satisfy the restriction

$$\mathcal{U}^* = \{U \in \mathcal{U} : (F^N, D)(U) \text{ is singleton}\}.$$

Thus, $\mathcal{U}^*$ contains all strictly convex problems. With this domain restriction, the next corollary follows immediately from the previous theorem. Let f^N be defined as in the proof of Theorem 1.

Corollary 2 *Under domain* $\mathcal{U}^*$, *F satisfies PAR, INV, ESYM and RIIA if and only if* $F = \{f^N\}$.

To verify this result one only needs to establish that if $U \in \mathcal{U}^*$ such that $(\tilde{F}, \tilde{D})(U) = \{((1,1),(-1,-1))\}$, then there is a symmetric problem $V \in \mathcal{U}^*$ that contains U, where V is *not* a rectangle. To see that such always exists, take $\mathring{V} := \{u \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 2\}$ (an origin-centered ball with radius $\sqrt{2}$), and a rectangle V^c that contains U. Then there is small enough $\lambda \in (0,1)$ such that $V^\lambda := \mathring{V}\lambda + V^c(1-\lambda)$ contains U. As $\mathring{V}$ is strictly convex, so is V^λ, for all $\lambda \in (0,1)$. Thus $V^\lambda \in \mathcal{U}^*$.

3 Alternative characterizations

3.1 Bounded domain

For simplicity we focus on domain $\mathcal{U}^*$ where set valued solution is not needed. Admittedly, unlike IIA, RIIA does not have natural interpretation. Why should IIA be restricted only in the domain of RIIA and not in other problems, or why should the restriction bind in *all* the problems defined by RIIA? It is clear, however, that if one finds the logic of IIA appealing, and wants to remove a fixed disagreement point but not to give up the other Nash axioms, then the domain of IIA must to be restricted somehow. Would there be any other way to weaken the IIA condition and still obtain a unique solution? The aim of this section is to show that under plausible conditions the answer to this question is negative.

Take a partial order $\Sigma \subseteq \{(U,V) \in \mathcal{U}^* \times \mathcal{U}^* : U \subset V\}$, and assume that IIA binds only on Σ. Thus we get the Σ-restricted version of the IIA.

$\Sigma-$RIIA If $f(V) \in U$, then $f(V) = f(U)$ for all $(U,V) \in \Sigma$.

Instead of RIIA, we assume that a natural solution meets $\Sigma-$RIIA, for some Σ, and is continuous in the following sense.

CON (Continuity) If sequence $\{U^k\}_{k=1}^\infty \subset \mathcal{U}$ converges in the Hausdorf metric, then $\{f(U^k)\}_{k=1}^\infty \subset \mathbb{R}^2$ converges in the Euclidean metric.

We say $\mathcal{U}'$ is *bounded* if there is $c \in \mathbb{R}_+$ such that $|u| \leq c$ for all $\mathcal{U}'$.

Lemma 3 *Let bargaining solution f be unique and satisfy PAR, INV, ESYM, CON and $\Sigma-$RIIA on bounded $\mathcal{U}^*$. Then $f = f^N$.*

Proof. Take U, identify $f(U), f(-U)$ and, by INV, adopt normalization $f(U) = f(-U) = (1,1)$.[3] Choose $V^0 = U$. Uniqueness of $f(V^0)$ is a consequence

[3]For any T, use scales $U = aT + b$ such that $a_i := 2(f_i(T) - f_i(-T))^{-1}$, $b_i := -(f_i(T) + f_i(-T))(f_i(T) - f_i(-T))$, for $i = 1,2$.

of either ESYM and PAR, or $\Sigma-$RIIA and PAR: either there is $(V^0, V^1) \in \Sigma$ such that $f(V^1) = (1,1)$, or V^0 is symmetric, in which case choose $V^0 = V^1$. Sequence $\{V^k\}$ constructed this way has the property that $V^k \subset V^{k+1}$ and $f(V^k) = (1,1)$ for all $k = 0, 1, ...$. Since $\mathcal{U}^*$ is bounded, every sequence of $\{V^k\}$ converges in the Hausdorf metric to some $V^* \in \mathcal{U}^*$. By CON, $f(V^*) = (1,1)$. Since $f(V^*)$ is unique, and since there is no $V' \in \mathcal{U}^*$ such that $(V^*, V') \in \Sigma$ and $f(V') = (1,1)$, it follows that V^* must be symmetric. Since $V^k \subset V^*$ for all k, we have $U \subset V^*$.

