Complete Information All-Pay Auctions: Closed form Characterizations and Comparisons[*]

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Abstract

Nash equilibria of all-pay auctions are studied when players' cost functions are known but nonlinear. A complete closed form characterization of the first price all-pay auction and a partial closed form characterization of the second price all-pay auction is provided. In both cases, a closed form formula of seller's revenues is derived. With linear cost functions the revenue maximizing equilibrium of the first price all-pay auction is at least as profitable as that of the second price all-pay auction, and the regular action lies in between. In asymmetric case this order is strict. With quadratic cost function the order of the first price all-pay action and the second price all-pay action may be reversed, and both generate a higher revenues than the regular auctions.

Keywords: All-pay auctions, closed form characterization, revenue comparisons.

JEL: D44, D72.

1 Introduction

An all-pay auction is an auction game where the player putting forth the greatest effort wins the prize while the others go unrewarded. All-pay auctions capture the essential features of a contest, and they are pertinent to a large class of economically interesting situations, e.g. tournaments,

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rent-seeking, R&D-race, lobbying, advertising, political campaining, animal conflicts. Understanding the structure of equilibria of all-pay auctions is important from the perspective of general economic modelling. This paper collects together a number of new results on all-pay auctions.

The two most prominent auctions in this class are the *first pay all-pay auction* and the *second price*[1] *all-pay auction*. In the former case, the winner pays his own bid, and in the latter, he pays the second highest bid. All other bidders pay their own bids. Whereas the analysis of "standard" auction models is trivial in the complete information case, equilibria of the all-pay auctions are much more difficult to characterize since the resulting equilibria are in nondegenrate mixed strategies. Even if the equilibria in the standard linear case are well studied,[2] little is known about structure of the Nash equilibria when players' payoff functions are nonlinear. Also revenue properties of the auctions are not completely understood. This paper focuses on all-pay auctions under complete information. Equilibria of the all-pay auctions with more general payoff functions are characterized, and characterizations are used to compare auctions revenue-wisely.

First, a complete closed form characterization of the first price all-pay auction is provided. Let there be $n \geq 2$ bidders bidding for an object of value 1. Costs functions are of form $\psi_i y(\cdot)$, where y is an increasing and continuous function[3] and $\psi_i > 0$, $i = 1, ..., n$, such that $\psi_1 \leq \psi_2 \leq ... \leq \psi_n$. Then there is a unique equilibrium if and only if $\psi_2 < \psi_3$. In such equilibrium, 1 and 2 randomize continuously on $[0, y^{-1}(\psi_2^{-1})]$, and bidders $3, ..., n$ bid 0 with probability one. If $\psi_2 = \psi_3$, then there is a continuum of equilibria. We give explicit characterizations of these in terms of function y and parameters $(\psi_1, ..., \psi_n)$. The characterization builds on Baye *at.al.* (1993, 1996), who focus on the case where y is linear.

Second, we construct a partial closed form characterization of the second price all-pay auction. As argued by Hendricks *et.al.* (1988), only Nash equilibria with nondegenerate mixed strategies meet the additional restriction of the subgame perfection in the war of attrition game. We focus on such equilibria. We characterize a necessary condition for completely mixed equilibria, and also a distinct but closely connected sufficient condition.

We also establish a closed form formula of seller's revenues in the first and second price all-pay auctions. Together with the equilibrium charcterizations, these permits us to conduct revenue comparisons. We conjecture that the revenue rankings of auctions is the main contribution of the paper.[4]

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[1] Whose dynamic version is also known as the *war of attrition*.

[2] See Baye *et.al.* 1993, 1996, and references therein. Important contributions include Hillman and Riley 1989, Hendricks *et.al.* 1988, and Moulin 1986.

[3] Equivalently, we could assume that bidders share the same cost function γ but disagree of the value of the prize c_i^{-1}. With the restriction $\gamma(b) = b$, this would coincide with Baye *at.al.* (1986) assumptions.

[4] Krishna and Morgan (1997) analyze all-pay auction in the incomplete information

It is shown that with linear cost functions the revenue maximizing equilibrium of the first price auction is always at least as profitable as that of the second price auction, and that the standard actions lie in between. In the asymmetric case (bidders' marginal costs differ), this order is *strict*. Thus the *revenue equivalence* between the auctions does not hold.[5] From the perspective of Myerson (1981), this is due to the non-efficiency of all-pay auctions: equilibrium strategies contain randomization and, hence, cannot guarantee the most efficient allocation of the prize. To our knowledge, this revenue ranking of auctions under complete information has gone unnoticed in the previous literature.

We also show that with convex cost functions (or, equivalently, risk averse bidders) the revenue ordering of the first price all-pay action and the second price all-pay action may be reversed. In particular, with common quadratic cost function, the two-player first price all-pay auction generates higher revenues than the second price all-pay auction, *and* both the all-pay auctions generate higher revenues than the standard auctions. Heuristically, the reason for the latter effect is that for all bids $b \leq \beta$, where $\psi y(\beta) = 1$, it follows from convexity of y that $y(b) \leq b$. This implies, by the zero-profit constraint of the bidders, that the probability of player i winning with bid b in the first price all-pay auction must be lower under convex than linear y. Roughly, this can be true only if other bidders' strategies (first order) stochastically dominate in the convex case those in the linear case. Stochastic dominance in turn implies higher expected values of the bids, and hence greater expected revenues.

Finally, we show that equilibrium profits of the first price all-pay auction under large number of symmetric bidders and exponential cost functions can be characterized in a parsimonious way: as the number of bidders become large, sellers profits tend to the exponential of the cost function. This result should be handy in applications.

The paper is organized as follows: Section 2 introduces the set up. In Section 3, the characterizations of the first and second price all-pay auctions are derived. Section 4 conducts some comparisons between the auctions. In Section 5, some additional remarks are made on the properties of the auctions, and concluding lines are provided.

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scenario á la Milgrom and Weber (1982). Krishna-Morgan focus on ex ante symmetric case, and impose conditions to the bidders' signal structure that prevent signals being "too affiliated". These properties allow them focus equilibria with pure stratgies. However, they restrict us from applying the Krishna-Morgan results to the current set up.

[5] Baye *et.al.* (1986) show that the equilibria of the first price all-pay auction are not revenue equivalent, and they specify the most profitable equilibrium. We show that the argument extends to the non-linear case as well.

2 The Set Up

There is an indivisible object to be allocated to $\{1, ..., n\} := N$ players, "bidders". Bidder i's utility depends two ways on all bidders' actions: allocation of the object and transfers are contingent on all bidder' actions. Bidder i's action space is R_+ with a typical element b_i. Define allocation rule $s = (s_1, ..., s_n) : \mathsf{R}^n_+ \to \{0,1\}^n$ such that $\sum s_i(b) = 1$, for all $b = (b_1, ..., b_n) \in \mathsf{R}^n_+$. Given b, the object is devoted to bidder i iff $s_i(b) = 1$. A transfer rule is a mapping $t = (t_1, ..., t_n) : \mathsf{R}^n_+ \to \mathsf{R}^n_+$, specifying a transfer from each bidder which is contingent on the joint action b. Pair (s,t) is called an *auction*.

We focus on auctions that allocate the cake to the highest bidder. Let

$$M(b) := \arg\max_i \{b_i\}.$$

Then

$$\begin{aligned} s_i(b) &= \frac{1}{\#M(b)}, \quad \text{if } i \in M(b), \\ s_i(b) &= 0, \quad \text{if } i \notin M(b). \end{aligned}$$

The two all-pay auctions differ in terms of how transfers are determined. Denote by $b^{(2)}$ the second order statistics of sample $b_1, ...b_n$.

