Emission Policy in an Economic Union with Poisson Technological Change

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Discussion Paper No. 625: 2006
ISBN 952-10-2768-1,   ISSN 1459-3696

October 17, 2006

Abstract

This study examines optimal emission policy in a union of countries. In each country, labor is used in either production or research and development (R&D) which increases the probability of the improvement of production technology. The production of goods in any country incurs emissions that are spread all over the union. A household's utility in any country depends positively on his personal consumption and negatively on total emissions in the union. This study constructs the Pareto-optimal emission taxes for the countries.

Journal of Economic Literature: 041, H23, F15

Keywords: Emissions, Technological change, Economic Intergration

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[*] This paper is prepared for the presentation in the IIASA/IFAC Workshop, December 5-7, 2006, in IIASA, Laxenburg, Austria.

1 Introduction

In this study, I examine optimal emission policy in a union of several countries. In each country, labor can be used in either production or research and development (R&D) which increases the probability of the improvement of production technology. The production of goods in any country incurs emissions that are spread all over the union. A household's utility in any country depends positively on his personal consumption and negatively on total emissions in the union. This study attempts to find out the Pareto-optimal emission taxes for the countries.

To solve the problem, I introduce a local government for each country and a central government for the union as a whole. A local government maximizes the welfare of a typical household in the country and it has enough instrument to control the allocation of resources in the country. The central government maximizes the welfare of a typical household in the whole union and attempts to control the emissions of the countries by emission taxes. There is policy externality between the countries in a dynamic environment.

The impact of any environmental policy depends crucially on the existence of uncertainty. Because governmental activities influence not only the expected values of economic variables, but also their volatility, many papers consider public policy within a stochastic growth model where productity shocks follows a Wiener process.[1] Soretz (2003) applies this approach to environmental policy. In this study, I choose a different approach and assume that uncertainty is directly embodied in technological change.

There is some literature on policy externality with pollution in an economic union. Philippopoulos and Economides (2003) consider a union composed of a number of countries as follows. In each country, private agents consume, save in domestic capital and produce goods from capital with constant returns to scale. Pollution occurs as by product of output produced and decreases welfare. Philippopoulos and Economides show that the type of policy externality from one country to another changes from positive into negative, when capital accumulation is introduced into the model: the tax on externality is too low without, but too high with capital accumulation. In this study, I show that the tax on externality remains too low, after dynamics

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[1] Cf. Turnovsky (1993, 1995, 1999), Smith (1996) and Corsetti (1997).

is introduced in the form of R&D.

Reis (2001) examines the case where welfare depends on emissions and R&D increases the probability that at some moment in the future a technology is discovered that will eliminate emissions.[2] She shows that the hope of discovering such a technology (measured by the probability of the discovery) increases the optimal rate of growth. However, the optimal rate of growth remains smaller than in an economy without emissions. In this study, I show that the optimal growth rate is an increasing function of the productivity of labor in R&D that improves finding a new emission-saving technology.

Beltratti et al. (1994) introduce a growth model where an environmental asset is a source of utility and depleted by a pollution process which is linked to consumption. They define the concept of the Green Golden rule as the best sustainable configuration, i.e. the path that gives the highest maintainable level of instantaneous utility. Ayong Le Kama (2001) transforms this model by linking the pollution process to production. Following these papers, I search for the Green Golden for the economic union.

Sections 2 and 3 present the basic structure of the model. Section 4 ignores the welfare effects of pollution and examines the Cournot-Nash case where the local government maximizes take each other's emissions as given. In section 5, this model is generalized for the case where local governments form expectations on each other behavior. Finally, in section 6, the basic model is extended for the case where emissions have long term effects through pollution. In all cases, the solution method is dynamic programming of Poisson jump processes. These models provides a lot of challenges for both economists and mathematicians.

2 The union

I consider an economic union that consists of fixed number n of similar countries. Each country $j \in \{1, ..., n\}$ produces a different good. Competitive firms produce a consumption good from all n goods through Cobb-Douglas

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[2] Reis calls emissions as "the flow of pollution".

technology:

$$y = \prod_{j=1}^{n} y_j^{1/n}, \tag{1}$$

where y is total consumption in the union and y_j is output in country j.[3] Given this technology, the true consumption price is the minimum unit cost of consumption, $p = \prod_{j=1}^{n} p_j^{1/n}$, where p_j is the output price in country j. Normalizing the consumption price at unity yields

$$1 = p = \prod_{j=1}^{n} p_j^{1/n}. \tag{2}$$

The union employs a_j labor units j in country in abatement activities. The level of abatement, X, is a Cobb-Douglas function of these labor inputs, $X = \prod_{j=1}^{n} a_j$.[4] The absorbtion rate of pollution, h, is the higher, the more there are abatement activities relative to the level of pollution:

$$h(x), \quad x \doteq \frac{X}{P} = \frac{1}{P}\prod_{j=1}^{n} a_j^{1/n}, \quad h' > 0. \tag{3}$$

Following Michel and Rotillon (1995), I assume that emissions m accumulate pollution P, but pollution absorbtion takes place at the rate $h(x)$:

$$\dot{P} \doteq \frac{dP}{dt} = m - h(x)P, \tag{4}$$

where t is time. The elasticity of the absorbtion rate of pollution is given by

$$\varepsilon \doteq \frac{X}{h}\frac{dh}{dX} = \frac{X}{hP}h' > 0. \tag{5}$$

Assume that (*i*) all households in the union share the same preferences, and (*ii*) total emissions in the union, m, and the degree of pollution in the union, P, decrease a household's welfare in all countries. In country j, the

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[3] With some complication, the same results can be generalized for any neoclassical production function with constant returns to scale.