Conversely, use the same argument to identify symmetric $\bar{V}^*$ such that $-U \subset \bar{V}^*$ and $f(\bar{V}^*) = f(-\bar{V}^*) = (1,1)$. Then also $V^* \cap (-\bar{V}^*)$ is symmetric and, by multiplying by (-1), $U \subset V^* \cap (-\bar{V}^*)$. Since $f(U) = f(-U) = (1,1)$, we have $f(U) = f^N(U)$. ■

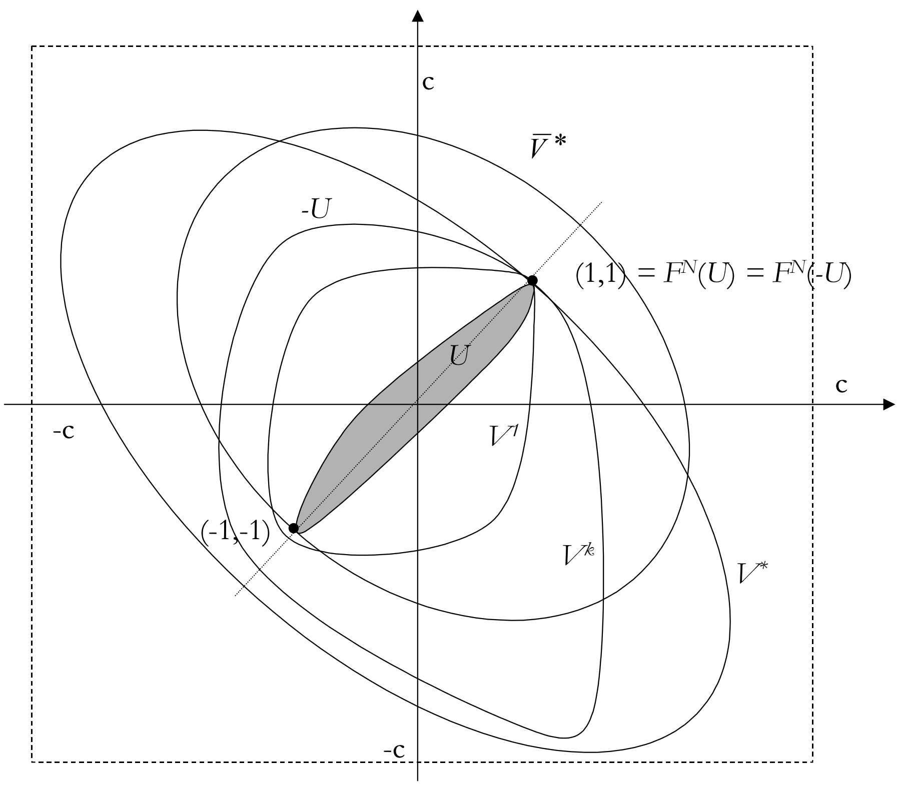

Figure 2

The proof is based on the idea that any uniquely determined solution on U must use ESYM and PAR, or $\Sigma-$RIIA to pin down the solution. The latter implies that there is a utility set containing U with uniquely determined solution. Thus, inductively, we get a sequence of nested utility sets $U, V^1, ..., V^k, ...$ (see Fig. 2) that ordered by RIIA. By boundedness of the utility space, such sequence converges. The convergent point V^* must be a symmetric utility set. By continuity, the solution of U then equals $f^N(V^*)$. As the same applies to the inverse problem we can, after normalizing U such that $f^N(U) = f^N(-U) = f^N(V^*)$, identify the intersection of the two convergent utility sets. As both of them are symmetric, the intersection is symmetric as well. Thus the solution of both U and $-U$ equals $f^N(V^*)$.

