A Noncooperative Solution to the Bargaining Problem[*]

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Abstract

Binmore *at.al.* (1987) show that the Nash (1950) solution emerges as a limit point of a two player alternating offers bargaining game when the time difference between offers goes to zero. Krishna and Serrano (1996) establish the same result in the $n-$player *cake sharing* set up. Kultti and Vartiainen (2003) argue that noncooperative bargaining behavior á la Krishna-Serrano can be compactly described by means of von Neumann-Morgenstern stable set. This paper analyses the *general* problem. We show that a stable set exists and converges to the Nash solution in *any* smooth, compact and convex problem. A connection to the generalized Krishna-Serrano game is also established.

1 Introduction

An $n-$player bargaining problem is defined by an $n-$dimensional compact, comprehensive, and convex utility possibility set U. Nash's (1950) solution is without doubt the most commonly accepted cooperative solution to the problem.[1] But this constitutes only part of the story; a more complete theory would explain why the Nash solution emerges also in the *noncooperative* framework.

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[1] For an authorative discussion on n-player bargaining theory, see Thomson and Lensberg (1989).

Binmore, *at. al.* (1986) show that the Nash bargaining solution has a noncooperative interpretation: the unique equilibrium outcome of the two-player Rubinstein (1982) alternating offers bargaining game converges to the Nash solution when the time difference between offers becomes small. However, Binmore *et al* are silent about the general $n-$player bargaining context, which is known to be qualitatively different from its two-player counterpart (see e.g. Osborne and Rubinstein, 1990).

Krishna and Serrano (1996) [KS] tackle this problem by focusing on model of $n-$player alternating offers bargaining in the *cake sharing* context. The decisive feature of KS is that any player may exit the game after he has accepted an offer regardless of the other players' acceptance decisions. Thus unanimity is not needed for an allocation-offer to become executed. KS shows that the game induces a unique subgame perfect equilibrium, and that this equilibrium converges to the Nash bargaining solution as the time intervall becomes small.

Kultti and Vartiainen (2003) [KV] point out that noncooperative bargaining behavior á la KS can be captured in reduced form by means of vonNeumann-Morgenstern (1944) stability concept.[2] They demonstrate that a *stable set* of the $n-$player cake sharing problem is geometrically intimitely related to the KS equilibrium allocations that are indexed by the first-moving player. It is proven that a stable set exists, is unique, and converges to the Nash bargaining solution as the time interval becomes small.

KS and KV crucially rely on the physical structure of the cake division problem that allows two players to compare utilities without affecting a third player's payoff. This property, which precludes all kinds of externalities at the outset, together with standard assumptions on players' utility functions imply *resource monotonicity* of a solution to the problem. Resource monotonicicity, in turn, drives the uniqueness result. For example in the presence of externalities, this need not be the case. The relation between stable set and the Nash bargaining solution in the general case is still an open question.

Our aim is to tackle the general problem. First we show that a stable set always exists and converges to the Nash bargaining solution. Then we demonstrate the link between stable set and the Krishna-Serrano (1996) bargaining framework. As no assumptions are made as regards to the underlying physical environment,[3] a most general noncooperative foundation for the Nash bargaining is solution is provided.

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[2] For review, see Owen (1989).

[3] Except that it induces a smooth bargaining problem.

Outline of the Argument Specifically, the *stable set* solution is derived by assuming the following. Any player may impose an objection to a division of utilities by demanding a new division. It takes one period before any such demand may materialize. By this simple structure, a dominance relation over the divisions of utilities is created: division u is dominated by division v if and and only if the discounted value of v exceeds the current value u, for some player.

With short enough period, it is clear that there are no undominated divisions. What do reasonable, or stable, outcomes then look like? We focus on a subset of all divisions, the stable set, which has the properties that, first, any element of the stable set can only be dominated by an element outside the set and, second, any element outside the stable set is dominated by some element of the set.

First we characterize a stable set. A stable set is characterized by a minimal point $\underline{u}$ such that points in U above $\underline{u}$ constitute a stable set. It also contains n "maximal points" $u^1, ..., u^n$ that induce the highest possible payoff in the stable set for each $1, ..., n$. Player i's maximal point satisfies $u^i = (\delta_i^{-1}\underline{u}_i, \underline{u}_{-i})$. Then we prove the existence of a stable set. Finally, we prove that any stable set converges to the (asymmetric) Nash bargaining solution as the time interval becomes small.

To understand the relationship between stable set and a noncooperative bargaining framework,[4] note that a key feature of any bargaining model is that before any tentative outcome is implemented, players should be able to resume in one form or another the negotiation process. Thus any (equilibrium) outcome of the bargaining process must be such that players would rather implement this outcome than induce another outcome that they know would be implemented. On the other hand, if a particular outcome cannot be implemented (in equilibrium), then there *must* be an (equilibrium) outcome that some player wants to, and is able to, induce. The remaining question is which players should be in the position to resume the negotiation process. In this respect we do not want to be too dogmatic. Assuming that *any* player can resume the negotiation process, it becomes clear that the conditions for inducable outcomes are isomorphic under stable set and noncooperative approcahes. Thus, also the set of inducable outcomes must coincide.