[4] With some complication, the same results can be generalized for any neoclassical production function with constant returns to scale.

representative household's utility from an infinite stream of its consumption c_j, emissions m and pollution P beginning at time T is then given by

$$U(c_j, m, P, T) = E\int_T^\infty u(c_j, m, P)e^{-\rho(t-T)}dt \text{ with}$$
$$\rho > 0,\ \frac{\partial u}{\partial c_j} > 0,\ \frac{\partial u}{\partial m} < 0,\ \frac{\partial u}{\partial P} < 0, \tag{6}$$

where E is the expectation operator, ρ the constant rate of time preference and u the level of instantaneous utility. Because it is impossible to find any analytical solution for the general case (6) in Bellman's dynamic programming, we specify the instantaneous utility function in the form

$$u(c_j, m, P) \doteq c_j^\sigma m^{-\delta} P^{-\nu},\ \ 0 < \sigma < 1,\ \ \delta > 0,\ \ \nu \geq 0, \tag{7}$$

where σ, δ and ν are constants. The ratio $1/(1-\sigma)$ can be interpreted also as the household's constant rate of risk aversion. Following Beltratti et al., (1994) and Ayong Le Kama (2001), I define the *Green Golden Rule (GGR)* as the path of the union that gives the highest maintainable level of instantaneous utility (7) in all countries j.

3 A single country

In country j, there is a fixed labor supply L, of which the amount l_j is used in production, the amount a_j in abatement activities and the rest z_j in R&D:

$$L = l_j + a_j + z_j. \tag{8}$$

The productivity of labor in production, l_j, in country j is A^{γ_j}, where $A > 1$ is a constant and γ_j is the serial number of technology. In the advent of technological change, this productivity increases from A^{γ_j} to A^{γ_j+1}. The total output of the consumption good in country j is therefore given by

$$y_j = A^{\gamma_j} l_j. \tag{9}$$

In equilibrium, the wage w_j is equal to the marginal product of labor:

$$w_j = p_j \frac{\partial y_j}{\partial l_j} = p_j A^{\gamma_j}. \tag{10}$$

Emissions in country j, m_j, are in fixed proportion to labor input in production in that country, l_j. By a proper choice of units,

$$m_j = l_j \tag{11}$$

holds true and total emissions in the whole union are given by

$$m \doteq \sum_{j=1}^{n} m_j = \sum_{j=1}^{n} l_j. \tag{12}$$

The improvement of technology in country j depends on the labor z_j devoted to R&D. I assume that in a small period of time dt, the probability that R&D leads to development of a new technology is given by $\lambda z_j dt$, while the probability that R&D remains without success is given by $1 - \lambda z_j dt$, where λ is the productivity of labor in R&D. This defines a Poisson process

$$dq_j = \begin{cases} 1 & \text{with probability } \lambda z_j dt, \\ 0 & \text{with probability } 1 - \lambda z_j dt, \end{cases} \tag{13}$$

where dq_j is the increment of the process q_j. Technological change (13) generates economic growth. Noting (8), the average growth rate of productivity A^{γ_j} in the stationary state is equal to

$$E\big[\log A^{\gamma_j+1} - \log A^{\gamma_j}\big] = (\log A)\lambda z_j, \tag{14}$$

where E is the expectation operator.[5] Because in the stationary state labor devoted to production, l_j, is constant, the average growth rate of consumption (9) in the stationary state is also given by (14). Because this growth rate is in fixed proportion $(\log A)\lambda$ to R&D, z_j, one can use labor devoted to R&D, $z_j = L - l_j - a_j$, as a measure for the rate of economic growth.

There are governments at two levels: a central government for the union as a whole, and a local government for each country. The central government attempts to control the emissions of the countries by emission taxes. To keep the model simple, I specify taxation as follows.[6] The central government imposes the tax $w_j\tau$ on all emissions m_j and pays the subsidy $w_j b$ to all labor L and the wage w_j for labor in abatement, a_j, in country j, where τ

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[5] For this, see Aghion and Howitt (1998), p. 59.

[6] With this specification, the growth rates of different countries are independent.

and b are policy parameters and w_j is the wage in that country. Thus, noting (9), (10) and (11), I obtain the household's budget constraint in country j as:

$$c_j = p_j y_j + w_j a_j - w_j \tau m_j + w_j bL = p_j A^{\gamma_j}[(1-\tau)l_j + a_j + bL]. \tag{15}$$

The central government's budget constraint is given by

$$b\sum_{j=1}^{n} w_j L + \sum_{j=1}^{n} w_j a_j = \tau \sum_{j=1}^{n} w_j m_j = \tau \sum_{j=1}^{n} w_j l_j, \tag{16}$$

where $b\sum_{j=1}^{n} w_j L$ is total labor subsidies, $\sum_{j=1}^{n} w_j a_j$ total costs in abatement activities and $\tau \sum_{j=1}^{n} w_j m_j$ total emission taxes in the union.

