On the Tree-Cutting Problem under Interest Rate and Timber Value Uncertainty

Luis H. R. Alvarez[1] and Erkki Koskela[2]

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Department of Economics, University of Helsinki

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[1] Department of Economics, Quantitative Methods in Management, Turku School of Economics and Business Administration, FIN-20500 Turku, Finland, e-mail: luis.alvarez@tukkk.fi

[2] Department of Economics, University of Helsinki, FIN-00014 University of Helsinki, Finland, the Research Department of the Bank of Finland, P.O. Box 160, 00101 Helsinki, Finland, CESifo, München, and IZA, Bonn, e-mail: erkki.koskela@helsinki.fi

Abstract

The current literature on optimal forest rotation makes the unrealistic assumption of constant interest rate even though harvesting decisions of forest stands are typically subject to long time horizons under which interest rates fluctuate and are subject to shocks. We apply the Wicksellian single rotation framework to cover the unexplored case of variable and stochastic interest rate. By modelling the stochastic interest rate as a parametrized mean-reverting model a lá Cox-Ingersoll-Ross and the timber value as a geometric Brownian motion we provide an explicit solution for the two-dimensional path-dependent rotation problem. We show that increased interest rate volatility prolongs the optimal rotation period. Numerical illustration indicates that higher interest rate volatility lengthens the optimal rotation period at an increasing rate.

Keywords: forest rotation, stochastic interest rates, optimal stopping.

JEL Subject Classification: Q23, G31, C61

1 Introduction

In forest economics the Faustmannian ongoing rotations framework has been the most often used starting point in the analyzes of optimal rotation period of forest stands. Under the assumption of constant timber price, constant total cost of clear-cutting and replanting as well as constant interest rate and perfect capital markets the basic deterministic model leads to a constant optimal rotation period for an even age stand, which maximizes the present value of forest stand (see. e.g. Samuelson 1976). The rotation age depends on timber price, total cost of clear-cutting and replanting, nature of forest growth as well as the interest rate. The perfect foresight assumption has been relaxed in studies focusing the implications of stochastic timber price (see e.g. Insley 2002), risk of forest fire (see e.g. Reed 1984), and stochastic timber value growth on optimal rotation age (see e.g. Clarke and Reed 1989, 1990 , Willassen 1998 and Alvarez 2003). When forest fire risk is modelled as a Poisson process the optimal rotation age becomes shorter. This is because forest fire risk increases the appropriate risk adjusted discount rate for forestry (cf. Reed 1984). However, under timber price and forest growth risk usually the reverse happens; higher volatility will lengthen the optimal rotation period. The reason for this finding is that even though increased volatility increases the expected net present value of the harvesting yield, it also raises the value of waiting by increasing the expected net present value of future harvesting opportunities. Since the latter effect dominates the former, the net impact of increased volatility on the length of the optimal rotation period is unambiguously positive (cf. Clarke and Reed 1989, 1990 , Insley 2002, Willassen 1998 and Alvarez 2003).

To our knowledge all the research has, however, used the specification of a constant interest rate. This is both an unrealistic and problematic assumption because forest rotation periods are usually quite long and interest rates fluctuate over time. Hence, harvesting rules which are based on a constant discount factor may be highly biased. In this paper we use a Wicksellian single rotation framework to analyze the important unexplored issue of how an inter-temporally variable and stochastic interest rate will affect the optimal forest rotation period, when timber value is also assumed to be stochastic. It is known on the basis of extensive empirical research in financial economics (see e.g. Cochrane 2001, Ch. 20) that in the long run interest rates follow mean-reverting processes. This means that although there will be a long-run steady state value for interest rates, shocks will cause their volatility. In accordance with this finding we model the interest rate as a parametrized mean-reverting process by using the well-known Cox-Ingersoll-Ross 1985 model. On the other hand, we specify the timber value as a geometric Brownian motion. It is important to notice that these assumptions imply that the considered valuation constitutes a two-dimensional and path-dependent optimal stopping problem.

In this paper we provide three important new results. First, we present the explicit solution for the tree-cutting problem under interest rate and timber value uncertainty by expressing the original valuation as an associated ordinary path-independent optimal stopping problem. Second, we demonstrate that higher interest rate volatility unambiguously increases the optimal exercise threshold of the harvesting opportunity and, therefore, lengthens the optimal rotation period. Finally, numerical illustration indicates that higher interest rate volatility raises the optimal threshold and prolongs the expected rotation period at an increasing rate. Thus, our results clearly indicate that even relatively small increases in the volatility of the underlying interest rate dynamics have a significant impact on the length of the optimal rotation period.