This intuitive connection between stable set and the noncooperative equilibria of a bargaining game is established in a bargaining model that is moti-

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[4] Greenberg (1990) offers a comprehensive analysis of the connections between noncooperative models and stable set.

vated by Krishna-Serrano (1996). The game is defined recursively as follows. Players make sequential offers (say, in ascending order) in terms of utility allocations. Any player is permitted to implement the utility that is proposed to him by accepting the offer. A rejection leads to a subgame where the next player of those who reject the previous offer makes a proposal. A proposer's utility allocation is implemented only if all reponders accept the offer.

We show that any Markov equilibrium outcome of the bargaining game coincides with a maximal point of a stable set. On the other hand, there always exists a stable set whose maximal point coincides with a Markov equilibrium outcome of the noncooperative game. Thus we become to prove that a Markov equilibrium of the Krishna-Serrano bargaining game exists and that any Markov equilibrium converges to the Nash bargaining solution even when no restrictions are placed on the structure of the environment.

First we give a characterization of any stable set. A stable set is convex valued, and it has a simple and intuitive geometric representation. We then show that a stable set always exists for any $n-$player problem. Uniqueness cannot be established without further assumptions. However, we show that any stable set shrinks to the asymmetric Nash bargaining solution as the time interval goes to zero.

2 The set up

There is set $N = \{1, ..., n\}$ of players and a compact, convex and comprehensive utility possibility set $U \subset \mathbb{R}^n_+$.[5][6] Vector of realized utilities is denoted by $u = (u_1, ..., u_n)$, or $u = (u_i, u_{-i})$. For any $u \in U$, let $D(v)$ be the points that Pareto dominate $v \in U$:

$$D(v) := \{u \in U : u \geq v\}. \tag{1}$$

For any $u \in U$, $D(u)$ is a compact and $u-$comprehensive set. Weakly Pareto-optimal outcomes P are then defined by $P := \{u \in U : D(u) = \{u\}\}$.

Bargaining takes place through objections against potential division of utilities. An objection contains a specification for new division. However, there is one period dealy before an objection may become effective. Delay is

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[5] Vector notation: $x \geqq y$ if $x \geq y$ for all i, $x \geq y$ iff $x \geqq y$ and not $x_i = y_i$ for all i, and $x > y$ iff $x_i > y_i$ for all i.

[6] $X \subset \mathbb{R}^k$ is $d-$comprehensive if $x \in X$ and $x \geq y \geq d$ imply $y \in X$. If $d = 0$, then X is comprehensive.

costly: The present value of player i's next period utility u_i is $u_i\delta_i^{\Delta}$, where $0 < \delta_i < 1$ is the discount rate and $\Delta \geq 0$ is the length of the period.[7]

Stabe set consists of a domain alternatives and dominace relation on this set. In the current set up, we let the domain be U. Dominance relation $\succ$ is defined as follows: $u \succ v$ iff $u_i\delta_i^{\Delta} > v_i$, for some $i \in N$, for $u, v \in U$. Set $G \subset U$ is *stable* if

- (External stability) $u \notin G$ implies there is $v \in G$ s.t. $v \succ u$,
- (Internal stability) $u \in G$ implies there is *no* $v \in G$ s.t. $v \succ u$.

3 Characterization and Existence

3.1 Case $\Delta = 0$.

If $\Delta = 0$ then $\delta^{\Delta} = 1$, and the family of stable sets has a simple structure.

Theorem 1 *Let* $\Delta = 0$. *Then* G *is a stable set if and only if* $G = \{u\}$ *for* $u \in P$.

Proof. "If". Take $u \in P$ and assume $G = \{u\}$. Since G is singleton, internal stability is met. External stability is met by the definition of Pareto-optimality. Thus G is a stable set.

"Only if". Assume G is a stable set. By internal stability, G is singleton, say $G = \{u\}$. Suppose there is v such that $v \geq u$. Then $v \notin G$ and, by external stability there is $v' \in G$ such that $v' \succ v$. But then also $u \succ v$, a contradiction. Thus $u \in P$. ■

3.2 Case $\Delta > 0$.

Without loss of generality, fix $\Delta = 1$. Take $u = (u_1, ..., u_n)$, and call $(\delta_i^{-1}u_i, u_{-i})$ the δ_i-extension of $u \in U$.[8] Denote by $\underline{u}$ a point whose all δ_i-extensions, for $i = 1, ..., n$, lie in the Pareto-frontier:

$$\underline{u} :\in \{u \in U : (\delta_i^{-1}\underline{u}_i, \underline{u}_{-i}) \in P, \text{ for all } i \in N\}. \tag{2}$$

Occasionally, such allocation is called a "minimal point".

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[7] Discount factor δ_i can be interpreted as probability of termination $p_i\Delta$, where p_i is a Poisson rate. Now it is nonproblematic to assume that players' preferences obey von Neumann-Morgenstern assumptions.

[8] The concepts are taken from Thomson - Lensberg (1989), Ch 8.