4 The basic model

Assume for a while that utility (7) is independent of pollution P, $\nu = 0$. In such a case, there is no need for abatement activities and $a_j = 0$ for all j.

4.1 The local governments

I assume that the local government in the union behave in Cournot-Nash manner. The government of country j then maximizes the utility of country j's representative household (6) subject to (7), (8), (12), (13) and (15), holding the output price p_j, the tax parameter τ, the subsidy parameter b and the emissions by the other countries, $l_{-j} \doteq \sum_{\iota \neq j} m_\iota = \sum_{\iota \neq j} l_\iota$, constant. The value of the optimal program starting at time T for country j is then

$$\begin{aligned}
\Gamma^j(l_{-j}, \gamma_j, \tau, b, p_j, T) &= \max_{(c_j, l_j, z_j) \text{ s.t. (8),(12),(13),(15)}} E\int_T^\infty c_j^\sigma m^{-\delta} e^{-\rho(t-T)} dt \\
&= \max_{(c_j, l_j, z_j) \text{ s.t. (8),(13),(15)}} E\int_T^\infty c_j^\sigma (l_j + l_{-j})^{-\delta} e^{-\rho(t-T)} dt \\
&= \max_{(l_j, z_j) \text{ s.t. (8),(13)}} E\int_T^\infty p_j^\sigma A^{\sigma\gamma_j}[(1-\tau)l_j + bL]^\sigma (l_j + l_{-j})^{-\delta} e^{-\rho(t-T)} dt. \qquad (17)
\end{aligned}$$

The Bellman equation corresponding to this optimal program obtains[7]

$$\begin{aligned}\rho\Gamma^j(l_{-j},\gamma_j,\tau,b,p_j,T) &= \max_{(l_j,z_j)\text{ s.t. (8) and } a_j=0} \Phi^j(z_j,l_j,l_{-j},\gamma_j,\tau,b,p_j,T)\\ &= \max_{l_j} \Phi^j(L-l_j,l_j,l_{-j},\gamma_j,\tau,b,p_j,T),\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\Phi^j(z_j,l_j,l_{-j},\gamma_j,\tau,b,p_j,T)\\ &\quad= p_j^\sigma A^{\sigma\gamma_j}[(1-\tau)l_j+bL]^\sigma(l_j+l_{-j})^{-\delta}\\ &\qquad+\lambda z_j\big[\Gamma^j(l_{-j},\gamma_j+1,\tau,b,p_j,T)-\Gamma^j(l_{-j},\gamma_j,\tau,b,p_j,T)\big].\end{aligned}$$

This leads to the first-order condition

$$\begin{aligned}\frac{\partial\Phi^j}{\partial l_j} &= \left[\frac{(1-\tau)\sigma}{(1-\tau)l_j+bL}-\frac{\delta}{l_j+l_{-j}}\right]p_j^\sigma A^{\sigma\gamma_j}[(1-\tau)l_j+bL]^\sigma(l_j+l_{-j})^{-\delta}\\ &\quad-\lambda[\Gamma^j(l_{-j},\gamma_j+1,\tau,b,p_j,T)-\Gamma^j(l_{-j},\gamma_j,\tau,b,p_j,T)]=0.\end{aligned} \tag{19}$$

To solve the dynamic program, I try the solution that the value of the program, Γ^j, is in fixed proportion $\vartheta_j>0$ to instantaneous utility $c_j^\sigma m^{-\delta}$:

$$\Gamma^j(l_{-j},\gamma_j,\tau,b,p_j,T)=\vartheta_j c_j^\sigma m^{-\delta}=\vartheta_j p_j^\sigma A^{\sigma\gamma_j}[(1-\tau)l_j+bL]^\sigma(l_j+l_{-j})^{-\delta}. \tag{20}$$

This implies

$$\Gamma^j(l_{-j},\gamma_j+1,\tau,b,p_j,T)/\Gamma^j(l_{-j},\gamma_j,\tau,b,p_j,T)=A^\sigma. \tag{21}$$

Inserting (20) and (21) into the Bellman equation (18) yields

$$1/\vartheta_j=\rho+(1-A^\sigma)\lambda(L-l_j)>0. \tag{22}$$

This is constant, if in equilibrium labor input l_j is constant. Inserting (20) and (21) into the first-order condition (19) yields

$$\begin{aligned}(A^\sigma-1)\lambda &= \left[\frac{(1-\tau)\sigma}{(1-\tau)l_j+bL}-\frac{\delta}{l_j+l_{-j}}\right]\frac{1}{\vartheta_j}\\ &= \left[\frac{\sigma}{l_j+bL/(1-\tau)}-\frac{\delta}{l_j+l_{-j}}\right]\big[\rho+(1-A^\sigma)\lambda(L-l_j)\big].\end{aligned} \tag{23}$$

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[7] Cf. Dixit and Pindyck (1994).

