Immaterial Property Rights, Product Cycles and Non-Diversifiable Risk

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Abstract

In this study, I examine immaterial property rights in an economy where R&D firms innovate and imitate and households face non-diversifiable risk. Some property rights postpone the expected time an innovation will be imitated (e.g. increase the "length" of an innovation), while the others protect the imitator's profits after a successful imitation (i.e. increase the "width" of an innovation). The main findings are as follows. Property rights that generate "short" and "wide" innovations also speed up economic growth. The smaller the households' rate of risk aversion, the "longer" and "narrower" the welfare-maximizing innovations.

Journal of Economic Literature: $L11$, $L16$, $O31$, $O34$

Keywords: Imitation, property rights, product cycles

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1 Introduction

In endogenous growth theory with uncertainty, the assumption of the diversifiable risk simplifies models considerably. Firms can then borrow any amount for R&D at a given market interest rate and households are protected from uncertainly through diversification in the market portfolio. There are however good reasons to suspect that such an assumption may distort results. First, it is in contradiction with the whole literature of venture capital where firms cannot borrow without collateral and use their immaterial property (e.g. new ideas) as collateral.[1] Second, Wälde (1999a, 1999b) shows that with non-diversifiable risk investment decisions are made by households rather than firms which changes the equilibrium conditions substantially. In this study, I assume that households cannot wholly diversify their investment risk. Firms must then finance their R&D through issuing shares and households purchasing these shares face the uncertainly associated with investment. It is instructive to see how the protection of immaterial property rights affects economic growth in this case.

In a growth model of creative destruction, firms can step forward in the quality ladders of technology by investing in R&D. It is assumed that a firm's technology is a random variable so that the probability of its improvement in any time is an increasing function of labor devoted to R&D.[2] If imitation is possible, then economic growth is subject to product cycles as follows. Through the development of new products, an innovator achieves a temporary advantage earning monopoly profits. This advantage ends when an imitator succeeds in copying the innovation, enters the market and starts competing with the innovator. The use of a product cycle model allows us to distinguish between two kind of property rights: those that (i) prolong the expected time an innovation will be imitated (e.g. the "length" of an innovation), and those that (ii) protect the imitator's profits after a successful imitation (i.e. the "width" of an innovation).

The basis ideas of this paper are the following. At the level of the whole economy, innovations generate economic growth, but at the level of a single firm they are mainly a vehicle of taking over the market. In that case,

[1]A nice summary of this literature is given by Gompers and Lerner (1999).

[2]Cf. Grossman and Helpman (1991) (in ch. 4), and Aghion and Howitt (1998).

oligopolists benefit, but monopolies do not benefit from an innovation, and a bigger proportion of oligopoly industries leads to greater innovative R&D and faster economic growth. If the probability of a successful imitation is small (i.e. innovations are "long"), or if an imitator's profit is small (i.e. innovations are "narrow"), then the expected profit for an imitative investment is low. In that case, households invest in innovative rather than imitative R&D projects, firms spend more time as a monopoly and less time as an innovating oligopoly in their product cycle, and the growth rate decreases.

The literature on the length and width of patents usually assumes diversifiable risk.[3] In that case, the trade-off between the length and width of patents can be judged by the present value of investment projects and households' rate of risk aversion has no significance. In a product-cycle model with non-diversifiable risk, there is a trade-off between length and width of innovations through the proportions of innovating and imitating firms in the entire product cycle. In that case, the socially optimal form of innovations is a function of the rate of risk aversion.

The analysis is based on my earlier work on competition policy [Palokangas (2006)] that combines Wälde's (1999a, 1999b) growth model with non-diversifiable risk with a product cycle model with cumulative technology.[4] The rest of this paper is organized as follows. Section 2 presents the structure of the model and section 3 proves the existence of the equilibrium. Section 4 considers the growth effects of property rights and section 5 the socially optimal length and width of innovations.

2 The model

The basis structure of the model is the following:

1. Labor is homogeneous. There exists a fixed number N of households, each of which supplies one labor unit. With a competitive labor market, labor supply N is equal to labor in production, x, and in R&D, l:

$$N = x + l. \tag{1}$$

[3] For the survey of this literature, cf. Denicolo (1996), Takalo (1998), and Mukherjee and Pennings (2004).

[4] Cf. Segerstrom (1991) and Mukoyama (2003).

2. Because in the model there is no money that would pin down the nominal price level at any time, it is convenient to normalize the households' total spending in consumption at unity:

$$Py = 1, \quad y \doteq \sum_{\iota=1}^{N} C_\iota, \tag{2}$$

where y aggregate consumption, P the consumption price and C_ι consumption by household $\iota \in \{1, ..., N\}$.

3. Because R&D firms finance their expenditure by issuing shares and the households save only in these shares, aggregate income is equal to the value of consumption, Py, plus wages paid in R&D, wl, where w is the wage and l labor devoted to R&D. Given (2), it is then true that

$$\sum_{\iota=1}^{N} A_\iota = wl + Py = wl + 1, \tag{3}$$

where A_ι is the income of household $\iota \in \{1, ..., N\}$ and $\sum_{\iota=1}^{N} A_\iota$ aggregate income.