Define $\Sigma^N = \{(U,V) \in \mathcal{U}^* \times \mathcal{U}^* : U \subset V,\ f^N(V), f^N(-V) \in U \cap (-U)\}$. Then, by Theorem 1, f^N satisfies PAR, INV, ESYM and Σ^N-RIIA on $\mathcal{U}^*$. Thus, from Lemma 3 we have the following.

Theorem 4 *Unique single valued solution f can be characterized by PAR, INV, ESYM, CON and $\Sigma-$RIIA on bounded $\mathcal{U}^*$, for some Σ, if and only if $f = f^N$.*

Boundedness of $\mathcal{U}^*$ is clearly a questionable restriction given that players' behavior should be independent on their vNM scales. However any condition that guarantees that V^k sequence, V^k being symmetric with $(1,1) \in P(V^k) \cap P(-V^k)$, does not enlarge without a bound could be used instead. For example, ruling out too "thin" U's would suffice (e.g. by requiring that relation $-\left[\max_{u,v\in U} \prod(u_i - v_i)\right]\left[\min_{u,v\in U} \prod(u_i - v_i)\right]^{-1}$ lies above some strictly positive number, for all U). Such restriction is clearly independent of the utility scales.

3.2 Endogenous reference point

Another way to get rid of boundedness of $\mathcal{U}^*$ is to appeal the axiom of *independency of nonindividually rational alternatives*, where the level of minimally individually rational outcomes can be arbitrary (but equal between the players). Of course, such condition would require the existence of a reference outcome. In this subsection, we develop a model where a reference outcome is derived endogenously, and together with the actual solution. Thus, the difference between the reference outcome and a disagreement outcome is that the reference outcome need not be fixed.

First we define an endogenoud reference point. Such point is then used only to rule out the nonindividually rational outcomes. Let the reference point be a function $d : \mathcal{U}^* \to \mathbb{R}^2$ such that $d(U) \in U$ for all $U \in \mathcal{U}^*$. Then pair $(f,d)(U)$ constitutes the solution and the reference point for U. First we extend some of the preceding conditions to take into account the existence of a reference point.

The most acceptable extension concern INV, ESYM and CON: formation of the reference point should be governed by the same principles than the solution (add D to all conditions to refer d function).

DINV $(f,d)(aU+b)=a(f,d)(U)+b$, for $a\in\mathbb{R}^2_{++}, b\in\mathbb{R}^2$, for all $U\in\mathcal{U}$.

DESYM $U=U'$ implies $(f_1,d_1)(U)=(f_2,d_2)(U)$, for all $U\in\mathcal{U}$.

DCON If sequence $\{U^k\}_{k=1}^{\infty}\subset\mathcal{U}$ converges in the Hausdorf metric, then $\{(f,d)(U^k)\}_{k=1}^{\infty}\subset\mathbb{R}^2$ converges in the Euclidean metric.

The following condition is, perhaps, more subtle. It says that if IIA binds for f under (U,V), given restriction Σ, then IIA should bind for d also.

$\Sigma-$**DRIIA** If $f(V)\in U$, then $(f,d)(V)=(f,d)(U)$ for all $(U,V)\in\Sigma$.

Write $U_d=\{u\in U: u\geq d\}$. Now we state the condition that differentiates this model from the previous subsection.

INIR $(f,d)(U)=(f',d')$ implies $(f,d)(U_{d'})=(f',d')$.