- First price all-pay auction (FPAA):

$$t_i(b) = b_i, \quad \text{for all } i \in N.$$

- Second price all-pay auction (SPAA):[6] .

$$\begin{aligned} t_i(b) &= \frac{b^{(2)}}{\#M(b)}, \quad \text{if } i \in M(b), \\ t_i(b) &= b_i, \quad \text{if } i \notin M(b). \end{aligned}$$

The corresponding *standard* auction forms are similar with the exception that $t_i(b) = 0$ if $i \notin M(b)$. Thus, the difference between all-pay and standard auctions is that in the former case bidders are required to pay even if they do not win the auction whereas in the latter case they are not.

Function $c_i : [0,1] \to \mathsf{R}_+$ describes the value of utility loss from transfer $t_i \in \mathsf{R}_+$. We mostly focus on the case where

$$c_i(t) = \psi_i y(t), \text{ for all } t \in \mathsf{R}_+, \text{ and for all } i = 1, ..., n, \tag{1}$$

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[6] The second price all-pay auction is known also as *the war of attrition.*

where $\psi_1, ..., \psi_n$ are positive scalars, and $y(\cdot)$ is a nondecreasing, continuous, and differentiable function, and satisfies $y(0) = 0$, $\lim_{t\to\infty} y(t) = \infty$. Without loss, assume $\psi_1 \leq \psi_2 \leq ... \leq \psi_n$. Given payment vector $t = (t_1, ..., t_2)$, bidder i's payoff is of quasilinear form $u_i(b) = s_i(b) - c_i(t_i)$. Hence, bidder's payoff from wealth is separable from the consumption of the prize. Possible nonlinearity of $y(\cdot)$ allows for risk aversion: *convexity* of y implies *concavity* of u_i in $-t$.[7] Thus, if y is convex (linear, concave) then i is risk *averse* (neutral, loving resp.) with respect to his wealth.

Denote by $\Sigma_1, ..., \Sigma_n$ a collection of independent cumulative distribution functions on R_+^n that are interpreted as bidders' strategies. Let S_i be the support of Σ_i.[8] If $S_i = \mathsf{R}_+$, then strategy Σ_i is *completely mixed.* With bid b_i and other bidders' strategies $\Sigma_{-i} = (\Sigma_j)_{j\neq i}$, bidder i's expected payoff is

$$\begin{aligned}\mathsf{E}u_i(b_i, \Sigma_{-i}) &= \int_{\mathsf{R}_+^{n-1}} [s_i(b) - c_i(t_i(b))]\, d\Sigma_{-i}(b_{-i}) \\ &= \prod_{j\neq i} \Sigma_j(b_i) - \int_{\mathsf{R}_+^{n-1}} c_i(t_i(b)) d\Sigma_{-i}(b_{-i}).\end{aligned}$$

Choices are made simultaneously. Strategy $\Sigma = (\Sigma_1, ..., \Sigma_n)$ constitutes a Nash equilibrium if and only if

$$\mathsf{E}u_i(\Sigma) \geq \mathsf{E}u_i(b_i, \Sigma_{-i}), \quad \text{for all } b_i \in \mathsf{R}_+,\ i \in N.$$

With bids $b = (b_1, ..., b_n)$, seller's payoff is $v(b) = \sum t_i(b)$. Since strategies are independent, his expected revenues are

$$\mathsf{E}v(\Sigma) = \sum_{i=1}^{n} \int_{\mathsf{R}_+} t_i(b) d\Sigma_i(b).$$

3 Characterizations

3.1 First price all-pay auction

First we focus on the first price all-pay auction. Hillman and Riley (1989) and Baye *et.al* (1996), provide a thorough analysis of the linear cost functions case. They show that in any Nash equilibrium buyer 1 extracts payoff $(\psi_2 - \psi_1)\psi_2^{-1}$,[9] while all other bidders get zero. In particular, Baye *et.al.* (1996) show that there is a continuum of equilibria, and that these equilibria differ revenue-wisely.

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[7] Suppose γ is twisely differentiable. Let $w = w_0 - t$ denote agents wealth, with initial wealth w_0 and transfer t. Then $du_i(s, w)/dw = c_i\gamma'(w_0 - w)$ and $d^2u_i(s, w)/(dw)^2 = -c_i\gamma''(w_0 - w)$, for all w. This implies that u_i is concave in w only if γ is convex in t.

[8] The smallest closed set S_i such that $\Sigma_i(b) - \Sigma_i(b + \varepsilon) > 0$, for all $\varepsilon > 0$, for all $b \in S_i$.

[9] Baye *at.al.* (1986) assume identical (linear) cost functions but allow different reservation valuations. Their and our approaches are isomorphic.

We show that the Baye *et.al.* characterization extends in a natural way to the non-linear case. The main contribution of the section is to characterize the equilibria and revenues in a closed form, i.e. in terms of buyers' cost functions only. This facilitates revenue comparisons of the auctions, conducted in the next section.

Under (1), denote by $\beta = y^{-1}(\psi_2^{-1})$ the break-even bid for bidder 2, assuming this bid wins with probability one. Let m be the largest integer such that $\psi_m \leq \psi_2$, and write

$$\tilde{c}_i(b) = \begin{cases} c_i(b) + \dfrac{\psi_2 - \psi_1}{\psi_2}, & \text{for } i = 1, \\ c_i(b), & \text{for all } i = 2, ..., n. \end{cases}$$

If $\psi_1 = \psi_2$, then $(\tilde{c}_1, ..., \tilde{c}_n) = (c_1, ..., c_n)$. The following proposition (see appendix for the proof) completely characterizes the set of equilibria under these conditions.

Proposition 1 *Assume (1). Strategy* $(\Sigma_i)_{i=1}^n$ *constitutes a NE of the FPAA if and only if there is a permutation of agents* $\{2, ..., m\}$ *and numbers* $0 = \lambda_1 = \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_m \leq \lambda_{m+1} = ... = \lambda_n = \beta$ *such that, for all* $k = 2, ..., m$,

$$\text{for all } b \in (\lambda_k, \lambda_{k+1}], \qquad \Sigma_i(b) = \left(\frac{\tilde{c}_i(b) \prod_{j=1}^{k-1} \dfrac{\tilde{c}_j(b)}{\tilde{c}_i(b)}}{\prod_{j=k+1}^{m} \alpha_j(0)} \right)^{\frac{1}{k-1}}, \qquad \text{for all } i = 1, ..., k,$$

$$\text{for all } b \in [0, \lambda_{k+1}], \qquad \Sigma_i(b) = \alpha_i(0), \qquad \text{for all } i = k+1, ..., n, \tag{2}$$

where the size of i's atom $\alpha_i(0)$ *at* 0, $i = 2, ..., n$, *is defined recursively by*

$$\alpha_k(0) = \tilde{c}_1(\lambda_{m+1})^{\frac{1}{m}} = 1, \qquad \text{for all } k = m+1, ..., n, \tag{3}$$

$$\alpha_k(0) = \left(\frac{\tilde{c}_1(\lambda_k)}{\prod_{j=k+1}^{m} \alpha_j(0)} \right)^{\frac{1}{k-1}}, \qquad \text{for all } k = 2, ..., m.$$

By Proposition 1, any player i imposes an atom $\alpha_i(0)$ on bid 0, and mixes continuously on $(\lambda_i, \beta]$. In any equilibrium, bidder 1 earns payoff $(\psi_2 - \psi_1)/\psi_2$ and others earn zero payoff. To see where the precise functional form (2) of the strategies comes from, note that if all $1, 2, ..., k$ bidders mix at b, then

$$\prod_{j \neq i} \Sigma_j(b) - \tilde{c}_i(b) = 0, \quad \text{for all } i \in \{1, ..., k\}. \tag{4}$$

Consequently

$$\Sigma_j(b) = \left(\frac{\tilde{c}_i(b)}{\tilde{c}_j(b)}\right)\Sigma_i(b), \quad \text{for all } i, j \in \{1, ..., k\}.$$

Inserting back to (4)

$$\prod_{j \neq i} \Sigma_j(b) = \Sigma_i(b)^{k-1} \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{\tilde{c}_i(b)}{\tilde{c}_j(b)} \prod_{j=k+1}^{n} \Sigma_j(b) = \tilde{c}_i(b), \quad \text{for all } i \in \{1, ..., k\}.$$

Dividing and rearranging

$$\Sigma_i(a) = \left(\tilde{c}_i(b) \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{\tilde{c}_j(b)}{\tilde{c}_i(b)} \prod_{j=k+1}^{n} \frac{1}{\Sigma_j(b)}\right)^{\frac{1}{k-1}}, \quad \text{for all } i \in \{1, ..., k\}.$$

By noting that if $j = k+1, ..., n$ does not mix on $(0, \lambda_i]$, then $\Sigma_j(b)$ is equivalent to the atom $\alpha_j(0)$. Thus (2) is induced.