4.2 Equilibrium

Because of the symmetry over all countries $j = 1, ..., n$, it is evident that in equilibrium $l_j = l$ and $l_{-j} = \sum_{\iota \neq j} l_\iota = (n-1)l$ for all $j \in \{1, ..., n\}$. The more countries (i.s. the bigger n), the more decentralized decision making in the union. From the budget constraint (16), $l_j = l$ and $a_j = 0$ it follows that

$$b = \tau \sum_{j=1}^{n} w_j l_j \Big/ \sum_{j=1}^{n} w_j L = \tau \frac{l}{L}. \tag{24}$$

Given this, the equilibrium condition (23) takes the form

$$\begin{aligned} 1 &= \left[\frac{\sigma}{l + bL/(p_j - \tau)} - \frac{\delta}{nl}\right]\left[\frac{\rho/\lambda}{A^\sigma - 1} - L + l\right] \\ &= \left[\frac{\sigma}{1 + \tau/(1-\tau)} - \frac{\delta}{n}\right]\left[\left(\frac{\rho/\lambda}{A^\sigma - 1} - L\right)\frac{1}{l} + 1\right] \\ &= \left[(1-\tau)\sigma - \frac{\delta}{n}\right]\left[\left(\frac{\rho/\lambda}{A^\sigma - 1} - L\right)\frac{1}{l} + 1\right]. \end{aligned}$$

Solving for l leads to the equilibrium level of emissions as:

$$l = \left[\frac{1}{(1-\tau)\sigma - \delta/n} - 1\right]^{-1}\left(\frac{\rho/\lambda}{A^\sigma - 1} - L\right), \quad \frac{dl}{dn} > 0. \tag{25}$$

Given (1) and (9), the consumption good is produced as follows:

$$c \doteq \frac{1}{n}\sum_{j=1}^{n} c_j = \frac{1}{n} A^\gamma \prod_{j=1}^{n} l_j^{1/n} = \frac{l}{n} A^\gamma, \quad \gamma \doteq \frac{1}{n}\sum_{j=1}^{n} \gamma_j, \tag{26}$$

where the serial number of the consumption-good technology. Because the improvement of productivity in country j follows the Poisson process (13), noting (14) and (26), I obtain

$$\begin{aligned} E\big[\log A^{\gamma+1} - \log A^\gamma\big] &= \frac{1}{n}\sum_{j=1}^{n} E\big[\log A^{\gamma_j+1} - \log A^{\gamma_j}\big] = \log A \frac{1}{n}\sum_{j=1}^{n} \lambda z_j \\ &= (\log A)\lambda z. \end{aligned}$$

This means that in the production of the consumption good the improvement of productivity follows the Poisson process q with[8]

$$dq = \begin{cases} 1 & \text{with probability } \lambda z \, dt, \\ 0 & \text{with probability } 1 - \lambda z \, dt. \end{cases} \tag{27}$$

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[8] Cf. also Wälde (1999).

4.3 The central government

Noting (12) and (26), the welfare of the representative household in the union takes the form

$$U(c,m,T)=\int_T^\infty c^\sigma m^{-\delta}e^{-\rho(t-T)}dt=\int_T^\infty A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}e^{-\rho(t-T)}dt, \qquad (28)$$

Because the central government is able to control labor input in production, l, by the tax τ, it maximizes welfare (28) by l subject to technological change (27). Noting (26), the value of its optimal program starting at time T is

$$\Gamma(\gamma,T)=\max_{l\text{ s.t. (27)}} E\int_T^\infty A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}e^{-\rho(t-T)}dt. \qquad (29)$$

The Bellman equation corresponding to this optimal program obtains

$$\rho\Gamma(\gamma,T)=\max_l\bigl\{A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}+\lambda(L-l)\bigl[\Gamma(\gamma+1,T)-\Gamma(\gamma,T)\bigr]\bigr\}. \qquad (30)$$

This leads to the first-order condition

$$\frac{\partial\{\}}{\partial l}=(\sigma-\delta)A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta-1}-\lambda[\Gamma(\gamma+1,T)-\Gamma(\gamma,T)]=0. \qquad (31)$$

To solve the dynamic program, I try the solution that the value of the program, Γ, is in fixed proportion $\vartheta>0$ to instantaneous utility:

$$\Gamma(\gamma,T)=\vartheta A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}. \qquad (32)$$

This implies

$$\Gamma(\gamma+1,T)/\Gamma(\gamma,T)=A^\sigma. \qquad (33)$$

Inserting (32) and (33) into the Bellman equation (18) yields

$$1/\vartheta=\rho+(1-A^\sigma)\lambda(L-l)>0. \qquad (34)$$

Inserting (32) and (33) into the first-order condition (31) yields

$$(A^\sigma-1)\lambda=\frac{\sigma-\delta}{l}\frac{1}{\vartheta}=\frac{\sigma-\delta}{l}\bigl[\rho+(1-A^\sigma)\lambda(L-l)\bigr]. \qquad (35)$$

Given (35), I solve for the Pareto-optimal solution

$$l^*\doteq\Bigl(\frac{1}{\sigma-\delta}-1\Bigr)^{-1}\Bigl(\frac{\rho/\lambda}{A^\sigma-1}-L\Bigr). \qquad (36)$$

Note that, given (8), (36) and $a = 0$, labor input to production, l^*, falls and that to R&D, $z = L - l^*$, rises, when the productivity of labor in R&D, λ, increases. This result is in line with Reis (2001).