The contents of this paper proceed as follows. In Section 2 we present and solve the considered optimal rotation problem and illustrate our results numerically. Finally, section 3 concludes our study.

2 Optimal Forest Rotation: A Solvable Model

In this section we establish the following results. First, we characterize the optimal rotation problem under stochastic interest rate and timber value and show that under a set of plausible assumptions the two-dimensional path-dependent rotation problem can be re-expressed as an ordinary path-independent optimal stopping problem. Second, we demonstrate that the transformed rotation problem is explicitly solvable and provide an analytic characterization. Third, a numerical illustration about the relationship between the optimal rotation threshold and interest rate volatility is also presented.

Consider the following (path-dependent) Wicksellian optimal rotation problem

$$V(x,r) = \sup_{\tau} \mathbf{E}_{(x,r)} \left[e^{-\int_0^\tau r_s ds} X_\tau \right], \tag{2.1}$$

where the underlying timber value and interest rate processes (X_t, r_t) evolve according to the dynamics described by the following stochastic differential equations

$$dr_t = (a - br_t)dt + c\sqrt{r_t}dW_t, \quad r_0 = r \tag{2.2}$$

and

$$dX_t = \mu X_t dt + \sigma X_t d\hat{W}_t, \quad X_0 = x, \tag{2.3}$$

where $a, b, c, \sigma, \mu \in \mathbb{R}_+$ are known exogenously given constants and W_t and $\hat{W}_t$ are two stochastically independent Wiener processes (under the objective probability measure $\mathbb{P}$). The interest rate r_t follows a mean-reverting process while the timber value X_t follows a geometric Brownian motion. It is worth emphasizing – as we mentioned earlier – that the interest rate model (2.2) is known in financial economics as the Cox-Ingersoll-Ross model of the interest rate which can be supported theoretically (cf. Cox, Ingersoll, and Ross 1985) and which lies in conformity with empirics (cf. Björk 1998, chapter 17, and Cochrane 2001, chapters 19, 20). It is also worth pointing out that if $a \geq c^2/2$, then the interest rate process r_t converges towards a long run stationary (*Gamma*-) distribution with density (cf. Borodin and Salminen 2002, pp. 35–37)

$$p(r) = (b\eta)^{a\eta} \frac{r^{a\eta - 1} e^{-b\eta r}}{\Gamma(a\eta)},$$

where $\eta = 2/c^2 > 0$. Especially, we find that if $a \geq c^2/2$ then the expected long-run interest rate can be expressed as $\lim_{t\to\infty} \mathbf{E}[r_t] = a/b > 0$ which coincides with the long run stationary steady state interest rate in the absence of uncertainty.

Before proceeding in the analysis of the stochastic valuation, we first establish the following.

Theorem 2.1. *In the absence of volatility of interest rate and timber value, i.e. when $c = \sigma = 0$ and assuming that $\mu < a/b$, which guarantees the finiteness of the value of the optimal policy, the optimal rotation date is*

$$t^* = \ln\left(\frac{a - b\min(\mu, r)}{a - b\mu}\right)^{1/b}$$

and the value of the optimal rotation strategy is

$$\hat{V}(x, r) = \sup_{t \geq 0}\left[e^{-\int_0^t r_s ds} X_t\right] = \begin{cases} x & r \geq \mu \\ xe^{-(r-\mu)/b}\left(\frac{a-b\mu}{a-br}\right)^{(a/b-\mu)/b} & r < \mu. \end{cases} \tag{2.4}$$

Proof. See Appendix A. □

Having characterized the underlying stochastic dynamics in (2.2) and (2.3) and the optimal single rotation problem (2.1) in the absence of volatility we can now state the following important result