A major part of the proof is devoted to arguing that (i) bidder 1 randomizes continuously on interval $[0, \beta]$, (ii) some bidder i randomizes continuously on interval $(0, \beta]$ and imposes an atom of size $\alpha_i(0)$ on 0, (iii) any player $j \in \{2, ..., m\} \setminus \{i\}$ randomizes continuously on some interval $(\lambda_i, \beta]$, $\lambda_i \in [0, 1)$, and imposes atom of size $\alpha_j(0)$ on 0, (iv) all bidders get zero profit. Baye *et.al.* (1996) proved in the linear case that the equilibria meets these conditions. The key contribution of Proposition (1) is the *closed form* characterization of the strategies.

By Proposition 1, the generic case of $m = 2$ induces a unique equilibrium where players 1 and 2 randomize on $[0, \beta]$ such that

$$\Sigma_1(b) = c_2(b) \text{ and } \Sigma_2(b) = \tilde{c}_1(b), \qquad \text{for all } b \in [0, \beta]. \tag{5}$$

Player 2's atom at 0 satisfies $\alpha_2(0) = \Sigma_2(0) = \tilde{c}_1(0) = \dfrac{\psi_2 - \psi_1}{\psi_2}$, and player i's, $i = 3, ..., n$, atom at 0 satisfies $\alpha_i(0) = \Sigma_i(0) = 1$. Thus the latter ones bid 0 with probability 1.

Now we turn to the seller's revenues. Denote seller's expected revenues under the first price all-pay auction by $\mathsf{E}v^f$

Proposition 2 *Let $(\Sigma_i)_{i=1}^n$ constitute a NE of FPAA. Then the expected payoff to the seller is*

$$\mathsf{E}v^f = \beta m - \sum_{i=1}^{m} \int_0^{\beta} \Sigma_i(b) db. \tag{6}$$

Proof. Recall that Σ_i is differentiable almost everywhere and hence admits density σ_i on $(0, \beta]$. The expected transfer from i is now obtained by integrating by parts (note that bid $b = 0$ results in 0 payment),

$$\begin{aligned}
\mathsf{E}t_i &= \int_0^\beta \sigma_i(b) b db + 0 \cdot \Sigma_i(0) \\
&= \beta \int_0^\beta \sigma_i(b) db - \int_0^\beta \int_0^a \sigma_i(b) da db \\
&= \beta(\Sigma_i(\beta) - \Sigma_i(0)) - \int_0^\beta (\Sigma_i(b) - \Sigma_i(0)) da \\
&= \beta\Sigma_i(\beta) - \int_0^\beta \Sigma_i(b) da.
\end{aligned}$$

Noting that $\Sigma_i(\beta) = 1$ for all $i = 2, ..., m$, we have, by summing over bidders,

$$\begin{aligned}
\mathsf{E}v^f &= \sum_{i=1}^m \mathsf{E}t_i \\
&= \beta m - \sum_{i=1}^m \int_0^\beta \Sigma_i(b) da.
\end{aligned}$$

■

Property (6) reveals that if Σ_i first order stochastically dominates Σ_i' for all $i = 1, ..., n$, then the seller prefers $(\Sigma_i)_{i=1}^n$ over $(\Sigma_i')_{i=1}^n$. Effects if the parameter changes may be surprising. Rewrite 5 in the form

$$\Sigma_1(b) = \psi_2 y(b) \text{ and } \Sigma_2(b) = \frac{\psi_2 - \psi_1}{\psi_2} + \psi_1 y(b) = 1 + \psi_1(y(b) - y(\beta)), \qquad \text{for all } b \in [0, \beta].$$

Since $y(b) < y(\beta)$ for all $b < \beta$, an increase in ψ_1 induces a new strategy for 2 which stochastically dominates the original strategy while leaving the other strategies unaffected. Thus such change unambigiously increases seller's payoffs. The explanation for this effect is that higher marginal costs of the low cost bidder indeces more fierce competition between 1 and 2 and induces them to bid more agressively. However, while an increase in ψ_2 induces a new strategy for 2 which stochastically dominates the original strategy while leaving the other strategies unaffected it also induces a new strategy for 1 which is stochastically dominated by the original strategy. By Proposition 2, the total effect remains unclear. More formally, the after some manipulation, sellers payoffs in the generic case can be written in the form

$$\mathsf{E}v^f = (\psi_1 + \psi_2) \int_0^\beta (y(\beta) - y(b))\, db$$

An important corollary of this is that reducing the number of randomizing bidders increases seller's profits. Thus in the optimal equilibrium only bidders 1 and 2 randomize. This is established in the following proposition.[10]

Proposition 3 *In the most profitable NE of the FPAA only* 1 *and* 2 *are active.*

Proof. Let bidders $2, ..., m$ randomize in the Nash equilibrium. Then $\lambda_{k-1} < \lambda_k < \lambda_{k+1} = \beta$. We show that $\mathsf{E}v^f$ increases in λ_k. As all Σ_i's are continuous, increase in λ_k only affects through the direct effect on $\alpha_k(0) = \tilde{c}_1(\lambda_k)^{k-1}$. We have

$$\frac{d\alpha_k(0)}{d\lambda_m} = (k-1)c_1 y'(\lambda_k)\tilde{c}_1(\lambda_k)^{k-2} > 0.$$

On the other hand, by (2),

$$\frac{d\Sigma_i(b)}{d\alpha_k(0)} < 0, \text{ for all } i = 1, ..., k.$$

Thus,

$$\frac{d\mathsf{E}v^f}{d\lambda_m} = -\frac{d}{d\lambda_m}\sum_{i=1}^{m}\int_0^\beta \Sigma_i(b)da = -\sum_{i=1}^{m}\int_0^\beta \frac{d\alpha_k(0)}{d\lambda_m}\frac{d\Sigma_i(b)}{d\alpha_k(0)}da > 0$$

Since this is true for any $k > 2$, and in any NE k is at least 2, it follows that in the optimum $k = 2$. ■

Thus in the most profitable equilibrium of the first price all-pay auction, bidders 1 and 2 completely mix on $[0, \beta]$, and all the other bidders bid 0. When comparing maximally profitable equilibria of the two all-pay auctions, it suffices to confine attention to such simple equilibrium.

3.2 Second price all-pay auction

Now we focus on the second price all-pay auction, whose equilibria are much more difficult to characterize than those of the first price all-pay auction. In the $n \geq 3$ bidders case, only a partial characterization is provided. First we characterize a necessary and a sufficient condition for any *completely mixed* equilibrium (proof in the appendix).