Lemma 5 *Let f be unique and satisfy PAR, DINV, DESYM, DCON, $\Sigma-$DRIIA and INIR. Then $f=f^N$.*

Proof. As in Theorem 3, let $U=V^0$ and construct sequence $\{V^k\}$ such that $V^k\subset V^{k+1}$ and $f(V^k)=(1,1)$ for all $k=0,1,\ldots$. By $\Sigma-$DRIIA, $d(V^k)=\delta$ for all k. Sequence $\{V^k_\delta\}$ converges in the Hausdorf metric to some $V^*_\delta\in\mathcal{U}^*$. By INIR, $(f,d)(V^k_\delta)=((1,1),\delta)$ for all k and, by DCON, $(f,d)(V^*_\delta)=((1,1),\delta)$. Since $f(V^*_\delta)$ is unique, and since there is no $V'\in\mathcal{U}^*$ such that $(V^*_\delta,V')\in\Sigma$ and $f(V')=(1,1)$, it follows that V^*_δ must be symmetric. Since $V^k_\delta\subset V^*_\delta$ for all k, we have $U_\delta\subset V^*_\delta$.

Conversely, identify $d(-V^k)=\bar\delta$ for all k and symmetric $\bar V^*_{\bar\delta}$ such that $-U_{\bar\delta}\subset\bar V^*_{\bar\delta}$ and $f(\bar V^*_{\bar\delta})=(1,1)$. Then $\text{co}\{V^*_\delta\cap(-\bar V^*_{\bar\delta})\}$ is symmetric and $\text{co}\{U_\delta\cap(-U_{\bar\delta})\}\subset\text{co}\{V^*_\delta\cap(-\bar V^*_{\bar\delta})\}$. Since $f(U_\delta)=f(-U_{\bar\delta})=(1,1)$, we have $f^N(\text{co}\{U_\delta\cap(-U_{\bar\delta})\})=f^N(-\text{co}\{U_\delta\cap(-U_{\bar\delta})\})=(1,1)$. As $\text{co}\{U_\delta\cap(-U_{\bar\delta})\}\subset U$, we have, by the same token, $f(U)=f^N(U)$. ■

Hence, we have shown that boundedness of the utility space can be replaced with INIR defined with repect to endogenous reference point to obtain that RIIA is unique admissable restriction on IIA, if one wants the single valued solution to satisfy the other standard Nash axioms. Again, defining $d(U)=-f^N(-U)$ for all $U\in\mathcal{U}^*$ we have, by Theorem 1, that f^N satisfies PAR, DINV, DESYM and Σ^N-DRIIA on $\mathcal{U}^*$.

Theorem 6 *Unique single valued solution f can be characterized by PAR, DINV, DESYM, DCON, $\Sigma-$DRIIA on $\mathcal{U}^*$, for some Σ, if and only if $f = f^N$.*

4 The Extended Kalai-Smorodinsky

In this section, we construct in a similar spirit a solution that is based on restricted version of monotonicity by Kalai and Smorodinsky (1975). There is no need to allow set valued solutions, hence we focus on single valued solution f on $\mathcal{U}$. Define $\overline{m}_i(U) = \max\{u_i : u \in U\}$ and $\underline{m}_i(U) = \min\{u_i : u \in U\}$, and let $\overline{m}(U) = (\overline{m}_1(U), \overline{m}_2(U))$, $\underline{m}(U) = (\underline{m}_1(U), \underline{m}_2(U))$, for $i \in \{1, 2\}, U \in \mathcal{U}$. Note that necessarily $\overline{m}(U) = -\underline{m}(-U)$.

RMON (Restricted Individual Monotonicity) $\overline{m}_i(V) = \overline{m}_i(U)$, $\underline{m}_j(V) = \underline{m}_j(U)$ and $U \subseteq V$ implies $f_j(V) \geq f_j(U)$, for all $i \neq j$, $U, V \in \mathcal{U}$.

Again, RIIA is *weaker* condition than the usual MON, which binds only when $\overline{m}_i(V) = \overline{m}_i(U)$ and $U \subseteq V$. Thus, if one views MON acceptable, then RMON should be acceptable as well. Note that condition RMON can equivalently be stated as follows: $\overline{m}_i(V) = \overline{m}_i(U)$, $\underline{m}_j(V) = \underline{m}_j(U)$ and $U \subseteq V$ implies $f_i(-V) \geq f_i(-U)$, or $\overline{m}_j(-V) = \overline{m}_j(-U)$, $\underline{m}_i(-V) = \underline{m}_i(-U)$ and $U \subseteq V$ implies $f_j(V) \geq f_j(U)$, for $i \neq j$, $U, V \in \mathcal{U}$.