4. All households are risk averters and share the same preferences. The utility for a single household $\iota \in \{1, ..., N\}$ from an infinite stream of consumption C_ι beginning at time T is given by

$$U(C_\iota, T) = E \int_T^\infty C_\iota^\sigma e^{-\rho(t-T)} dt \text{ with } 0 < \sigma < 1 \text{ and } \rho > 0, \tag{4}$$

where t is time, E the expectation operator, ρ the rate of time preference and $(1 - \sigma)$ is the constant relative risk aversion.

5. Competitive firms produce the consumption good from a great number of intermediate goods that are evenly placed over the limit $[0, 1]$. Each intermediate good $j \in [0, 1]$ is a composite good of the products of a number n_j of firms in industry $j \in [0, 1]$. The first firm is always an innovator, while the rest $\kappa = 2, ..., n_j$ are imitators. The entry of new firms through successful imitations decreases the innovator's market share.[5] Aggregate consumption is then produced from the products $x_{j\kappa}$

[5] I ignore the possibility that firms $3, ..., n_j$ crowd out the market share of the second firm, for simplicity. Since in equilibrium there will be only two producers per industry, this would complicate the model without any change in the results.

of all intermediate-good firms $\kappa \in \{1, ..., n_j\}$ in all industries $j \in [0, 1]$ through Cobb-Douglas technology:

$$\log y = \int_0^1 \log[B_j x_j] dj + \log \int_0^1 J(n_j) dj, \quad J'(n_j) > 0, \quad J(1) = 0,$$
$$\log x_j = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right) \log x_{j1} + \sum_{\kappa=2}^{n_j} \epsilon_\kappa \log x_{j\kappa}, \quad \epsilon_\kappa > \epsilon_{\kappa+1} \text{ for all } \kappa, \tag{5}$$

where ϵ_κ is a constant, B_j is the productivity parameter, x_j the quantity of intermediate good j and n_j the number of firms in industry j. The market proportion of the first firm is characterized by $1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa$ and that of firm $\kappa > 1$ by ϵ_κ. The function $J(n_j)$ with $J' > 0$ characterizes the property that a wider variety of products (i.e. a bigger n_j) in any industry provides more services to households and thereby increases a household's welfare.[6]

6. The productivity parameter in industry j [Cf. (5)] is determined by

$$B_j \doteq \mu^{\tau_j}, \quad \mu > 1, \tag{6}$$

where μ is a parameter and τ_j an index of technology in industry j. The invention of a new technology in industry j raises the index τ_j by one and the level of productivity by $\mu > 1$.

7. Each firm doing innovative R&D obtains a spillover of technological knowledge that is in fixed proportion to total labor devoted to R&D in the economy, l.[7] When firm κ in industry j innovates, its technological change follows a Poisson process $q_{j\kappa}$ in which the arrival rate of innovations, $\Lambda_{j\kappa}$, is given by

$$\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\nu} l^\nu, \quad \lambda > 0, \quad 0 < \nu < 1, \tag{7}$$

[6] In general, the property that product variety increases welfare is commonly established through a CES production function. In this study, the replacement of the Cobb-Douglas function (5) by a CES function would excessively complicate the analysis.

[7] This spillover effect ensures the existence of an equilibrium. If there were no spillover effect, then $\nu = 0$ holds in (7), the average product of labor would be constant both in innovative R&D and in imitative R&D, and there would be no interior solution for a household's maximization problem (see section 3.5 and Appendix, especially equations (56) and (57)). With spillover effect $\nu > 0$, the average product of labor is decreasing in innovative R&D and a household has an interior solution.

where $l_{j\kappa}$ is the firm's own labor input, l the spillover of technological knowledge, and λ and ν are constants. During a short time interval dt, there is an innovation $dq_{j\kappa} = 1$ in firm κ with probability $\Lambda_{j\kappa}dt$, and no innovation $dq_{j\kappa} = 0$ with probability $1 - \Lambda_{j\kappa}dt$. The parameter λ characterizes the quality of environment for innovation: the higher λ, the higher the productivity of investment in innovative R&D.

8. When firm κ in industry j imitates, its technological change follows a Poisson process $Q_{j\kappa}$ in which the arrival rate of imitations is in fixed proportion γ to the firm's own labor input $l_{j\kappa}$:

$$\Gamma_{j\kappa} = \gamma l_{j\kappa} \text{ for } j \in \Theta, \quad \gamma > 0. \tag{8}$$

During a short time interval dt, there is an imitation $dQ_{j\kappa} = 1$ with probability $\Gamma_{j\kappa}dt$, and no imitation $dQ_{j\kappa} = 0$ with probability $1 - \Gamma_{j\kappa}dt$.

9. The "width" and "length" of the innovations can be measured as follows. The larger the innovator's market share $1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa$, the less its profit falls in the advent of a successful imitation and the "wider" the innovations. The lower the productivity of imitative R&D – i.e. the smaller γ in (8) – the more time a successful imitation will take and the "longer" the innovations. Through the regulation of immaterial property rights, the government determines both ϵ and γ.