Proposition 4 *Assume (1). There is a NE of SPAA where* m *bidders completely mix only if these bidders constitute set* $\{k+1, ..., k+m\}$ *for*

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[10] Baye *et. al.* (1996) prove the result in the linear case.

some $k \in \{1, ..., n-m\}$ *such that* $\psi_k = ... = \psi_{k+m-1} \leq \psi_{k+m}$*, and*

$$\Sigma_i(b) = \left((1 - e^{-c_i(b)}) \prod_{j \neq i} \frac{1 - e^{-c_j(b)}}{1 - e^{-c_i(b)}} \right)^{\frac{1}{m-1}}, \quad \textit{for all } b \in \mathsf{R}_+, \textit{for all } i = k{+}1, ..., k{+}m. \tag{7}$$

In particular, such NE exists for $k = 0$.

Hendricks *at.al.* (1988) point out that the second price all-pay auction always hosts an asymmetric *pure strategy* equilibrium where bidder 1 bids $b \geq \beta$ and all other players bid 0. However, they argue that such Nash equilibrium is never *subgame perfect* in the dynamic version of the game, where bidders continue to raise their bids until they are the sole contestants (the war of attrition). We do not argue

We now turn to seller's revenues. Denote the expected payoff for the seller by $\mathsf{E}v^s$.

Proposition 5 *Let* $(\Sigma_i)_{i=1}^n$ *constitute a completely mixed NE of the SPAA. The expected revenue to the seller is*

$$\mathsf{E}v^s = \sum_{i=1}^n \int_0^\infty e^{-c_i(b)} \left(1 - \Sigma_i(b)\right) db. \tag{8}$$

Proof. The expected transfer of bidder i submitting bid a_i is

$$\mathsf{E}t_i(a_i) = \int_0^{a_i} b dG_i(b) + a_i(1 - G_i(a_i)).$$

On the other hand, since $\mathsf{E}t_i(a_i)$ is differentiable almost everywhere,

$$\mathsf{E}t_i(a_i) = \int_0^{a_i} d\mathsf{E}t_i(b) + \mathsf{E}t_i(0).$$

Nothing that $\mathsf{E}t_i(0) = 0$ and combining the other two expressions,

$$\begin{aligned} \mathsf{E}t_i(a_i) &= \int_0^{a_i} (1 - G_i(b))db \\ &= \int_0^{a_i} e^{-c_i(b)} db. \end{aligned}$$

The expected transfer of bidder i is then

$$\begin{aligned} \mathsf{E}t_i &= \int_0^\infty \mathsf{E}t_i(a_i) d\Sigma_i(a_i) \\ &= \int_0^\infty \int_0^a e^{-c_i(b)} db \sigma_i(a) da. \end{aligned}$$

This yields, by integrating by parts,

$$\begin{aligned}\mathsf{E}t_i &= \int_0^\infty e^{-c_i(b)} db \Sigma_i(\infty) - \int_0^\infty e^{-c_i(b)} \Sigma_i(b) db \\ &= \int_0^\infty e^{-c_i(b)} \left(1 - \Sigma_i(b)\right) db.\end{aligned}$$

Since bids are independent,

$$\begin{aligned}\mathsf{E}v^s &= \sum_i \mathsf{E}t_i \\ &= \sum_i \int_0^\infty e^{-c_i(b)} \left(1 - \Sigma_i(b)\right) db,\end{aligned}$$

as required by (8). ■

In the general case, seller's payoffs under SPAA are difficult to analyse and compare to FPAA.. However, in special cases this can be done. This is the theme of the next section.

4 Revenue comparisons

4.1 Linear cost functions

In this section, we make the assumption of linear cost functions. However, we allow asymmetries. Assume that $y(b) = b$ for all i. Then $\beta = 1/\psi_2$ and $c_1(\beta) = \psi_1/\psi_2$.

Proposition 6 *With linear cost function, the revenue maximizing NE of the SPAA is at least as profitable to the seller as the revenue maximizing NE of the FPAA, with strict inequality when* $\psi_1 < \psi_2$.

Proof. Recall that $\beta = 1/\psi_2$. By Propositions 7 and 3, the expected payoff from the most profitable NE of the FPAA is

$$\begin{aligned}\mathsf{E}v^f &= \beta(1 + c_1(\beta)) - \int_0^\beta (\psi_1 + \psi_2) b db \\ &= \frac{2}{\psi_2} - \int_0^{1/c_2} \left((\psi_1 + \psi_2) b + \left(1 - \frac{\psi_1}{\psi_2}\right)\right) db \\ &= \frac{1}{\psi_2}\left(2 - \frac{\psi_1 + \psi_2}{2\psi_2}\right) - \frac{\psi_2 - \psi_1}{\psi_2^2} \\ &= \frac{\psi_1 + \psi_2}{2\psi_2^2}. \qquad (9)\end{aligned}$$

By Propositions 4 and 5, there is a completely mixed NE of the SPAA, whose expected revenue can be written

$$\begin{aligned} \mathsf{E}v^s &= 2\int_0^\infty e^{-(\psi_1+\psi_2)b}db \\ &= \frac{2}{\psi_1+\psi_2}. \end{aligned} \tag{10}$$

Denote the average marginal cost by $\bar{\psi} = (\psi_1 + \psi_2)/2$. Then $\psi_2 \geq \bar{\psi} \geq \psi_1$. Now

$$\mathsf{E}v^s = \frac{1}{\bar{\psi}} \geq \beta \geq \beta\frac{\bar{\psi}}{\psi_2} = \mathsf{E}v^f, \tag{11}$$

with strict inequalities when $\psi_2 > \psi_1$. ∎

Thus, with linear cost function the revenue maximizing second price all-pay auction is at least as profitable for the seller as the first price all-pay auction, and strictly more profitable if the lowest marginal cost is strictly lower than the second smallest. Roughly, the reason for this is that the first price all-pay auction necessarily permits bidder number 1 to gain surplus of value $(\psi_2 - \psi_1)/\psi_2$ whereas the second price all-pay auction extracts all the surplus from all bidders. To our notice, this revenue difference has gone unnoticed in the previous literature.

From (9) and (10) it is easy to deduce that an increase in c_1 contributes *positively* to the expected revenues of the first price all-pay auction but *negatively* to those of the second price all-pay auction. Thus *decrease* in c_1 increases the revenue gap between the two auctions. When $\psi_1 = 0$, the expected revenue from the second price all-pay auction is 2β, and from the first price all-pay auction $\beta/2$, constituting a revenue gap of $3\beta/2$. This means that the revenue difference between the auctions always lies in $[0, 3\beta/2]$. Note that as ψ_2 goes to zero, the upper bound of this interval goes to infinity.

It is interesting to compare the all-pay auctions to the standard ones. Under full information, the analysis of standard auctions is straightforward. In the unique (trembling hand) perfect equilibrium the good is sold to the buyer who is willing to pay the most for the object, i.e. the bidder with the lowest marginal cost equal to the break-even price of bidder with the second lowest marginal cost. Such outcome constitutes the unique Nash equilibrium of the first price auction, and is equivalent with the *dominant* strategy Nash equilibrium of the latter[11] The payoff to the seller from the two auctions is β. In a symmetric case, the seller extracts all the surplus. In such equilibrium the seller gains the monetary value of the prize. This paper argues that such revenue equivalence does not extend to the all-pay auctions. On the other hand, it is easy to see that the any (trembling hand)

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[11] Assuming that in the first price auction ties are broken in favor of the player with the lowest marginal cost.

perfect equilibrium of the standard first price and second price auction forms generate seller payoff equal to β. Thus, from (11) it follows that the auctions can be ranked by their profitability as follows:

Corollary 1 *With linear cost functions, FPAA, SPAA, and standard auction(s) are ranked by their most profitable (trembling hand) NE in the following order: 1. FPAA, 2. standard auctions, 3. SPAA. Decrease in* ψ_1 *increases the revenue differences between the auctions.*

Intuitively, the reason why the first price all–pay auction generates lower profit than the standard ones is due to the fact that in the first price all-pay auction randomization causes inefficiencies when marginal costs differ. Hence the extractable payoff is lower under the first price all-pay auction than under standard auctions.