Theorem 2 *Set $G \subset S$ is stable if and only if $G = D(\underline{u})$.*

Proof. For any G, for any $i \in N$, let $u_i^i \in \mathbb{R}$ satisfy[9]

$$u_i^i = \sup\{u_i : u \in G\}.$$

”If”: Assume that $G = D(\underline{u})$. By construction, $u_i \geq \underline{u}_i = u_i^i \delta_i \geq v_i \delta_i$, for all $i \in N$, for all $u, v \in D(\underline{u})$. Thus internal stability is met. Take $u \notin D(\underline{u})$. Then there is i such that $\underline{u}_i > u_i$. This implies that also $\underline{u}_i = u_i^i \delta_i > u_i$. Since $u_i^i \in D(\underline{u})$, also external stability is met.

”Only if”: Suppose G is a stable set. Then, by internal stability,

$$\bigcup_{i \in N} \{u \in U : u_i^i \delta_i > u_i\} \subseteq U \setminus G. \tag{3}$$

On the other hand, by external stability,

$$\bigcap_{i \in N} \{u \in U : u_i^i \delta_i \leq u_i\} \subseteq G. \tag{4}$$

Since

$$\begin{aligned} & \bigcup_{i \in N} \{u : u_i^i \delta_i > u_i\} \cup \bigcap_{i \in N} \{u : u_i^i \delta_i \leq u_i\} \\ = \; & G \cup (U \setminus G) \\ = \; & U, \end{aligned}$$

we have that, in fact, (3) and (4) hold as equality. By (4), it follows that G is a compact set. Then there is $u^i \in P \cap G$ such that $(u_i^i, u_{-i}^i) = u^i$. Since U is a comprehensive set, vector $u\delta = (u_1^1 \delta_1, ..., u_n^n \delta_n)$ is an element of U. Thus, there is $\underline{u} = (\underline{u}_1, ..., \underline{u}_n) \in U$ such that $u_i^i \delta_i = \underline{u}_i$ for all $i \in N$. By construction, then

$$G = \{u : u \geq \underline{u}\}.$$

But then $G = D(\underline{u})$ for D meeting (1), and $\underline{u}$ meeting (2). ■

Thus now we know that any stable set has a particular structure. A stable set is characterized by a minimal point $\underline{u} = (\underline{u}_1, ..., \underline{u}_n)$: points in U above $\underline{u}$ constitute a stable set. Stable set contains n "maximal points" $u^1, ..., u^n$ that induce the highest possible payoff in the stable set for each $1, ..., n$. Player i's maximal point satisfies $u^i = (\delta_i^{-1} \underline{u}_i, \underline{u}_{-i})$. Stable set is a convex set.

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[9] Note that stable set cannot be empty.

The characterization leaves open the question of existence and uniqueness of the solution. Indeed, it is well known that in many scenarios there are many stable sets, and in others it fails to exist (see e.g. Owen 1995).

Denote the $i-$maximal Pareto-optimal point given payoffs u_{-i} of players $j \neq i$ by

$$u_i^i(u) = \max\{u_i' : (u_i', u_{-i}) \in U\}.$$

Note that if $u \in P$, then $u_i^i(u) = u_i$.

Theorem 3 *Stable set exists.*

Proof. Define function $\hat{g}_i : U \to \mathbb{R}_+$

$$g_i(u) := \delta_i u_i^i(u), \text{ for all } (u_i, u_{-i}) \in U, \text{ for all } i \in N. \tag{5}$$

By convexity of U, g_i is a continuous function. Let $g(\cdot) := (g_1(\cdot), ..., g_n(\cdot))$, and define function $\bar{x} : U \to \mathbb{R}_+$ such that

$$\bar{x}(u) := \max\{x \in \mathbb{R} : xg(u) \in U\}, \text{ for all } u \in U.$$

By compactness of U, $\bar{x}$ is well defined. Construct function $\hat{g}_i : U \to \mathbb{R}_+$

$$\hat{g}_i(u) := g_i(u) \min\{\bar{x}(u), 1\}, \text{ for all } u \in U.$$

If $\min\{\bar{x}(u), 1\} = 1$, then $\hat{g}(u) \in U$, and if $\min\{\bar{x}(u), 1\} = \bar{x}(u)$, then $\hat{g}(u) = \bar{x}(u)g(u) \in U$. Thus,

$$\hat{g}(u) = (\hat{g}_1(u), ..., \hat{g}_n(u)) : U \to U.$$

By convexity of U, function $\bar{x}$ is continuous. Thus $\hat{g} : U \to \mathbb{R}_+^n$ is a continuous function. By Brouwer's Theorem, there is $\underline{u} \in U$ such that

$$\hat{g}(\underline{u}) = \underline{u}. \tag{6}$$

If also

$$g(\underline{u}) \in U, \tag{7}$$

then, $g(\underline{u}) = \underline{u}$. This implies that $\underline{u}$ satisfies condition (2), and that $D(\underline{u})$ is a stable set. Thus condition (7) needs to be checked.