The central government sets the tax τ so that this Pareto optimum is established, $l = l^*$. Noting (25) and (36), one then obtains $(1-\tau)\sigma - \delta/n = \sigma - \delta$ and $\tau = (1 - 1/n)\delta/\sigma$. This result can be rephrased as follows:

Proposition 1 *The optimal emission tax for country j is given by*

$$\tau w_j = \Big(1 - \frac{1}{n}\Big)\frac{\delta}{\sigma} w_j,$$

where w_j is the wage in that country. This tax is the smaller, the more decentralized the union (i.e. the bigger n). When the number of countries is large, $n \to \infty$, the tax is equal to $\frac{\delta}{\sigma} w_j$.

Note that if dynamics is based on R&D, the tax on externality (here emissions) is too low and decreases with a larger number of countries. This reverses the result of Philippopoulos and Economides (2003) who introduce dynamics in the form of capital accumulation.

5 Flexible strategic dependence

In the basic model, the countries took the levels of each other's decision variables as given in optimization. This Cournot-Nash assumption is of course a simplification. Dixit (1986) generalizes this setting through the assumption that each agent forms expectations on the others' prospective responses to its action. This idea can be applied for emission games as follows.

Assume that country j anticipates the other countries $\ell \neq j$ to increase their emissions m_ℓ by the constant $\beta \in (-\infty, 1)$ percentages, when the country itself increases its emissions by one percentage. This and (11) imply

$$\frac{l_j}{l_\ell}\frac{dl_\ell}{dl_j} = \frac{m_j}{m_\ell}\frac{dm_\ell}{dm_j} = \beta \text{ for all } \ell \neq j. \tag{37}$$

Given this, $l_{-j} \doteq \sum_{\ell \neq j} l_\ell$ and (18), the first-order condition (19) for country

j changes into

$$\frac{d\Phi^j}{dl_j} = \frac{\partial\Phi^j}{\partial l_j} + \sum_{\ell\neq j}\frac{\partial\Phi^j}{\partial l_\ell}\frac{dl_\ell}{dl_j} = \frac{\partial\Phi^j}{\partial l_j} + \beta\sum_{\ell\neq j}\frac{\partial\Phi^j}{\partial l_\ell}\frac{l_\ell}{l_j} = \frac{\partial\Phi^j}{\partial l_j} + (n-1)\beta\frac{\partial\Phi^j}{\partial l_\ell}\frac{l_\ell}{l_j}$$
$$= \frac{\partial\Phi^j}{\partial l_j} - (n-1)\beta\delta_j p_j^\sigma A_j^{\sigma\gamma_j}[(1-\tau)l_j + bL]^\sigma (l_j + l_{-j})^{-\delta-1}\frac{l_\ell}{l_j}$$
$$= \left\{\frac{(1-\tau)\sigma}{(1-\tau)l_j + bL} - \frac{[1+(n-1)\beta]\delta}{l_j + l_{-j}}\right\}\frac{p_j^\sigma A^{\sigma\gamma_j}}{(l_j+l_{-j})^\delta}[(1-\tau)l_j + bL]^\sigma$$
$$- \lambda[\Gamma^j(l_{-j},\gamma_j+1,T) - \Gamma^j(l_{-j},\gamma_j,T)] = 0. \tag{38}$$

Inserting (20) and (21) into the first-order condition (38) yields

$$(A^\sigma - 1)\lambda = \left\{\frac{\sigma}{l_j + bL/(p_j-\tau)} - \frac{[1+(n-1)\beta]\delta}{l_j + l_{-j}}\right\}\big[\rho + (1-A^\sigma)\lambda(L - l_j)\big]. \tag{39}$$

Noting (24), the equilibrium condition (39) takes the form

$$1 = \left\{(1-\tau)\sigma - \left[\frac{1}{n} + \Big(1 - \frac{1}{n}\Big)\beta\right]\delta\right\}\left[\left(\frac{\rho/\lambda}{A^\sigma - 1} - L\right)\frac{1}{l} - 1\right].$$

Solving for l leads to the equilibrium level of emissions as:

$$l = \left\{\frac{1}{(1-\tau)\sigma - [1/n + (1-1/n)\beta]\delta} - 1\right\}^{-1}\left(\frac{\rho/\lambda}{A^\sigma - 1} - L\right). \tag{40}$$

The central government sets the tax parameter τ so that this Pareto optimum is established, $l = l^*$. From (40) and (36) it then follows that

$$(1-\tau)\sigma - [1/n + (1-1/n)\beta]\delta = \sigma - \delta$$

and $\tau = (1-\beta)(1-1/n)\delta/\sigma$. This result can be rephrased as follows:

Proposition 2 *The optimal emission tax for country j is given by*

$$\tau w_j = (1-\beta)\Big(1 - \frac{1}{n}\Big)\frac{\delta}{\sigma}w_j,$$

where w_j is the wage in that country. This tax is the lower (higher) than in the tax in proposition 1, if a local government anticipates that the others will respond to its increase in emissions by increasing (decreasing) their emissions, $\beta \in (0,1)$ ($\beta \in (-\infty, 0)$).

6 Pollution

In the basic model, emissions had no long-term effects on welfare. If the level of emissions is increased for a period, then it decreases welfare only for that period but no longer. I now introduce long-term effects through pollution and assume $\nu > 0$.