4.2 Two bidders - convex cost functions

It is easy to verify that the aforementioned characterizations can be extended to cover *general* increasing and differentiable (a.e.) cost functions when there are two bidders.

Proposition 7 *Let* $n = 2$ *and let* c_1 *and* c_2 *be continuous and increasing functions. (i) In the unique NE of the FPAA,* $\Sigma_i(b) = \tilde{c}_j(b)$*, for* $i, j = 1, 2$*,* $i \neq j$*. (ii) In the unique completely mixed NE of the SPAA,* $\Sigma_i(b) = 1 - e^{-c_j(b)}$*, for* $i, j = 1, 2$*,* $i \neq j$*.*

For proof of part (ii), the reader is referred to Hendricks *et.al.* (1988). They also show that the completely mixed Nash equilibrium is *only* one that satisfies subgame perfection in the war of attrition -version of the game. Denote by $\mathsf{E}v_2^f$ and $\mathsf{E}v_2^s$ the expected revenue generated to the seller under the two auctions when the two bidders obey Nash equilibria characterized by Proposition 7.

The expected revenues of the seller can now be written:

Corollary 2 *If* $n = 2$*, then*

$$\mathsf{E}v_2^f = \beta(1 + c_1(\beta)) - \int_0^\beta (c_1(b) + c_2(b))db,$$
$$\mathsf{E}v_2^s = 2\int_0^\infty e^{-c_1(b) - c_2(b)}db.$$

Thus the seller's payoff under the two auction depends only of the *average* cost function of the two bidders. General revenue comparisons are hard but in the particular case of quadratic cost functions we can say more.

Example 1 *Let* $c_1(b) = c_2(b) = b^2$ *for all* b*. Then* $\beta = 1$ *and*

$$\mathsf{E}v_2^f = 2 - \int_0^1 2b^2 db = \frac{4}{3},$$
$$\mathsf{E}v_2^s = 2\int_0^\infty e^{-2b^2} db = \sqrt{\frac{\pi}{2}}.$$

Therefore, $\mathsf{E}v_2^f > \mathsf{E}v_2^s$.

Thus the first price all-pay auction is more profitable in the symmetric quadratic case, to the contrary of the asymmetric linear case. The intuition is as follows. With symmetric buyers the buyers receive the object with the same probability in the two auctions. In the first price auction, this is the only uncertainty a buyer faces. However, in the second price auction he is also unaware of the price he is about to pay. Since he is risk averse, the same average expected payment from a bidder is more costly for the bidder in the second price auction. Thus the zero-payoff condition implies he is willing to bid less aggressively in the second price auction.

Note that with quadratic cost functions á la Example 1, the expected revenue of the seller in the standard auctions is equal to 1. This implies that the ranking of the actions in this case is: 1. FPAA, 2. SPAA, 3. the standard auctions. This observation sheds new light to the voluminous literature on over/under dissipation of rents.[12] In the current set up, when the value of the prize is independent of bidders' wealth, bidders' risk aversion leads to over-dissipation in the expected terms (see also the next section); on the average, bidders consume more than they receive.

5 Remarks

5.1 FPAA under symmetric bidders

Assume that there are n symmetric players: $c(\cdot) = c_1(\cdot) = ... = c_n(\cdot)$. From Propositions 2 and 3, the following corollary is immediate.

Corollary 3 *Assume symmetric payoff functions. In the maximal NE of the FPAA generates revenue*

$$\mathsf{E}v^f = n\left(\beta - \int_0^\beta c(b)^{\frac{1}{n-1}} db\right). \qquad (12)$$

It is clear that in the standard auction the (unique) symmetric equilibrium generate bids $a_i = \beta$ for all $i = 1, ..., n$ (in the first price case this is the *unique* equilibrium, in the second price case there are asymmetric

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[12] See Konrad and Schlesinger (1997) and references therein.

equilibria). Thus seller's payoff is 1 under standard auction. By (12) it is now straightforward to compare the all-pay auctions to the standard ones. Note that if $c(\cdot)$ is linear, then $\mathsf{E}v^f = \beta$, which is equivalent to the equilibrium payoff for the seller under the standard auction forms. This, of course, follows also from the revenue equivalence theorem (e.g. Myerson 1981).

If $c(\cdot)$ is to interpreted as bidder's utility function over wealth, then we can say that bidders are risk averse, neutral, or loving if c is convex, linear, or concave, respectively (note the sign of c). Note that with convex $c(\cdot)$ we have $c(b) \leq b$ for all $b \in [0, \beta]$, and with concave $c(\cdot)$ the direction of the inequality is reversed. Thus, from (12) it is easy to deduce the following.

Corollary 4 *The seller prefers, is indifferent to, or does not prefer the standard auctions over the FPAA if buyers are risk averse, neutral, or loving, respectively.*

It may be counter-intuitive that risk aversion makes the value of a randomized transfer *lower*, not higher, in the expected terms. By the zero-profit constraint of the bidders, this can be true only if other bidders' strategies in the convex case (first order) stochastically dominate those in the linear case. This implies higher expected values of the bids, and hence greater expected revenues for the seller.

What about the effect of the number of bidders? The next example shows that seller's revenues always increases in the number of bidders, if the cost functions are of Cobb-Douglas form, but not unboundedly.

Example 2 *Suppose symmetric cost function satisfies* $c_1(b) = ... = c_n(b) = b^r$, $r > 0$. *Then* $\beta = 1$ *and*

$$\mathsf{E}v^f = n\left(1 - \int_0^1 b^{\frac{r}{n-1}} db\right) = \frac{nr}{r+n-1}.$$

As n *becomes large,*

$$\lim_{n\to\infty} \mathsf{E}v^f = \frac{r}{\lim_{n\to\infty}(\frac{r}{n} + \frac{n-1}{n})} = r.$$

Thus with large number of symmetric bidders and Cobb-Douglas cost functions, the exponential gives a good estimate of the revenues. This property of the first price all-pay auction can be very handy in applications. Note also that as the number of bidder grows, the seller's revenues become less risky.

5.2 Winning bid

Sometimes the social benefit from the allocation of the prize is dependent of the winner's investment. Then it may be of the interest of the seller to maximize the expected bid of the *winner*. Suppose the value of a worker to a company is dependent on his work and educational history. Then it is in the interest of a company to encourage potential employees to gain as much relevant experiece as possible. The company prefers to employ a winner with higher educational effort b. Should the company allocate a job via the first or the second price all-pay auction?

Denote the expected winner's action in a symmetric $n-$bidder first and second price all-pay auctions, respectively, by $\mathsf{E}b^f$ and $\mathsf{E}b^s$. Assume $\beta = 1$. We are interested in the case where the number of bidders is large.

Proposition 8 *As the number of bidders become large, the expected bid of the winner in the symmetric FPAA is*

$$\lim_{n\to\infty} \mathsf{E}b^f = 1 - \int_0^1 c(b)db,$$

and in the symmetric SPAA

$$\lim_{n\to\infty} \mathsf{E}b^s = \int_0^\infty e^{-c(b)}db.$$

Proof. Denote by $\Sigma^{(k)}$ the cdf of the k^{th} order statistic. By definition, $\Sigma^{(1)}(b) = \Sigma^n(b)$ for all $b \in [0, 1]$. Thus, in the FPAA

$$\Sigma^{(1)}(b) = c(b)^{\frac{n}{n-1}}.$$

Integrating by parts gives

$$\begin{aligned}
\mathsf{E}b^f &= \int_0^1 bd\Sigma^{(1)}(b) \\
&= \lim_{a\to\infty} \int_0^a (\Sigma^{(1)}(a) - \Sigma_{(1)}(b))db \\
&= \int_0^1 (1 - \Sigma^{(1)}(b))db \\
&= \int_0^1 (1 - c(b)^{\frac{n}{n-1}})db.
\end{aligned}$$

Taking the limit $n \to \infty$ gives the result.