Suppose (7) does not hold. Then

$$\bar{x}(\underline{u}) < 1. \tag{8}$$

By (6) and (8),

$$\underline{u} = g(\underline{u})\bar{x}(\underline{u}) \in P. \tag{9}$$

This implies that $u_i^i(\underline{u}) = \underline{u}_i$, for all $i \in N$. By (5) and convexity of U we have

$$\begin{aligned} g(\underline{u}) &= (\delta_1 u_1^1(\underline{u}), ..., \delta_n u_n^n(\underline{u})) \\ &= (\delta_1 \underline{u}_1, ..., \delta_n \underline{u}_n) \\ &= \delta \underline{u} \\ &\in U, \end{aligned}$$

a contradiction. Thus $g(\underline{u}) \in U$, as required. ■

The Existence Theorem is based on the convexity of U.

Kultti and Vartiainen (2003) show that in the cake division problem, which imposes a degree of independency on players' payoffs, the stable set is unique.[10] The method of proof is to first show that in any two-player bargaining problem the solution is unique. Then we show that such solution is monotonic w.r.t. to the size of the cake. This in turn implies that if there are two distinct solutions for the general problem, then the minimal point of one of them Pareto dominates the minimal point of the other, which cannot be the case.

4 Relationship with the Nash solution

We now argue that there is a particular relation between a consistent partition and the Nash bargaining solution. Denote by G_Δ a stable set when the length of the period is Δ. We are mainly interested in the limit behavior of G_Δ when Δ becomes small.

First, introduce a vector of weights $\alpha = (\alpha_1, ..., \alpha_n)$ where

$$\alpha_i = \frac{-1}{\log \delta_i}, \quad \text{for all } i \in N.$$

Denote the $\alpha -$ *weighted Nash solution* by

$$u^\alpha := \arg\max_{u \in U} \prod_{i \in N} u_i^{\alpha_i}. \tag{10}$$

For any $\Delta > 0$, let $\underline{u}(\Delta)$ be the minimal point and $u^1(\Delta), ..., u^n(\Delta)$ the maximal points of players $1, ..., n$ of a stable set G_Δ. Denote by

$$H(c) := \left\{ (u_1, ..., u_n) : \prod_i u_i^{\alpha_i} = c \right\},$$

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[10] The result relies on the Fisburn-Rubinstein (1980) specification of time consistent preferences.

a $\alpha-$weighted hyperbola, indexed by $c > 0$.

Lemma 4 *$u^1(\Delta), ..., u^n(\Delta)$ lie in the same hyperbola, for any Δ.*

Proof. Recall that

$$\prod_i u_i^j(\Delta)^\alpha = \prod_i \delta_i^{-\Delta\alpha} \underline{u}_i(\Delta)^{\alpha_i}$$

Thus,

$$\begin{aligned} \prod_i u_i^j(\Delta)^\alpha &= \exp\left\{ -\Delta\alpha_j \log\delta_j + \sum_i \alpha_i \log \underline{u}_i(\Delta) \right\} \\ &= \exp\left\{ \Delta + \sum_i \alpha_i \log \underline{u}_i(\Delta) \right\}, \end{aligned}$$

which is independent of $j \in N$. Thus there is $c = \left\|u^i(\Delta)\right\|$[11] such that

$$\{u^1(\Delta), ..., u^n(\Delta)\} \subset H(c).$$

■

First we establish that in the two-player case the Nash solution always belongs to G_Δ.

Theorem 5 *Let $n = 2$. Then $u^\alpha \in G_\Delta$, for all $\Delta > 0$.*

Proof. By Lemma (4), $u^1(\Delta), u^2(\Delta) \in H(c)$ for some $c \in \mathbb{R}_{++}$. Since G_Δ is a convex set, $u \in H(c') \cap U$ for any $c' \geq c$ implies $u \in G_\Delta$. Since $u^\alpha \in H(c') \cap U$ for some $c'' > c$, also $u^\alpha \in G_\Delta$.. ■

An immediate corollary of the previous theorem is that since the stable set necessarily becomes "small" when Δ tends to zero, and since the Nash solution always belongs to the stable set, it follows thet the stable set actually shrinks to the Nash solution as Δ tends to zero.

Corollary 6 *Let $n = 2$. Then $\bigcap_{\Delta > 0} G_\Delta = \{u^\alpha\}$.*

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[11] Where $\|\cdot\|$ is the norm.

Proof. The distance between $u^i(\Delta)$ and $\underline{u}(\Delta)$ is $(\delta_i^{-\Delta}-1)\underline{u}_i(\Delta)$. Since $\underline{u}_i(\Delta)$ is bounded, this number goes to zero as Δ becomes small. Since $u^i(\Delta)\in P$ for all Δ, $\underline{u}(\Delta)$ converges to some u^*. Since $u^\alpha\in G_\Delta=D(\underline{u}(\Delta))$ for all Δ, we have $\bigcap_{\Delta>0}G_\Delta=\{u^\alpha\}$. ■

Unfortunately, it need not be the case that $x^\alpha\in G_\Delta$ when $n>2$; $G_\Delta\cap G_{\Delta'}$ may be empty, for some Δ,Δ' in such case. However, we are able to establish a weaker convergence result.