Lemma 2.2. *Under the stochastic interest rate and timber value dynamics (2.2) and (2.3) the path-dependent optimal rotation problem (2.1) can be re-expressed as an ordinary path-independent optimal stopping problem*

$$V(x, r) = xe^{Ar} \sup_{\tau} \mathbf{E}_r\left[e^{(\mu+aA)\tau - A\hat{r}_\tau}\right], \tag{2.5}$$

where

$$A = \frac{b}{c^2} - \sqrt{\frac{b^2}{c^4} + \frac{2}{c^2}} < 0$$

denotes the negative root of the quadratic equation $c^2z^2 - 2bz - 2 = 0$ and the interest rate $\hat{r}_t$ evolves (under $\mathbb{P}$) according to the dynamics described by the stochastic differential equation

$$d\hat{r}_t = \left(a - (b - Ac^2)\hat{r}_t\right)dt + c\sqrt{\hat{r}_t}dW_t, \quad \hat{r}_0 = r. \tag{2.6}$$

Proof. See Appendix B. □

Lemma 2.2 is crucial in the sense that using this we can demonstrate that under the assumptions concerning the stochastic processes modelling the interest rate and the timber value we get an ordinary and solvable one-dimensional optimal stopping problem. It is worth observing that our finding is essentially based on a technique known as a change of numeraire (cf. Björk, 1998 chapter 19). More precisely, instead of tackling the original valuation directly, we simplify the analysis by expressing the value of the project in terms of the price of a zero coupon bond maturing at exercise. Our main new result is now summarized in the following

Theorem 2.3. *Assume that the absence of speculative bubbles condition $\mu + aA < 0$, guaranteeing the finiteness of the value of the optimal policy, is satisfied. Then the value of the single rotation problem (2.1) reads as*

$$V(x, r) = xe^{Ar}\psi(r) \sup_{y \geq r}\left[\frac{e^{-Ay}}{\psi(y)}\right] = \begin{cases} x, & r \geq r^* \\ xe^{A(r-r^*)}\frac{\psi(r)}{\psi(r^*)}, & r < r^* \end{cases}$$

where the increasing fundamental solution

$$\psi(r) = \int_0^1 e^{2(b-Ac^2)rt/c^2} t^{\rho-1}(1-t)^{2a/c^2-\rho-1} dt$$

is known as Kummer's confluent hypergeometric function (see e.g. Abramowitz and Stegun 1968, pp. 503–535) and $\rho = (\mu + aA)/(Ac^2 - b) > 0$*. The optimal interest rate exercise threshold* r^* *is the unique root of the ordinary first order condition* $\psi'(r^*) = -A\psi(r^*)$*. It has the following properties:* $r^* > \mu$ *for* $c > 0$ *and* $r^* = \mu$ *when* $c = 0$*.*

Proof. See Appendix C. □

Theorem 2.3 demonstrates that the path-dependent optimal rotation problem (2.5) is explicitly solvable whenever the absence of speculative bubbles condition $\mu + aA < 0$ – guaranteeing the finiteness of the value of the optimal policy – is satisfied. It is worth observing that since

$$\frac{\partial A}{\partial c} = \frac{cA^2}{b - c^2 A} > 0 \tag{2.7}$$

and $A \downarrow -1/b$ as $c \downarrow 0$ we find that the absence of speculative bubbles condition can be satisfied only if the inequality $\mu < a/b$ holds. This means that the expected percentage growth rate of the timber value has to be smaller than the long run steady state interest rate. If this is the case, then there is a critical volatility c^*, satisfying the condition

$$\mu c^{*2} = ab - \sqrt{a^2b^2 + 2a^2c^{*2}},$$

above which the the absence of speculative bubbles condition $\mu + aA < 0$ is violated so that the value of the optimal policy becomes then unbounded. As is clear from (2.7), the condition $\mu + aA < 0$ is strengthened by the higher volatility described by the parameter c. Hence, an increase in the interest rate volatility coefficient c increases the required exercise premium and, thus, prolongs the optimal rotation period. An economic interpretation of this finding goes as follows. Higher interest rate volatility increases the certainty-equivalent interest rate and thereby lengthens the rotation period. In financial terms, increased interest rate volatility increases the value of the harvesting opportunity $V(x, r)$ (by increasing the value of zero-coupon bonds maturing at the exercise date τ) while leaving the exercise payoff x unaffected. However, since the option to harvest is lost at exercise (by the usual balance identity $V(x, r^*) = x$ stating that at the optimum the project value should be equal to its full cost which in this case is the lost option value), we observe that increased interest rate volatility raises the required exercise premium and, therefore, prolongs the expected length of the optimal rotation period.