Let $I(u, v)$ be the line segment joining $u, v \in \mathbb{R}^2$. Solution $f^{KS} : \mathcal{U} \rightarrow \mathbb{R}^2$ is then defined as follows

$$f^{KS}(U) = I(\overline{m}(U), \underline{m}(U)) \cap P(U), \quad \text{for all } U \in \mathcal{U}.$$

Note that f^{SK} is single valued, and that

$$f^{KS}(U), -f^{KS}(-U) \in I(\overline{m}(U), \underline{m}(U)) \cap U. \tag{3}$$

Theorem 7 *f satisfies PAR, INV, SYM and RMON if and only if $f = f^{KS}$.*

Proof. First we argue that f^{KS} satisfies PAR, INV, SYM and RMON. Seeing that f^{KS} meets PAR, INV and SYM is obvious. We check RMON. Suppose $\overline{m}_i(V) = \overline{m}_i(U)$, $\underline{m}_j(V) = \underline{m}_j(U)$ and $U \subseteq V$, for some $U, V \in \mathcal{U}$. Hence, it follows that $\overline{m}_i(V) \geq \overline{m}_i(U)$, $\underline{m}_j(V) \leq \underline{m}_j(U)$ and, therefore $f_j^{KS}(V) \geq f_j^{KS}(U)$.

Now we argue that if f satisfies PAR, INV, SYM and RMON, then $f(U) = f^{KS}(U)$, for all $U \in \mathcal{U}$. Take U, identify $(\overline{m}, \underline{m})(U) \in \mathbb{R}^2 \times \mathbb{R}^2$, and adopt normalization $\overline{m}(U) = -\underline{m}(U) = (1, 1)$.[4] By INV, it suffices to show that

[4] For any T, use scales $U = aT + b$ such that $a_i := 2(\overline{m}_i(T) - \underline{m}_i(T))^{-1}$, $b_i := (\overline{m}_i(T) + \underline{m}_i(T))(\overline{m}_i(T) - \underline{m}_i(T))^{-1}$, $i = 1, 2$.

PAR, SYM, and RMON imply that $f(U) = f^{SK}(U)$. Construct set V :=co$\{U$, $(1,-1)$, $(-1,1)\}$. Note that, by construction, $P(V) = P(U)$ and $(\overline{m}, \underline{m})(V) = (\overline{m}, \underline{m})(U)$ implying that $f^{KS}(U) = f^{KS}(V)$. Moreover, $U \subseteq V$. Then, by RMON, $f(V) \geq f(U)$. As, by PAR, $f(V) \in P(V)$, it follows that $f(U) = f(V)$. Thus we only need to show that necessarily $f(V) = f^{KS}(V)$. Note that $f^{KS}(-U) = f^{KS}(-V)$. Construct set $\hat{V}$:=co$\{f^{KS}(V)$, $(1,-1)$, $(-1,1)$, $-f^{KS}(-V)\}$. Then, by (3), $\hat{V}$ is symmetric and $\hat{V} \subseteq V$. By SYM and PAR it follows that $f(\hat{V}) = f^{KS}(\hat{V}) = f^{KS}(V)$. By RMON it follows that $f(V) = f^{KS}(V)$, as required. ■