Fix U, P and the correponding u^α. Before establishing the main result of this paper, we characterize the Nash solution in a more general case in the spirit of Corollary 6. Define a triangular problem T by the property that there is specific $a\in\mathbb{R}^n_+$ and $c\in\mathbb{R}_+$ such that $T=\{u\in\mathbb{R}^n_+ : a\cdot u\le c\}$.[12] For given triangular problem T, denote by G^T_Δ the correponding stable set under Δ. Take $u\in P$. Given that P is smooth, there is one and only one triangular problem $T(u)$ such that $P\subset T(u)$ and u lies on the boundary of $T(u)$.

Proposition 7 *Assume P is smooth. Then $\bigcap_{\Delta>0}G^{T(u)}_\Delta=\{u\}$ if and only if $u=u^\alpha$.*

Proof. Let u^α_T be the Nash solution of a triangular problem T. Then $\bigcap_{\Delta>0}G^T_\Delta=\{u^\alpha_T\}$. Note that $u^\alpha_T=u^\alpha$ if and only if $T=T(u^\alpha)$. Thus $u=u^\alpha$ implies $\bigcap_{\Delta>0}G^{T(u)}_\Delta=\{u^\alpha_T\}$. On the other hand, $u\neq u^\alpha$ implies $u^\alpha_{T(u)}\neq u^\alpha$. But then $u^\alpha_{T(u)}\notin P$, and hence $\bigcap_{\Delta>0}G^{T(u)}_\Delta\neq\{u\}$. ■

Stable set G_Δ converges to $\{u\}$ in the Hausdorff metric as $\Delta\to 0$, denoted by $G_\Delta\to\{u\}$, if for any open ball with radius ε around $u\in S$, denoted by $B^\varepsilon(u)$, there is Δ_ε such that $G_\Delta\subset B^\varepsilon(u)$ for all $\Delta<\Delta^\varepsilon$. We show that in any situation the stable set converges to the asymmetric Nash solution as Δ becomes small.

Theorem 8 *Assume P is smooth. Then $G_\Delta\to\{u^\alpha\}$ as $\Delta\to 0$.*

Proof. For any $\Delta>0$, let $\underline{u}(\Delta)$ be the minimal point and $u^1(\Delta),...,u^n(\Delta)$ the maximal points of players $1,...,n$ of a stable set G_Δ. By construction,

$$\{u^1(\Delta),...,u^n(\Delta)\}\subset P.$$

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[12] Where "$\cdot$" denotes the inner product.

Take any sequence $\{\Delta^k\}$ such that $\Delta^k \to 0$. Since U is bounded and since the distance $(\delta_i^{-\Delta^k} - 1)\underline{u}_i(\Delta^k)$ between any $u^i(\Delta^k)$ and $\underline{u}(\Delta^k)$ tends to zero as Δ^k becomes small, there is subsequence $\{\Delta^l\}$ such that, for some $u^* \in P$,

$$\begin{array}{l} \underline{u}(\Delta^l) \to u^*, \\ u^i(\Delta^l) \to u^*, \text{ for all } i = 1, ..., n. \end{array} \tag{11}$$

It suffices to show that $u^* = u^\alpha$ for any convergence point u^*.

Suppose that $u^* \neq u^\alpha$. Since $H(\|u^*\|)$ and P are surfaces of n-dimensional convex sets in $\mathbb{R}^n$, and thus themselves $n-1$ dimensional manifolds, their intersection $P \cap H(u^*)$ is an $n-2$ dimensional manifold. Since $u^* \neq u^\alpha$, and P is smooth, there is a unique a $n-2$ dimensional hyperplane $L(u^*)$ that supports $P \cap H(u^*)$ at u^*.[13]

For any $v \in U$, denote by $B^\lambda(v)$ an open ball around v with radius λ. For any $c \in \mathbb{R}^n_+$, write $cx = (c_1 x_1, ..., c_n x_n)$ and $cX = \{cx : x \in X\}$, for any $x \in X \subset \mathbb{R}^n_+$. Note that $L(cv) = cL(v)$ and $H(\|cv\|) = cH(\|v\|)$.

By smoothness of P, for any $\varepsilon, \mu, \lambda > 0$ and there exists $k \in \mathbb{R}_+$ such that

$$\begin{array}{rcl} \varepsilon & > & \sup\Big\{ \|u - v\| : v \in B^{\lambda/k}(u^*) \cap L(u^*)\,, \\ & & \text{for all } \ u \in B^{\lambda/k}(u^*) \cap P \cap H(\|u'\|)\,, \\ & & \text{for all } \ u' \in B^{\mu/k}(u^*) \Big\}. \end{array}$$

Analogously, for any $\varepsilon, \mu, \lambda > 0$ there exists big enough $c \in \mathbb{R}^n_+$ such that

$$\begin{array}{rcl} \varepsilon & > & \sup\Big\{ \|u - v\| : v \in B^{\lambda}(cu^*) \cap L(cu^*)\,, \\ & & \text{for all } \ u \in B^{\lambda}(cu^*) \cap cP \cap H(\|cu'\|)\,, \\ & & \text{for.all } \ u' \in B^{\mu}(cu^*) \,\}. \end{array} \tag{12}$$