6.1 The local governments

Noting (3) and (12), the accumulation of pollution (4) takes the form

$$\dot{P} = l_j + l_{-j} - h(X/P)P, \quad X = \prod_{\iota=1}^{n} a_\iota^{1/n}. \tag{41}$$

The government in country j maximizes the utility of country j's representative household (6) subject to (7), (8), (12), (13), (15) and (41), holding the output price p_j, the tax parameter τ, the subsidy parameter b, abatement activities X, and the emissions by the other countries, $l_{-j} \doteq \sum_{\iota \neq j} m_\iota = \sum_{\iota \neq j} l_\iota$, constant. The value of the optimal program starting at time T for country j is then

$$\begin{aligned}
&\Gamma^j(l_{-j}, X, \gamma_j, P, \tau, b, p_j, T) \\
&= \max_{(c_j, l_j, z_j) \text{ s.t. } (8),(12),(13),(15),(4)} E \int_T^\infty c_j^\sigma m^{-\delta} P^{-\nu} e^{-\rho(t-T)} dt \\
&= \max_{(c_j, l_j, z_j) \text{ s.t. } (8),(13),(15),(4)} E \int_T^\infty c_j^\sigma (l_j + l_{-j})^{-\delta} P^{-\nu} e^{-\rho(t-T)} dt \\
&= \max_{(l_j, z_j) \text{ s.t. } (8),(13),(4)} E \int_T^\infty p_j^\sigma A^{\sigma\gamma_j} [(1-\tau)l_j + bL]^\sigma (l_j + l_{-j})^{-\delta} P^{-\nu} e^{-\rho(t-T)} dt.
\end{aligned} \tag{42}$$

The Bellman equation corresponding to the optimal program obtains

$$\begin{aligned}
&\rho\Gamma^j(l_{-j}, X, \gamma_j, P, \tau, b, p_j, T) \\
&= \max_{(l_j, z_j) \text{ s.t. } (8)} \Psi^j(z_j, l_j, l_{-j}, X, \gamma_j, P, \tau, b, p_j, T) \\
&= \max_{l_j} \Psi^j(L - l_j - a_j, l_j, l_{-j}, X, \gamma_j, P, \tau, b, p_j, T),
\end{aligned} \tag{43}$$

where

$$
\begin{aligned}
&\Psi^j(z_j, l_j, l_{-j}, X, a_j, \gamma_j, P, \tau, b, p_j, T) \\
&= \frac{p_j^\sigma A^\sigma \gamma_j}{(l_j + l_{-j})^\delta P^\nu}[(1-\tau)l_j + a_j + bL]^\sigma + \frac{\partial \Gamma^j}{\partial P}(l_{-j}, X, a_j, \gamma_j, P, \tau, b, p_j, T)\dot{P} \\
&\quad + \lambda z_j \big[\Gamma^j(l_{-j}, X, a_j, \gamma_j + 1, P, \tau, b, p_j, T) - \Gamma^j(l_{-j}, X, a_j, \gamma_j, P, \tau, b, p_j, T)\big]\Big\}.
\end{aligned}
$$

This and (41) lead to the first-order condition

$$
\begin{aligned}
\frac{\partial \Psi^j}{\partial l_j} &= \left[\frac{(1-\tau)\sigma}{(1-\tau)l_j + a_j + bL} - \frac{\delta}{l_j + l_{-j}}\right]\frac{p_j^\sigma A^{\sigma\gamma_j}}{(l_j + l_{-j})^\delta}[(1-\tau)l_j + a_j + bL]^\sigma P^{-\nu} \\
&\quad - \lambda[\Gamma^j(l_{-j}, X, a_j, \gamma_j + 1, P, \tau, b, p_j, T) - \Gamma^j(l_{-j}, X, a_j, \gamma_j, P, \tau, b, p_j, T)] \\
&\quad + \frac{\partial \Gamma^j}{\partial P} = 0. \qquad (44)
\end{aligned}
$$

To solve the dynamic program, I try the solution that the value of the program, Γ^j, is in fixed proportion $\vartheta_j > 0$ to instantaneous utility:

$$
\Gamma^j(l_{-j}, X, \gamma_j, P, \tau, b, p_j, T) = \vartheta_j p_j^\sigma A^{\sigma\gamma_j}[(1-\tau)l_j + a_j + bL]^\sigma (l_j + l_{-j})^{-\delta} P^{-\nu}. \qquad (45)
$$

This implies

$$
\begin{aligned}
&\Gamma^j(l_{-j}, X, a_j, \gamma_j + 1, P, \tau, b, p_j, T)/\Gamma^j(l_{-j}, X, a_j, \gamma_j, P, \tau, b, p_j, T) = A^\sigma, \\
&\partial \Gamma^j/\partial P = -\nu \Gamma^j/P. \qquad (46)
\end{aligned}
$$

Inserting (45) and (46) into the Bellman equation (43) yields

$$
1/\vartheta_j = \rho + (1 - A^\sigma)\lambda(L - l_j - a_j) + \nu \dot{P}/P > 0. \qquad (47)
$$