The second order statistic satisfies $\Sigma^{(2)}(b) = n\Sigma^{n-1}(b) - (n-1)\Sigma^n(b)$.[13] In the SPAA,

$$\Sigma^{(2)}(b) = n(1-e^{-c(b)})^{\frac{n-1}{n-2}}(1-(1-e^{-c(b)})^{\frac{1}{n-1}})+(1-e^{-c(b)})^{\frac{n}{n-1}}, \text{ for all } b \in \mathsf{R}_+. \tag{13}$$

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[13] See e.g. Krishna (2001), Appendix C.

Note that $\Sigma^{(2)}(b) \to 1 - e^{-c(b)}$ as $n \to \infty$, for all b. On the other hand, as above,

$$\begin{aligned} \mathsf{E}b^s &= \int_0^\infty b d\Sigma^{(2)}(b) \\ &= \int_0^\infty (1 - \Sigma^{(2)}(b))db. \end{aligned}$$

Inserting (13) and taking the pointwise limit as $n \to \infty$ gives the result. ■

Thus we obtain that even if the number of competitors becomes large, the action taken by the winner does not increase without a bound, which is to be contrasted with the standard auctions. In the risk neutral case $c(b) = b$ for all b, we have $\lim_n \mathsf{E}b^f = 1/2$ and $\lim_n \mathsf{E}b^s = 1$. In the quadratic case $c(b) = b^2$ for all b, we have $\lim_n \mathsf{E}b^f = 2/3$ and $\lim_n \mathsf{E}b^s = \sqrt{\pi}/2$. The following corollary is immediate.

Corollary 5 *The SPAA generates higher expected winner's bid than the FPAA when cost functions are linear or quadratic.*

This means that with linear or qudratic cost functions, it should be in the interest of a firm to select their workers through a war of attrition rather than through a blind contest.

It should be noted, however, that with standard auctions the winner's bid is always 1. This is at least as high as the expected winner's bid under of the all-pay auctions. Moreover, this bid is riskless whereas winner's bid in all-pay auctions contains risk.

6 Closing remarks

This paper has investigated equilibria in complete information all-pay auctions when cost functions of the bidder may be non-linear. Complete closed chararacterization of the equilibria of the first price all-pay auction and partial but closed form chararacterization of the equilibria of the second price all-pay auction is established. Also closed form expression of the seller's revenues from the two auctions are derived. These results are then used to conduct revenue comparisons, which is the main contribution of the paper.

It is shown that under linear cost functions the second price all-pay auction generates at least as high revenue as the standard auction forms, which in turn generate at least as high revenue as the first price all-pay auction. When marginal costs (or, equivalently, valuations for the object) differ, this ranking of auctions is strict. With quadratic cost functions, however, the ordering of the first and the second price all-pay auctions is reversed, and both of them dominate the standard auctions. This suggests that if the bidders are risk neutral, then the seller should prefer the second price all-pay auction. However, if the bidders are risk averse, then the first price all-pay auction might be recommendable.

A Appendix

Proof of Proposition 1. *Necessity*: Let $(\Sigma_i)_{i=1}^n$ constitute a NE, and let $(u_i^*)_{i=1}^n$ be the corresponding payoff. Denote $c_i^*(b) = c_i(b) + u_i^*$, for all b and i. First, bidding more than β is dominated action for all $i = 2, ..., n$. Since 1 can guarantee payoff $1 - \psi_1 y(\beta) = (\psi_2 - \psi_1)\psi_2^{-1}$ by bidding $\beta + \varepsilon$ for any $\varepsilon > 0$, we have $u_1^* \geq (\psi_2 - \psi_1)\psi_2^{-1}$. Denote the support[14] of Σ_i by $S_i \subseteq [0, \beta]$.

Claim 0: There are no gaps in $\cup_{j \in N} S_j$.

Proof: If there was $b \in (0, \max S_j)$ for some j, but $b \notin \cup_{j \in N} S_j$, then there is i that would strictly benefit from choosing b instead of $b' = \inf\{b'' \in S_i : b'' > b, i \in N\}$, as the lower bid would not affect her winning probability but would decrease her payments.

Claim 1: Let $K = \{j : b \in S_j\}$. Then K contains at least two elements.

Proof: By Claim 0, K is nonempty. If $K = \{i\}$, then, since a lower bid does not affect her winning probability but does decrease her payments, i would strictly benefit from downgrading her bid by some $\varepsilon > 0$ (note that S_i is a closed set).

Claim 2: Suppose there is nonempty $K' \subset N$ such that all Σ_k, $k \in K'$, contain an atom $\alpha_k(b) > 0$ at b. Then there is $i \notin K'$such that $\Sigma_i(b) = 0$.

Proof: Under the supposition, there is i such that bidding $b + \varepsilon$, for any $\varepsilon > 0$, increases his winning probability at least the amount

$$\prod_{j \in N \setminus K'} \Sigma_j(b) \sum_{M \subseteq K'} \frac{1}{\# M} \prod_{j \in M} \alpha_j(b), \tag{14}$$

whereas the increase in the cost is $c_i(b + \varepsilon) - c_i(b)$. By the continuity of c_i, the latter number goes to zero. Thus so does (14). This implies there is $i \notin K'$such that $\Sigma_i(b) = 0$.

Claim 3: $\inf \cup_{j \in N} S_j = 0$.

Proof: If $\inf \cup_{j \in N} S_j > 0$, then, by Claim 2, bidder i such that $\inf S_i = \inf \cup_{j \in N} S_j$ would strictly benefit from choosing $b = 0$ rather than $b \in S_i$, as this change would not affect his winning probability.

Claim 4: $\inf S_i = 0$ for all $i = 1, ..., n$.

Proof: Suppose there is i such that $\inf S_i > 0$. Then, since there are no gaps in $\cup_{j \in N} S_j$ and $\inf \cup_{j \in N} S_j = 0$, there is bidder j and bid b such that $b \in S_j$ and $b < \inf S_i$. But this implies that i would strictly benefit from bidding 0, as this change would not affect his winning probability.

Claim 5: $u_j^* = 0$ for all $j \in \{2, ..., n\}$.

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[14] The smallest closed set S such that $\Sigma_i(b) - \Sigma_i(b + \varepsilon) > 0$, for all $\varepsilon > 0$, for all $b \in S$.

Proof: By Claims 2 and 4, there is i such that $\Sigma_i(0) = 0$. By Claim 4 we have $u_j^* = 0$, for all $j \neq i$. Since $u_1^* \geq (c_2 - c_1)c_2^{-1} > 0$, it must be that $i = 1$.

Claim 6: If $b \in \cap_{j \in K} S_j \cap (0, \beta]$, then $K \subseteq \{1, ..., m\}$.

Proof: Suppose not. Then by Claim 5, for all $b \in \cap_{j \in K} S_j$,

$$\prod_{j \in K \setminus \{i\}} \Sigma_j(b) - c_i(b) = 0, \quad \text{for all } i \in \{2, ..., m\}, \text{ and}$$
$$\prod_{j \in K \setminus \{k\}} \Sigma_j(b) - c_k(b) = 0, \quad \text{for some } k \in \{m+1, ..., n\}.$$

Take $\bar{b} = \sup S_k$. Then, since $\Sigma_k(\bar{b}) = 1 \geq \Sigma_i(\bar{b})$ and $c_k > c_i$ for all $i = 2, ...m$, we have

$$\begin{aligned} \prod_{j \in K \setminus \{i\}} \Sigma_j(\bar{b}) - c_i(b) &> \prod_{j \in K \setminus \{i\}} \Sigma_j(\bar{b}) - c_k(\bar{b}) \\ &\geq \prod_{j \in K \setminus \{k\}} \Sigma_j(\bar{b}) - c_k(\bar{b}) \\ &= 0. \end{aligned}$$

This violates Claim 5.