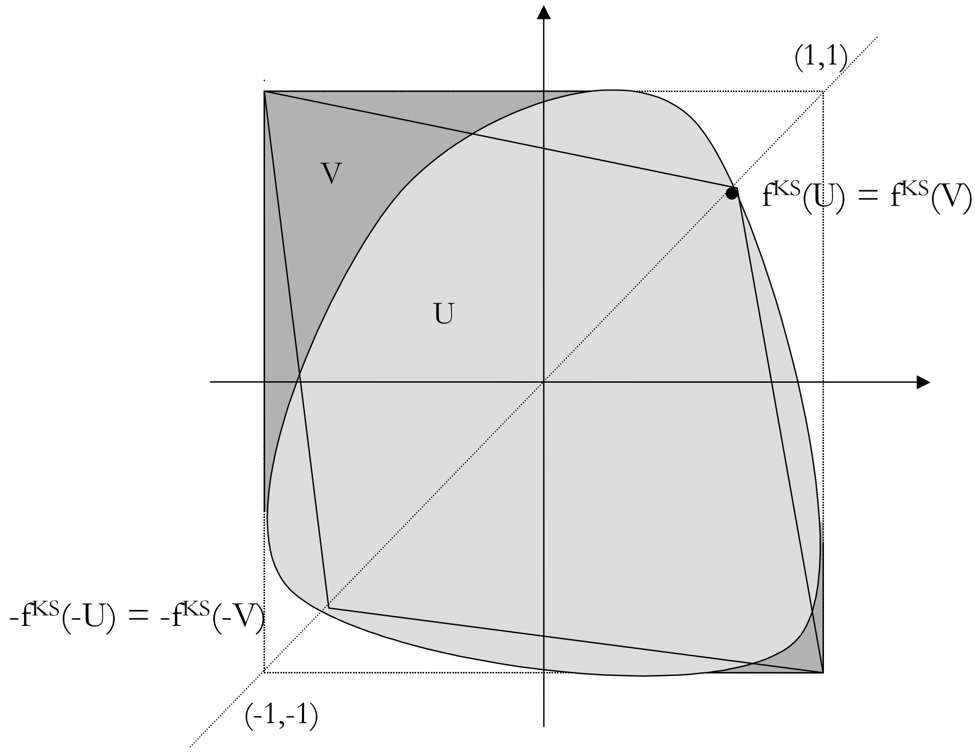

Figure 3.

5 Concluding Remarks

In this note we have argued that inverse symmetry can be exploited to remove the disagreement point from the characterization of the bargaining problem. The solutions have been derived by weakening the independence and monotonicity axioms of Nash and Kalai-Smorodinsky. Introduced solutions could have value in contexts where the position of a disagreement point cannot be justified.

The well know criticism against IIA, and the many substitute axiomatizations of the Nash solution, peg the question whether it is possible to remove the disagreement point and obtain a unique solution without appealing to IIA altogetheter.

Dagan *at. al.* (2002) show that IIA can be replaced with three conditions, independence of non-individually rational alternatives (INIR), twisting, which is a monotonicity requirement, and disagreement point convexity (DPC). INIR could be replaced with a condition that uses endogenous reference point, as in our Lemma 5. Disagreement point concavity is more difficult to replace with another, restricted, condition. In the end, the reference point should coincide with the inverse problem's solution, which leaves the condition with much less power.

Chun and Thomson (1990) in turn replace IIA with INIR, CON and DPC. Peters and van Damme (1991) assume INRA, individual rationality, DPC and a starshaped inverse condition (in the language of Thomson, 1991). In addition to DPC facing the difficulties described above, what is problemetic with these two characterizations is that they are defined with respect to the class of comprehensive problems. The extended Nash solution, on the other hand, is not defined in below unbounded utility space. Moreover, full domain of utility sets is also needed.

Lensberg (1988) shows that in the multiple players game IIA and symmetry can be replaced with consistency and anonymity. Again, his construction heavily relies on the comprehensiveness of the utility set. Nevertheless, as the underlying logic of consistency works much like IIA, it is plausible that a restricted version of it could be used to characterioze the extended Nash solution.

Mariotti (1999) introduces a maximal symmetry axiom which, in spirit, has connections to our construction. Maximal symmetry requires that a solution with relabeled player names should belong to the original utility space. Marriotti shows that maximal symmetry and strong individual rationality can be used to replace IIA. Moreover, his characterization does not require excessive domain assumptions. It is plausible that some (restricted) version of maximal symmetry could replace RIIA in our construction.

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