In Figure 1 we illustrate the optimal rotation threshold under the assumptions that $b = 0.1, a = 0.045b$, and $\mu = 0.03$ (implying that the critical volatility above which the absence of speculative bubbles condition $\mu + aA < 0$ is violated is $c^* = 12.25\%$). As one can immediately observe from Figure 1, increased volatility not only increases the optimal threshold, but does it at an increasing rate. Thus, close to the critical levels where the absence of speculative bubbles condition is compromised, a small increase in the volatility coefficient results into a relatively large increase in the required exercise premium so that the optimal rotation period will increase proportionally more than the

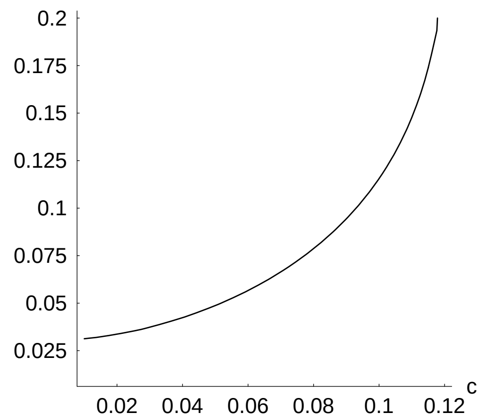

Figure 1: The optimal rotation threshold as a function of interest rate volatility c

volatility. This argument is explicitly illustrated in Table 1 where we characterize both the optimal rotation threshold r^* and the expected rotation length $E_\mu[\tau_{r^*}]$ starting from the state μ corresponding to the optimal rotation threshold in the absence of uncertainty.

c	0.01	0.03	0.05	0.07	0.09
r^*	0.031	0.037	0.049	0.066	0.094
$E_\mu[\tau_{r^*}]$	0.6	4.2	10.0	18.8	34.0

Table 1: The optimal rotation threshold r^* and expected rotation length $E_\mu[\tau_{r^*}]$

3 Conclusion

The research considering the determination of the optimal forest rotation strategy predominantly assumes a constant interest rate. This assumption, however, is problematic since forest rotation periods are long and interest rates fluctuate stochastically over time. In this paper we have used the Wicksellian single rotation framework to study the unexplored issue of forest rotation under variable and stochastic interest rate when timber value is also stochastic. In order to accomplish this task, we have modelled the stochastic interest rate as a parametrized mean-reverting process by applying the Cox-Ingersoll-Ross model of interest rate - which is well-known in financial economics and lies in conformity with empirics - and the forest value as a geometric Brownian motion. We have for the first time provided an explicit solution for two-dimensional path-dependent optimal stopping problem and shown analytically that higher interest rate volatility increases the optimal threshold and therefore prolongs the expected optimal rotation period. Moreover, and importantly, numerical illustration indicates that the optimal threshold is a strictly convex function of the volatility coefficient of the underlying interest rate process meaning that the optimal exercise threshold and the expected optimal rotation length becomes higher at an increasing rate as the interest rate volatility increases.

Whether our conclusions remain valid in the Faustmann's ongoing rotation framework is an open question beyond the scope of this paper. Given the close connection of impulse control problems and optimal stopping theory (see Alvarez 2003) we are tempted to conjecture that our conclusions will likely remain qualitatively valid in the Faustmann framework as well. The verification of this conjecture is an open issue which is left for future research.

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A Proof of Theorem 2.1

Proof. Integrating the ordinary differential equation $dr_t = (a - br_t)dt$ from 0 to t yields $r_t - r = at - b\int_0^t r_s ds$ implying that $e^{-\int_0^t r_s ds} = e^{(r_t - r)/b - at/b}$. On the other hand, since $X_t = xe^{\mu t}$ we find that

$$\hat{V}(x,r) = \sup_{t \geq 0}\left[e^{-\int_0^t r_s ds} X_t\right] = xe^{-r/b} \sup_{t \geq 0}\left[e^{(\mu - a/b)t + r_t/b}\right]. \tag{A.1}$$