Claim 7: Define correspondence $K : [0, 1] \to N$ such that

$$K(b) = \left\{ i \in N : \prod_{j \in N \setminus \{i\}} \Sigma_j(b) - c_i^*(b) = 0 \right\}, \text{ for all } b.$$

Then $K(\cdot)$ is upper hemi-continuous on $(0, \beta]$.

Proof: Take a converging sequence $b^\nu \to b$ and k such that $k \in K(b^\nu)$ for all ν.[15] We claim $k \in K(b)$. Now

$$\prod_{j \in N \setminus \{k\}} \Sigma_j(b^\nu) - c_k(b^\nu) = u_i^*$$

Since Σ_j contains no atoms on $(0, \beta]$, it is continuous in this range. Moreover, since c_k is continuous, the left hand side converges to u_i^*. Thus the equality holds for b, too, and hence $k \in K(b)$.

Claim 8: If $i \in K(b) \cap \{2, ..., m\}$, $b \in (0, \beta]$, then $i \in K(b')$, $b' \in (b, \beta]$.

Proof: Suppose there is an interval (b', b'') such that $i \in K(b') \cap K(b'') \cap \{2, ..., m\}$ but $i \notin K(b)$ for $b \in (b', b'')$. Then $\Sigma_i(b) = \Sigma_i(b') = \Sigma_i(b'')$ for all $b \in (b', b'')$. Note that, for any b,

$$\prod_{j \in N} \Sigma_j(b) - c_i^*(b)\Sigma_i(b) = 0, \quad \text{for all } i \in K(b). \tag{15}$$

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[15] Or equivalently a converging $k^\nu \to k$ such that $k^\nu \in K(b^\nu)$ for all ν.

Consequently

$$\Sigma_j(b) = \left(\frac{c_i^*(b)}{c_j^*(b)} \right) \Sigma_i(b), \quad \text{for all } i, j \in K(b). \tag{16}$$

In particular,

$$\Sigma_j(b) = \Sigma_i(b), \quad \text{for all } i, j \in K(b) \cap \{2, ..., m\}. \tag{17}$$

Take sequence b^ν converging to b' from upwards such that $k \in K(b^\nu) \cap \{2, ..., m\}$ and $b^\nu < b''$ for all ν. Then, since K is uhc by Claim 8, $k \in K(b')$. By (17), $\Sigma_k(b^\nu) \geq \Sigma_i(b^\nu) = \Sigma_i(b')$ for all ν. Since $i \notin K(b^\nu)$,

$$\prod_{j \in N \setminus \{i\}} \Sigma_j(b^\nu) - c_i(b^\nu) < 0 = \prod_{j \in N \setminus \{k\}} \Sigma_j(b^\nu) - c_k(b^\nu),$$

or $\Sigma_k(b^\nu) < \Sigma_i(b^\nu)$, a contradiction.

Now, since S_i contains no gaps on $(0, \beta]$, it can only have a gap of form $(0, \lambda_i]$. Thus $K(b) \subseteq K(b')$ for all $b' \geq b$. Since K contains at least two elements in $(0, \beta]$, there is $i \in \{2, ..., m\}$ such that $i \in \lim_{b \to 0} K(b)$. By (15) and (16),

$$\begin{aligned} \prod_{j \in N \setminus \{i\}} \Sigma_j(b) &= \Sigma_i(b)^{|K(b)|-1} \prod_{j \in K(b) \setminus \{i\}} \frac{c_i^*(b)}{c_j^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} \Sigma_j(b) \\ &= c_i^*(b), \quad \text{for all } i \in K. \end{aligned}$$

Dividing and rearranging

$$\Sigma_i(a) = \left(c_i^*(b) \prod_{j \in K(b) \setminus \{i\}} \frac{c_j^*(b)}{c_i^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} \frac{1}{\Sigma_j(b)} \right)^{\frac{1}{|K(b)|-1}}, \quad \text{for all } i \in K(b). \tag{18}$$

In particular, for $i \neq 1$, we have

$$c_i^*(b) \prod_{j \in K(b) \setminus \{i\}} \frac{c_j^*(b)}{c_i^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} \frac{1}{\Sigma_j(b)} = \frac{c_1^*(b)}{\prod_{j \in \{1, \ldots, m\} \setminus K(b)} \Sigma_j(b)},. \tag{19}$$

Claim 9: If $1 \in K(b') \cap K(b'')$, then $1 \in K(b)$ for all $b \in (b', b'')$, for all $b', b'' \in [0, \beta]$.

Proof: Suppose there is a $b' < b''$ such that $1 \in K(b') \cap K(b'')$ but $i \notin K(b)$ for $b \in (b', b'')$. Take sequence $b^\nu \in (b', b'')$ converging to b'. Since $1 \notin K(b^\nu)$, his payoff is, by (18),

$$\prod_{j \in N \setminus \{1\}} \Sigma_j(b^\nu) - c_1^*(b^\nu) = \frac{c_2(b^\nu)}{\Sigma_1(b^\nu)} - c_1^*(b^\nu) = \frac{c_2(b^\nu)}{\Sigma_1(b')} - c_1^*(b^\nu).$$

Recall that, by Claim 5, $c_j^*(b) = c_j(b)$ for all $j \in \{2, ..., m\}$ and that $c_1^*(b) = c_1 y(b) + u_1^*$. Since $1 \in K(b')$ and c_j's are continuous, this number converges to zero. Thus

$$\frac{c_2(b')}{c_1^*(b')} = \Sigma_1(b'). \tag{20}$$

Similarly, take sequence in (b', b'') converging to b''. Then, by continuity,

$$\frac{c_2(b'')}{c_1^*(b'')} = \Sigma_1(b''). \tag{21}$$

Since $\Sigma_1(b') = \Sigma_1(b'')$, we have

$$\frac{1}{c_1 + \frac{u_1^*}{y(b')}} = \frac{1}{c_1 + \frac{u_1^*}{y(b'')}}.$$

But this can hold only if $y(b') = y(b'')$. Since y is increasing, this implies $b' = b''$, a contradiction.

Claim 10: $\sup S_1 = \beta$ and $u_1^* = (c_2 - c_1)c_2^{-1}$.

Proof. Let $\sup S_1 = \bar{b}$. Since Σ_1 is a cdf, we have $\Sigma_1(\bar{b}) = 1$. Since $u_1^* \geq (c_2 - c_1)c_2^{-1}$, necessarily $\bar{b} \leq \beta$. Suppose $\bar{b} < \beta$. By (20) $c_2(\bar{b}) = c_1^*(\bar{b})$ or

$$c_2 y(\bar{b}) = c_1 y(\bar{b}) + u_1^*.$$

Therefore

$$y(\bar{b}) = \frac{u_1^*}{c_2 - c_1} \geq \frac{1}{c_2}.$$

Since y is an increasing function, this implies $\bar{b} \geq \beta$, a contradiction. Since $\bar{b} = \beta$, we have $u_1^* = (c_2 - c_1)c_2^{-1}$.

By Claims 5 and 10 we now have $c_i^* = \tilde{c}_i$ for all $i = 1, ..., n$. Rank bidders $\{2, ..., m\}$ according their $\inf S_i$'s. Rename the lowest ranked bidder 2, the second lowest ranked by 3, and so on. Choose $\lambda_1 = \inf S_1$, and $\lambda_j = \inf S_j$ for all $j = 1, 2, ..., m$. Then, by Claim 1, $\lambda_1 = \lambda_2 = 0 \leq \lambda_3 \leq ... \leq \lambda_m$. Thus, by (18) we have constructed strategies (Σ_i) of form (2).