Inserting (45) and (46) into the first-order condition (44) yields

$$
\begin{aligned}
(A^\sigma - 1)\lambda &= \left[\frac{(1-\tau)\sigma}{(1-\tau)l_j + a_j + bL} - \frac{\delta}{l_j + l_{-j}}\right]\frac{1}{\vartheta_j} - \frac{\nu}{P} \\
&= \left[\frac{\sigma}{l_j + (a_j + bL)/(1-\tau)} - \frac{\delta}{l_j + l_{-j}}\right]\left[\rho + (1 - A^\sigma)\lambda(L - l_j - a_j) + \nu\frac{\dot{P}}{P}\right] \\
&\quad - \nu/P. \qquad (48)
\end{aligned}
$$

With symmetry throughout $j = 1, ..., n$, $l_j = l$, $l_{-j} = (n-1)l$ and $a_j = a = X$ holds true. Noting this, the budget constraint (16) changes into

$$
b = \left(\tau \sum_{j=1}^n w_j l_j - \sum_{j=1}^n w_j a_j\right) \Bigg/ \left(L \sum_{j=1}^n w_j\right) = \frac{\tau l - a}{L},
$$

and the equilibrium condition (48) takes the form

$$1=\left[(1-\tau)\sigma-\frac{\delta}{n}\right]\left[\left(\frac{\rho/\lambda}{A^{\sigma}-1}-L+a\right)\frac{1}{l}+1+\frac{\nu/\lambda}{A^{\sigma}-1}\frac{\dot{P}}{Pl}\right]-\frac{\nu/\lambda}{A^{\sigma}-1}\frac{1}{P}. \tag{49}$$

I consider this equilibrium only in the stationary state where the union has attained its equilibrium level of resource, $\dot{P}=0$ and

$$hP=l_j+l_{-j}=nl. \tag{50}$$

Inserting this into (49), I obtain

$$\begin{aligned}\frac{1}{(1-\tau)\sigma-\delta/n}&=\left(\frac{\rho/\lambda}{A^{\sigma}-1}-L+a\right)\frac{1}{l}+1-\frac{h\nu/\lambda}{A^{\sigma}-1}\frac{1}{nl}\frac{1}{(1-\tau)\sigma-\delta/n}\\&=\left\{\frac{1/\lambda}{A^{\sigma}-1}\left[\rho-\frac{h\nu/n}{(1-\tau)\sigma-\delta/n}\right]-L+a\right\}\frac{1}{l}+1.\end{aligned}$$

Solving for l leads to the equilibrium level of emissions in the stationary state:

$$l=\left[\frac{1}{(1-\tau)\sigma-\delta/n}-1\right]^{-1}\left\{\frac{1/\lambda}{A^{\sigma}-1}\left[\rho-\frac{h\nu/n}{(1-\tau)\sigma-\delta/n}\right]+a-L\right\}. \tag{51}$$

6.2 The central government

Noting (12) and (26), the welfare of the representative household in the union takes the form

$$U(c,m,T)=\int_T^{\infty}c^{\sigma}m^{-\delta}P^{-\nu}e^{-\rho(t-T)}dt=\int_T^{\infty}A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}P^{-\nu}e^{-\rho(t-T)}dt, \tag{52}$$

The central government maximizes this welfare by labor inputs in production and abatement activities, (l,a), subject to technological change (27) and the dynamics of pollution (41). Noting (26), the value of the optimal program starting at time T for the central government is

$$\Gamma(\gamma,P,T)=\max_{(l,a)\text{ s.t. }(27),(41)}E\int_T^{\infty}A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}P^{-\nu}e^{-\rho(t-T)}dt. \tag{53}$$

The Bellman equation corresponding to this optimal program obtains

$$\begin{aligned}\rho\Gamma(\gamma,P,T) &= \max_{l,a}\Big\{A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}P^{-\nu}+\frac{\partial\Gamma}{\partial P}\dot{P} \\ &\qquad +\lambda(L-l-a)\big[\Gamma(\gamma+1,P,T)-\Gamma(\gamma,P,T)\big]\Big\} \\ &= \max_{l,a}\Big\{A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}P^{-\nu}+\frac{\partial\Gamma}{\partial P}\Big[nl-h\Big(\frac{a}{P}\Big)P\Big] \\ &\qquad +\lambda(L-l-a)\big[\Gamma(\gamma+1,P,T)-\Gamma(\gamma,P,T)\big]\Big\}. \qquad (54)\end{aligned}$$

This leads to the first-order conditions

$$\begin{aligned}\frac{\partial\{\}}{\partial l} &= (\sigma-\delta)A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta-1}P^{-\nu}+n\frac{\partial\Gamma}{\partial P}-\lambda[\Gamma(\gamma+1,T)-\Gamma(\gamma,T)]=0, \\ \frac{\partial\{\}}{\partial a} &= -\frac{\partial\Gamma}{\partial P}h'-\lambda[\Gamma(\gamma+1,T)-\Gamma(\gamma,T)]=0. \qquad (55)\end{aligned}$$

To solve the dynamic program, I try the solution that the value of the program, Γ, is in fixed proportion $\vartheta>0$ to instantaneous utility:

$$\Gamma(\gamma,T)=\vartheta A^{\sigma\gamma}n^{-\sigma-\delta}l^{\sigma-\delta}P^{-\nu}. \qquad (56)$$

This implies

$$\Gamma(\gamma+1,T)/\Gamma(\gamma,T)=A^{\sigma}, \quad \partial\Gamma/\partial P=-\nu\Gamma/P. \qquad (57)$$