Construct $c(\Delta) = (c_1(\Delta), ..., c_n(\Delta))$ such that

$$c_i(\Delta) = \frac{1}{(\delta^{-\Delta} - 1)\underline{u}_i(\Delta)}, \text{ for all } i \in N.$$

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[13] For let P_1 and P_2 be smooth surfaces of n dimensional manifolds. Suppose P_1 and P_2 intersect at u. Then P_1 and P_2 have at tangering hyperplanes of dimension $n-1$. If P_1 and P_2 intersect at u, then the intersection $L_1(u) \cap L(u_2)$ of the tangering hyperplanes of P_1 and P_2 at u has dimension $n-2$.

Note that since $\underline{u}(\Delta)$ converges to u^* and $\delta_i^{-\Delta}$ converges to 1, $c_i^i(\Delta)$ is monotonically increasing below some Δ'. Denote the symmetric $n-1$ dimensional standard simplex by[14]

$$T = \left\{ x \in \mathbb{R}_+^n : \sum x_i = 1 \right\}.$$

Now[15]

$$co\{c(\Delta)u^1(\Delta), ..., c(\Delta)u^n(\Delta)\} = c(\Delta)\underline{u}(\Delta) + T.$$

Thus the maximal points span an $n-1$ dimensional standard simplex with edge length $\sqrt{n-1}$.

Take any $\varepsilon, \mu > 0$. By (12), there is Δ_ε such that for all $\Delta < \Delta_\varepsilon$,

$$\begin{aligned} \varepsilon \quad > \quad & \sup \Big\{ \|u - v\| : v \in B^{\sqrt{n-1}}(c(\Delta)u^*) \cap L(c(\Delta)u^*)\ , \\ & \text{for all } \ u \in B^{\sqrt{n-1}}(c(\Delta)u^*) \cap c(\Delta)P \cap H(\|u'\|), \\ & \text{for all } \ u' \in B^{\mu}(c(\Delta)u^*) \Big\}\ . \end{aligned} \tag{13}$$

Invoke sequence $\{\Delta^l\}$. By construction,

$$c(\Delta)\underline{u}(\Delta) + T \subset B^{\sqrt{n-1}}(c(\Delta')u^i(\Delta^l)) \cap c(\Delta')P \cap c(\Delta')H(\left\|u^i(\Delta^l)\right\|), \text{ for all } \Delta^l.$$

By (11), there is $\Delta'_\varepsilon < \Delta_\varepsilon$ such that, for all $\Delta^l < \Delta'_\varepsilon$, $u^i(\Delta^l) \in B^\mu(c(\Delta')u^*)$ and

$$c(\Delta^l)\underline{u}(\Delta^l) + T \subset B^{\sqrt{n-1}}(c(\Delta')u^*) \cap c(\Delta')P \cap c(\Delta')H(\left\|u^i(\Delta^l)\right\|)\ .$$

By (13), for all $\Delta^l < \Delta'_\varepsilon$,

$$c(\Delta^l)\underline{u}(\Delta^l) + T \subset \{u : \|u - v\| < \varepsilon, v \in L(c(\Delta^l)u^*)\},$$

or

$$T \subset \{u : \|u - v\| < \varepsilon + \mu, \text{ for } v \in L(c(\Delta^l)u^* - u^*)\}.$$

Since this holds for all $\varepsilon, \mu > 0$ it follows that $n-1$ dimensional manifold is contained by an $n-2$ dimensional hyperplane, a contradiction. ■

The intuition of the proof can be summarized as follows. Assume that the number of players is 3 (n). Since a stable set shrinks as Δ becomes small, a convergence point u^* in the boundary exists. Suppose that u^* is not u^α. Then u^* lies in the intersection of some hyperbola H and P. Since both H

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[14] co$X$ is a convex hull of $X \subset \mathbb{R}^n$.

[15] Denote $y + X = \{y + x : x \in X\}$, for any $y \in \mathbb{R}^n$ and $X \subset \mathbb{R}^n$.

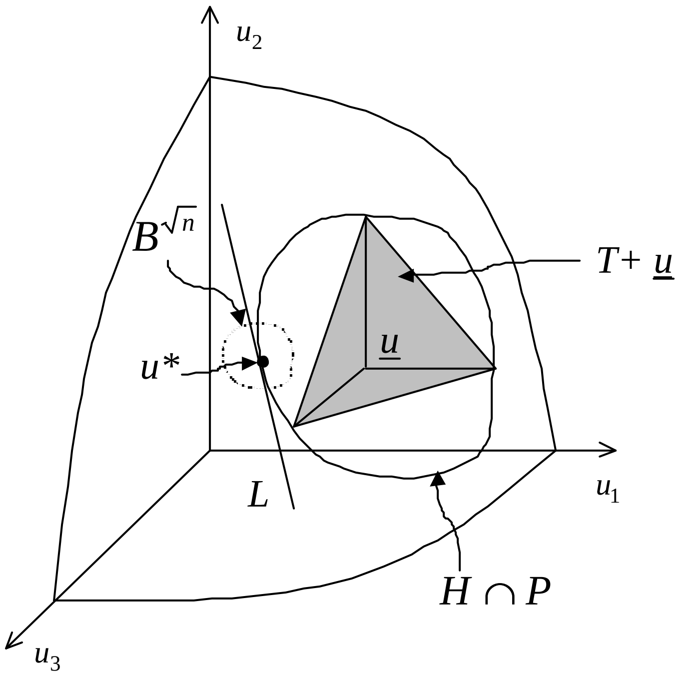

Figure 1:

and P are 2-dimensional ($n-1$ -dimensional) smooth surfaces of convex sets, their intersection constitutes a 1-dimensional ($n-2$ -dimensional) Jordan curve. Thus the intersection has a tangent ($n-2$ -dimensional supporting hyperplane). Normalize the situation such that for any Δ the distance between any two maximal points is $\sqrt{2}$ ($\sqrt{n-1}$). Let L denote the tangent at u^*. Identify triangle T that is spanned by the maximal points of G. Then the dimension of T is 2 ($n-1$). As Δ goes to zero, the maximal distance between T and L goes to zero. Thus T becomes embedded into L. But L's dimension is higher than T's, thus embedding is impossible.

5 Relation to Krishna-Serrano (1996)

We now construct a game that closely parallels Krishna-Serrano (1996) [KS]. The only difference is that we replace the "cake" structure of KS with the more general $U-$structure that allows externalities.

Take any $S \subset N$ and $u \in U$.Abusing the notation slightly, let V be the

$N \setminus S-$dimensional projection of U that keeps u_S fixed:

$$V(u_S) = \{u_{N\setminus S} \in \mathbb{R}^{N\setminus S} : (u_{N\setminus S}, u_S) \in U\}.$$

The extensive form of the bargaining procedure in which set S of players bargain over utility allocation in U will be denoted by $\Gamma_i(S, U)$, and is defined recursively.

Start by defining a two player bargaining game. For any two player set $\{i, j\}$, let $\Gamma_i(\{i, j\}, V(u_{N\setminus\{i,j\}}))$ denote the Rubinstein alternating offers game between i, j on projection $V(u_{N\setminus\{i,j\}})$ where i is the first mover.

For any $S \subset N$, $i \in N$ and $u \in U$, define $\Gamma_i(S, V(u_{N\setminus S}))$ as follows: In period 1 player i makes an offer $u_S \in V(u_{N\setminus S})$. All players $j \in S \setminus \{i\}$ respond simultaneously by accepting or rejecting u_S. If $j \in S$ accepts the offer $(u_k)_{k\in S}$, then he receives utility u_i. Let $A \subseteq S\setminus\{i\}$ be the set of players who accept the offer in period t. If $A = S \setminus \{i\}$, then all players, including i, receive their shares immediately. If $\varnothing \neq A \subset S \setminus \{i\}$, then in period 2, $\Gamma_j(S \setminus A, V(u_{N\setminus S}))$ is played where j is either the smallest index in $S \setminus A$ greater than i, or, if i is the highest index in $S \setminus A$, j is the smallest index in $S \setminus A$. If $A = \varnothing$, then $\Gamma_j(S, V(u_{N\setminus S}))$ is played in period 2.

For brevity, we concentrate on Markov equilibria of game $\Gamma_1(N, U)$ where i makes the same offer whenever it is his turn to make one, and any $j \in S$ responds to offer made by i in the same way, in any subgame $\Gamma_i(S, V)$ of $\Gamma_1(N, U)$. That is, i's offer is conditional only on the subgame and j's response is conditional only on the subgame and the offer.

Before establishing the result, we specify some concepts. Without loss, assume that $\delta = \delta_i$ for all i. For any $i \in S \subset N$ and $u \in U$, denote by $u^i(V(u_{N\setminus S}))$ the $i-$maximal point of a stable set related to $V(u_{N\setminus S})$. Set G is now a stable set if and only if $G = D(v)$ where v satisfies

$$v_j = \delta u^j_j(V(v_{N\setminus\{i,j\}})), \text{ for all } i \neq j.$$

Note that this implies that a stable set is consistent in the sense of Lensberg (1988).[16]

Proposition 9 *Any Markov equilibrium outcome of* $\Gamma_1(N, U)$ *is a* $1-$*maximal point of a stable set.*

Proof. In any Markov equilibrium, 1's offer is accepted. The set of offers that all $j \neq 1$ accept is written

$$V := \{v : v_j \geq \delta u^j_j(V(v_{N\setminus\{1,j\}}))\}.$$

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[16] Who calls the property "stability".

Let v^* be an offer in the closure of V such that

$$v_1^* = \sup\{v_1 : v \in V\}.$$

By continuity of U, $v_j^* = \delta u_j^j(V(v^1_{N\backslash\{1,j\}}))$ for all $j \neq 1$.[17] Then $D(\delta v_1^*, v_2^*, ..., v_n^*)$ constitutes a stable set, and v^* a $1-$maximal point of $D(\delta v_1^*, v_2^*, ..., v_n^*)$. ■

Proposition 10 *There is maximal point* u^1 *of a stable set that can be supported as a Markov equilibrium of* $\Gamma_1(N, U)$.