The remaining task is to construct the atoms at $b = 0$. Let k be the number of active bidders, i.e. $\lambda_k < \beta$. Then $\lambda_k = \max\{\lambda_j : \lambda_j < 1, j = 1, ..., m\}$. Then $\Sigma_j(0) = \alpha_j(0) = 1$ for all $j = k+1, ..., m$. Since $c_j^*(b) = c_2(b)$ for all $j = 2, ..., k$ and $\prod_{j \in k+1, ..., m} \Sigma_j(b) = 1$, we have, by (18),

$$\alpha_k(0) = \left(\tilde{c}_i(b) \prod_{j \in 1}^{k-1} \frac{\tilde{c}_j(b)}{c_i(b)} \right)^{\frac{1}{k-1}} = \tilde{c}_1(\lambda_k)^{k-1}.$$

Then

$$\alpha_{k-1}(0) = \left(\frac{\tilde{c}_1(\lambda_k)}{\alpha_k(0)} \right)^{\frac{1}{k-2}},$$

and, inductively,

$$\alpha_{k'}(0) = \left(\frac{\tilde{c}_1(\lambda_{k'})}{\prod_{j=k'+1}^{k} \alpha_j(0)} \right)^{\frac{1}{k'-1}}.$$

Sufficiency: Suppose that $(\Sigma_i)_{i=1}^n$ satisfies (18) for some K. It suffices to show there is no profitable deviation by $k \in N \setminus K$. Suppose there is a profitable bid $b > 0$ for k. Bidding over β is clearly dominated. Then

$$\prod_{j \in K \setminus \{i\}} \Sigma_j(b) - \tilde{c}_2(b) = 0.$$

Since k's deviation is profitable

$$\prod_{j \in K} \Sigma_j(b) - \tilde{c}_k(b) > 0.$$

By assumption $\tilde{c}_i(b) \leq \tilde{c}_k(b)$. But this implies

$$\prod_{j \in K} \Sigma_j(b) > \tilde{c}_k(b) \geq \prod_{j \in K \setminus \{i\}} \Sigma_j(b),$$

or $\Sigma_i(b) > 1$, a contradiction. ■

Proof of Proposition 4. We prove "only if" statement with $k = 0$, other cases are analogous. Let $\psi_1 = ... = \psi_l < \psi_{l+1} \leq \psi_m$, for some $l \in \{1, ..., m-1\}$, and suppose $(\Sigma_i)_{i=1}^n$ is a NE where $1, ..., m$ bidders completely mixes. We want to show $m = l + 1$.

For any $b \in \mathsf{R}_+$, define the probability of i winning

$$G_i(b) = \prod_{\substack{j=1 \\ j \neq i}}^{m} \Sigma_j(b).$$

Since the strategy is completely mixed,

$$\int_0^{a_i} (1-c_i(b)) dG_i(b) - c_i(a_i)(1-G_i(a_i)) = 0, \quad \text{for all } a_i \in \mathsf{R}_+, \text{ for all } i \in \{1, ..., m\}. \tag{22}$$

Taking the derivative,

$$G_i'(a_i) - c_i'(a_i)(1-G_i(a_i)) = 0, \quad \text{for all } a_i \in \mathsf{R}_+, \text{ for all } i \in \{1, ..., m\}. \tag{23}$$

Thus,

$$c_i(a_i) = \int_0^{a_i} \frac{G_i'(b)}{1 - G_i(b)} db = -\ln(1 - G_i(a_i)),$$

or

$$G_i(a_i) = 1 - e^{-c_i(a_i)}. \tag{24}$$

Thus,

$$\Sigma_j(a_i) = \left(\frac{1 - e^{-c_i(a_i)}}{1 - e^{-c_j(a_i)}} \right) \Sigma_i(a_i).$$

Inserting this into (24) gives

$$\Sigma_i(a_i) = \left(\frac{\prod_{j \neq i}(1 - e^{-c_j(a_i)})}{(1 - e^{-c_i(a_i)})^{n-2}} \right)^{\frac{1}{n-1}} = \left((1 - e^{-c_i(a_i)}) \prod_{\substack{i=1 \\ j \neq i}}^{m} \frac{1 - e^{-c_j(a_i)}}{1 - e^{-c_i(a_i)}} \right)^{\frac{1}{n-1}}, \quad \text{for all } i \in \{1, ..., m\}$$

establishing (7).

Since Σ_k is a cdf and right continuous, the derivative of Σ_k exists and is nonnegative everywhere. Thus, for all $a \in \mathsf{R}_+$ and for all $k = 1, ..., m$,

$$\frac{d}{da} \left(\frac{\prod_{j=1}^{m}(1 - e^{-\psi_j y(a)})}{(1 - e^{-\psi_k y(a)})^{m-1}} \right) \tag{25}$$

$$= \frac{\sum_{i=1}^{m} \psi_i y'(a) e^{-\psi_i y(a)} \prod_{j \neq i}(1 - e^{-\psi_j y(a)})}{(1 - e^{-\psi_k y(a)})^{m-1}} - (m-1)\psi_k y'(a) e^{-\psi_k y(a)} \frac{\prod_{j=1}^{m}(1 - e^{-\psi_j y(a)})}{(1 - e^{-\psi_k y(a)})^{m-2}}$$

$$= \frac{y'(a)}{(1 - e^{-\psi_k y(a)})^{m-2}} \left(\frac{\sum_{i=1}^{m} \psi_i e^{-\psi_i} \prod_{j \neq i}(1 - e^{-\psi_j y(a)})}{1 - e^{-\psi_k y(a)}} - (m-1)\psi_k e^{-\psi_k} \prod_{j=1}^{m}(1 - e^{-\psi_j y(a)}) \right)$$

$$= \frac{y'(a) \prod_{j=1}^{m}(1 - e^{-\psi_j y(a)})}{(1 - e^{-\psi_k y(a)})^{m-2}} \left(\sum_{i=1}^{m} \frac{\psi_i e^{-\psi_i y(a)}}{(1 - e^{-\psi_k y(a)})(1 - e^{-\psi_i y(a)})} - (m-1)\psi_k e^{-\psi_k y(a)} \right)$$

$$= \frac{y'(a) \prod_{j=1}^{m}(1 - e^{-\psi_j y(a)}) e^{-\psi_k y(a)}}{(1 - e^{-\psi_k y(a)})^{m-2}} \left(\sum_{i=1}^{m} \frac{\psi_i e^{(\psi_k - \psi_i) y(a)}}{(1 - e^{-\psi_k y(a)})(1 - e^{-\psi_i y(a)})} - (m-1)\psi_k \right) \geq 0$$

Letting $k = 1$ and $a \to \infty$ we have

$$\lim_{a \to \infty} \sum_{i=1}^{m} \frac{\psi_i e^{(\psi_1 - \psi_i) y(a)}}{(1 - e^{-\psi_1 y(a)})(1 - e^{-\psi_i y(a)})} = l\psi_1.$$

This number is at least $(m-1)c_1$ (if and) only if $l + 1 \geq m$. Since $l \in \{1, ..., m-1\}$, the weak inequality holds as equality.

To establish the "if" part, it suffices to check that (i) no bidder $m+1, ..., n$ benefits from deviation, and (ii) Σ_i is a cdf. For the first part, since $\psi_i \geq \psi_m$ for all $i = m, ..., n$, it is clear that

$$\int_0^{a_i} (1 - c_i(b)) dG_i(b) - c_i(a_i)(1 - G_i(a_i)) \leq 0, \quad \text{for all } a_i \in \mathsf{R}_+, \text{ for all } i \in \{m, ..., n\}.$$

Thus bidding zero is optimal for all $i = m, ..., n$. For the second part, by (25), Σ_i is increasing. Thus it suffices that $\lim_{b \to \infty} \Sigma_i(b) = 1$, which indeed is the case. ■

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