I consider this equilibrium only in the stationary state where $\dot{P}=0$ and (50) hold true. Inserting (32) and (57) into the Bellman equation (18) yields

$$1/\vartheta=\rho+(1-A^{\sigma})\lambda(L-l-a)>0. \qquad (58)$$

Inserting (50), (56) and (57) into the upper first-order condition (55), one obtains

$$\begin{aligned}(A^{\sigma}-1)\lambda &= \frac{\sigma-\delta}{l}\frac{1}{\vartheta}-\frac{n}{\Gamma}\frac{\partial\Gamma}{\partial P}=\frac{\sigma-\delta}{l}\big[\rho+(1-A^{\sigma})\lambda(L-l-a)\big]-n\frac{\nu}{P} \\ &= \Big\{(\sigma-\delta)\big[\rho+(1-A^{\sigma})\lambda(L-l-a)\big]-h\nu\Big\}\frac{1}{l}\end{aligned}$$

and

$$l=(\sigma-\delta)\bigg(\frac{\rho/\lambda}{A^{\sigma}-1}-L+a\bigg)-\frac{h\nu/\lambda}{A^{\sigma}-1}. \qquad (59)$$

Given (35), I solve for the Pareto-optimal solution

$$\begin{aligned} l^* &\doteq \Big(\frac{1}{\sigma-\delta}-1\Big)^{-1}\Big(\frac{\rho/\lambda}{A^\sigma-1}-L+a-\frac{1}{\sigma-\delta}\frac{h\nu/\lambda}{A^\sigma-1}\Big) \\ &= \Big(\frac{1}{\sigma-\delta}-1\Big)^{-1}\Big[\frac{1/\lambda}{A^\sigma-1}\Big(\rho-\frac{h\nu}{\sigma-\delta}\Big)-L+a\Big]. \end{aligned} \tag{60}$$

The central government sets the tax parameter τ so that this Pareto optimum is established, $l = l^*$. From (51) and (60) it then follows that

$$\tau = \Big(1-\frac{1}{n}\Big)\frac{1}{\sigma}\Big[\delta+(1+\delta-\sigma)h\nu\Big(\rho-h\nu-\lambda\frac{L-a}{A^\sigma-1}\Big)^{-1}\Big]. \tag{61}$$

Thus, the following result is obtained:

Proposition 3 *The optimal emission tax for country j that leads to the Green Golden Rule is given by τA^{γ_j}, where τ is determined by* (61)*. This tax is the smaller, the more decentralized the union is (i.e. the bigger n). When the number of countries is large, $n \to \infty$, the tax is equal to*

$$\frac{\delta}{\sigma}A^{\gamma_j}\Big[\delta+(1+\delta-\sigma)h\nu\Big(\rho-h\nu-\lambda\frac{L-a}{A^\sigma-1}\Big)^{-1}\Big].$$

Inserting (5), (50), (56) and (57) into the lower first-order condition (55), one obtains

$$(A^\sigma-1)\lambda = -\frac{1}{\Gamma}\frac{\partial\Gamma}{\partial P}h' = \frac{\nu}{P}h' = \nu\frac{\varepsilon h}{X} = \nu\frac{\varepsilon h}{a}.$$

Solving for a yields the following result:

Proposition 4 *The optimal labor input devoted to abatement activities is*

$$a^* = \frac{\nu\varepsilon h}{(A^\sigma-1)\lambda}.$$

7 Generalizations

The model can be extended in the following directions:

1. In the basic model, all countries are of equal size. One can assume that some of the countries are relatively bigger than the others. In such a case, the bigger countries can be considered as *Stackelberg leaders*, which take the optimal responses of the others into account in their optimization, and the smaller countries as *Stackelberg followers*, which take the levels of emissions in the bigger countries as given.

   Technical difficulty: Asymmetry in the model leads easily to multiple equilibria which complicates the interpretation of the results.

2. In the basic model, technological knowledge does not spill over to other countries. One can introduce technological diffusion so that investment in R&D in one country improves productivity also in the other countries. This creates an additional externality.

   Technical difficulty: The strategic inter-dependence of the countries becomes more complex.

3. In the basic model, all income is consumed. One can introduce an asset which is accumulated by private saving. This asset could be capital or internationally-traded bonds.

   Technical difficulty: the accumulation of the asset is a differential equation which enters as an constraint in the optimization and thereby complicates the construction of the value function.

4. In the basic model, output is made from labor only. Output can be produced from both labor and capital according to a neoclassical technology. Emissions can be complementary to labor, capital or output. Following Turnovsky (1993, 1995, 1999), Smith (1996), Corsetti (1997) and Soretz (2003), one can also introduce productivity shocks that follows a Wiener process.

   Technical difficulty: Nonlinearity in the production function complicates the construction of the value function. The optimization of a Poisson process will be replaced by that of a Poisson-Wiener process, to which it is challenging to form a value function.

8 Conclusions

This paper introduces a model that provides a variety of applications for mathematicians who are specialized in optimal control theory as well as means for economists to solve practical problems in economic integration and environmental policy. The results are expressed in the form of Pareto-optimal emission taxes. These leads to the best sustainable configuration (e.g. the Green Golden Rule) in the steady state. The optimal taxes can be estimated by a data over wages and household preferences.

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