Proof. Index each stable set by an element of an index set K.[18] Identify the $1-$maximal point $u^1(k)$ of stable set G_k for all $k \in K$. Then there is

$$c := \sup\{u_1^1(k) : k \in K\}. \tag{14}$$

Hence there alsois sequence $\{k_l\}$ such that $\{u^1(k_l)\}$ converges to $\bar{u}^1 \in U$ such that $\bar{u}_1^1 = c$. By construction,

$$u_j^1(k_l) = \delta u_j^j(V(u^1(k_l)_{N\backslash\{1,j\}})),\ \text{for all } k_l = 1, ...,\ \text{for all } j.$$

Coordinate-wise convergence of $u^1(k_l)$ implies that also

$$\bar{u}_j^1 = \delta u_j^j(V(\bar{u}^1_{N\backslash\{1,j\}})),\ \text{for all } j.$$

Thus there is $\bar{k} \in K$ such that $\bar{u}^1 = u^1(\bar{k})$.

We claim that 1's offer $\bar{u}^1$ can be supported as a Markov equilibrium of $\Gamma_1(N, U)$. Suppose not. By Rubinstein (1982), $u^i(V(u_{N\backslash\{i,j\}}))$ constitutes a Markov equilibrium of $\Gamma_i(\{i, j\}, V(u_{N\backslash\{i,j\}}))$. Since $\bar{u}^1 = u^1(\bar{k})$, no $j \neq 1$ benefits from unilaterally rejecting 1's offer. We thus need to show that 1 can profitably deviate from offer $\bar{u}^1$. Note that all $j \neq 1$ accept 1's offer v only if $v_j \geq \delta u_j^j(V(v_{N\backslash\{1,j\}}))$ for all such j. Define

$$V := \{v : v_j \geq \delta u_j^j(V(v_{N\backslash\{1,j\}}))\}.$$

Identify $v^1 \in V$ such that

$$v_1^1 = \sup\{v_1 : v \in V\}.$$

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[17] If this were not the case, then there would be player j for who the inequality would be strict. But this would allow improvement of 1's payoff *along* the $(1, j)-$projection of U.

[18] We are unable not to use the Axiom of Choice.

Since $\bar{u}^1$ cannot be supported as an equilibrium, V is nonempty and $v_1^1 > \bar{u}_1^1$. By continuity of U, $v_j^1 = \delta u_j^j(V(v_{N\setminus\{1,j\}}^1))$ for all $j \neq 1$. But then $D(\delta v_1^1, v_2^1, ..., v_n^1)$ constitutes a stable set, which contradicts (14). Thus $\bar{u}^1$ can be supported as a Markov equilibrium of $\Gamma_1(N, U)$. ■

Combining these propositions with our previous results gives some interesting corollaries. Since a stable set converges to the Nash bargaining solution u^α as the time intervall becomes small, it follows by Proposition 9 that also *all* Markov equilibria associated to $\Gamma_1(N, U)$ converge to u^α. On the other hand, since a stable set always exists, Proposition 10 implies that a Markov equilibrium associated to $\Gamma_1(N, U)$ exists, too. To sum up:

Corollary 11 *A Markov equilibrium of* $\Gamma_1(N, U)$ *exists, and any Markov equilibrium outcome of* $\Gamma_1(N, U)$ *converges to* u^α *as* Δ *tends to zero.*

References

[1] Binmore K., A. Rubinstein and A. Wolinsky 1986, The Nash bargaining solution in economic modelling, *Rand Journal of Economics 17*, 176-188.

[2] Fishburn, P. and Rubinstein, A. 1982, Time Preference, *International Economic Review 23*, 677-95.

[3] Greenberg J. 1990, *The Theory of Social Situations.* Cambridge University Press, Cambridge.

[4] Krishna V. and R. Serrano 1996, Multilateral bargaining, *Review of Economic Studies 63*, 61-80.

[5] Kultti, K. and H. Vartiainen 2003, Von Neumann-Morgenstern solution to the cake division problem, *manuscript.*

[6] Lensberg, T. 1988, Stability and the Nash Solution, *Journal of Economic Theory 54*, 101-16.

[7] Nash J. 1950, The bargaining problem, *Econometrica 18*, 155-162.

[8] Owen G. 1995, *Game theory,* New York: Academic Press.

[9] Rubinstein A. 1982, Perfect equilibrium in a bargaining model, *Econometrica 50*, 97-109.

[10] Rubinstein A., Safra, Z. and Thomson, W. 1992, On the Interpretation of the Nash Bargaining Solution and its Extension to Non-Expected Utility Preferences, *Econometrica 60*, 1171-1186.

[11] Thomson W. and T. Lensberg 1989, *Axiomatic theory of bargaining with a variable number of agents*. Cambridge University Press, Cambridge.

[12] von Neumann J. and O. Morgenstern 1944, *Theory of Games and Economic Behavior*. New York: John Wiley and Sons.