Given this observation, consider now the mapping $g(t) = e^{(\mu - a/b)t + r_t/b}$. Standard differentiation then yields $g'(t) = (\mu - r_t)g(t)$ implying that $g'(t) \gtreqless 0$ as long as $r_t \lesseqgtr \mu$. Combining this observation with the result $r_t = a/b + e^{-bt}(r - a/b)$ and assumption $\mu < a/b$ then finally shows that the rotation date

$$t^* = \inf\{t \geq 0 : r_t \geq \mu\} = \ln\left(\frac{a - b\min(\mu, r)}{a - b\mu}\right)^{1/b}$$

is optimal. Inserting this date in (A.1) then yields (2.4). □

B Proof of Lemma 2.2

Proof. It is well-known that the solution of the stochastic differential equation (2.3) reads as

$$X_t = x\exp((\mu - \sigma^2/2)t + \sigma\hat{W}_t).$$

Moreover, we find by applying Itô's theorem to the mapping $r \mapsto e^{zr}$ that

$$e^{-\frac{1}{2}(z^2c^2 - 2zb)\int_0^t r_s ds} = e^{z(r - r_t) + zat} M_t,$$

where

$$M_t = \exp\left(\int_0^t zc\sqrt{r_s}dW_s - \frac{1}{2}\int_0^t z^2c^2 r_s ds\right)$$

is a positive exponential martingale. Thus, choosing $z = A$ implies that the discount factor can be re-expressed as

$$e^{-\int_0^t r_s ds} = e^{A(r - r_t) + Aat} M_t.$$

Given this observation, we find that the present value of the forest stand can be expressed as

$$e^{-\int_0^t r_s ds} X_t = xe^{A(r - r_t) + Aat + \mu t}\hat{M}_t M_t,$$

where $\hat{M}_t = e^{\sigma \hat{W}_t - \frac{1}{2}\sigma^2 t}$ is a positive exponential martingale. Consequently, we find that the path-dependent optimal rotation problem (2.1) can be re-expressed as an ordinary path-independent optimal stopping problem

$$V(x,r) = xe^{Ar} \sup_{\tau} \mathbf{E}_r \left[e^{(\mu+aA)\tau - Ar_\tau} \hat{M}_\tau M_\tau \right]. \tag{B.1}$$

Defining the equivalent measure $\mathbb{Q}$ through the likelihood-ratio $\frac{d\mathbb{Q}}{d\mathbb{P}} = \hat{M}_t M_t$ we can now re-express (B.1) as

$$V(x,r) = xe^{Ar} \sup_{\tau} \mathbf{E}_r^{\mathbb{Q}} \left[e^{(\mu+aA)\tau - Ar_\tau} \right], \tag{B.2}$$

where the interest rate process r_t evolves according to the dynamics described by the stochastic differential equation

$$dr_t = \left(a - (b - Ac^2)r_t\right) dt + c\sqrt{r_t} d\tilde{W}_t, \quad r_0 = r,$$

where $\tilde{W}_t$ is a standard Brownian motion under the equivalent measure $\mathbb{Q}$. However, given the strong uniqueness of a solution for the stochastic differential equation above (cf. Øksendal, 1998, p. 66) we finally find that the rotation problem (2.1) can be rewritten in the path-independent form (2.5) defined under the objective measure $\mathbb{P}$. □

C Proof of Theorem 2.3

Proof. Since

$$L(r) = \mathbf{E}_r \left[e^{(\mu+aA)\tau - A\hat{r}_\tau} \right]$$

is an ordinary path-independent optimal stopping problem of a linear diffusion and, therefore, can be solved by relying on ordinary variational inequalities, the alleged result is a direct implication of Theorem 3 in Alvarez 2001. It is, therefore, sufficient to determine the increasing fundamental solution of the ordinary second-order differential equation

$$\frac{1}{2}c^2 r u''(r) + (a - (b - c^2 A)r)u'(r) + (\mu + aA)u(r) = 0.$$

Making the transformation $u(r) = v(\theta r)$,where $\theta \in \mathbb{R}$ is an unknown constant, and defining the variable $y = \theta r$ yields that

$$yv''(y) + \left(\frac{2a}{c^2} - \frac{2(b - Ac^2)}{c^2\theta} y \right) v'(y) + \frac{2(\mu + aA)}{\theta c^2} v(y) = 0.$$

Choosing $\theta = 2(b - Ac^2)/c^2$, then finally implies that the differential equation can equivalently be expressed as

$$yv''(y) + \left(\frac{2a}{c^2} - y \right) v'(y) - \frac{2(\mu + aA)}{Ac^2 - b} v(y) = 0,$$

which is Kummer's differential equation. □