10. Each R&D firm distributes its profit among those who had financed it in proportion to their investment in the firm. Because both innovation and imitation follow a Poisson process, the values of shares in R&D projects are random variables and household $\iota \in \{1, ..., N\}$ maximizes its utility (4) subject to the random development of these values.

3 The steady-state equilibrium

In this section, I prove the existence of the following equilibrium:

Definition. *The economy is in a stationary-state equilibrium, if the following properties are satisfied:*

(i) The industries j are run either by monopolies ($n_j = 1$) or duopolies ($n_j = 2$). In a monopoly industry outsiders not producing in the industry imitate, and in a duopoly industry the incumbent duopolists innovate.

(ii) The proportions of monopoly and duopoly industries in the economy (denoted α and β, respectively) are constants over time. Every time a new superior-quality product is discovered in some industry, changing this from a duopoly into a monopoly, imitation must occur in some other industry, changing this from a monopoly into a duopoly.

(iii) The profits of a typical monopoly and a typical duopolist are constant over time.

(iv) The wage w and the total output x of a typical industry are constants over time.

(v) A typical innovating firm's labor input in R&D, ℓ_β, and a typical imitating firm's labor input in R&D, ℓ_α, are constants over time.

3.1 The production of the consumption good

The representative consumption-good firm maximizes its profit

$$\Pi^c \doteq Py - \int_{j\in[0,1]} \sum_{\kappa=1}^{n_j} p_{j\kappa} x_{j\kappa} dj$$

subject to technology (5), given the output price P and the input prices $p_{j\kappa}$. Noting (2), this implies

$$\Pi^c = 0, \quad p_{j1} = P\frac{\partial y}{\partial x_{j1}} = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right) P\frac{y}{x_{j1}} = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right)\frac{1}{x_{j1}},$$
$$p_{j\kappa} = P\frac{\partial y}{\partial x_{j\kappa}} = \epsilon_\kappa P\frac{y}{x_{j\kappa}} = \frac{\epsilon_\kappa}{x_{j\kappa}} \text{ for } \kappa > 1. \tag{9}$$

3.2 The production of the intermediate goods

All intermediate-good firms produce one unit of their output from one labor unit. The product of the newest generation provides exactly the constant $\mu > 1$ times as many services as that of earlier generation. A firm of earlier generation earns the profit $\Pi_j^o = (p_j^o - w)x_j^o$, where p_j^o is its output price and x_j^o its output. Every firm with the newest technology pushes and keeps the firms with older technology out of the market by choosing its price p_j so that these earn no profit, $\Pi_j^o = 0$ and $p_j^o = w$. This yields $p_j/\mu = p_j^o = w$. Noting this and (9), one obtains the equilibrium conditions:

$$
\begin{aligned}
& p_j = \mu w \text{ for all } j \text{ and } \kappa, \quad x_{j\kappa} = \frac{\epsilon_\kappa}{p_{j\kappa}} = \frac{\epsilon_\kappa}{\mu w} \text{ and} \\
& \Pi_{j\kappa} = (p_{j\kappa} - w)x_{j\kappa} = (1 - 1/\mu)p_{j\kappa}x_{j\kappa} = \epsilon_\kappa \Pi \text{ for } \kappa > 1 \text{ and all } j, \\
& x_{j1} = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right)\frac{\epsilon_\kappa}{p_{j1}} = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right)\frac{\epsilon_\kappa}{\mu w} \text{ for all } j, \\
& \Pi_{j1} = (p_{j1} - w)x_{j1} = (1 - 1/\mu)p_{j1}x_{j1} = \left(1 - \sum_{\kappa=2}^{n_j} \epsilon_\kappa\right)\Pi,
\end{aligned}
\tag{10}
$$

where $\Pi \doteq 1 - 1/\mu > 0$ is a constant. Thus, *the property (iii) of a stationary-state equilibrium is proven.*

Noting (10), I conclude the following:

(a) The innovator will earn the constant profit Π as long as it remains the monopoly producer in the industry. Because a household holds the share of all firms in its same portfolio, it does not invest in innovative R&D in the monopoly industries.

(b) If a household invests in imitative R&D to enter a monopoly industry j, then its prospective profit is Π_{j2}, but if it does that (with the same cost) to enter an industry j with $\kappa > 1$ producers, then its prospective profit is $\Pi_{j,\kappa+1} < \Pi_{j2}$. Thus, it invest in imitative R&D only to enter a monopoly industry, but not to enter an oligopoly industry. This means that there can be at most two producers in an industry.

From (a) and (b) above it follows that in equilibrium there are only monopoly industries with imitative R&D or duopoly industries with innovative R&D. Thus, *the property (i) of a stationary-state equilibrium is proven.*

I denote the set of monopoly industries by $\Theta \subset [0,1]$. The relative proportion of duopoly industries, β, and the relative proportion of monopoly industries, α, are then given by

$$
\beta \doteq \int_{j \notin \Theta} dj, \quad \alpha = \int_{j \in \Theta} dj = 1 - \beta. \tag{11}
$$

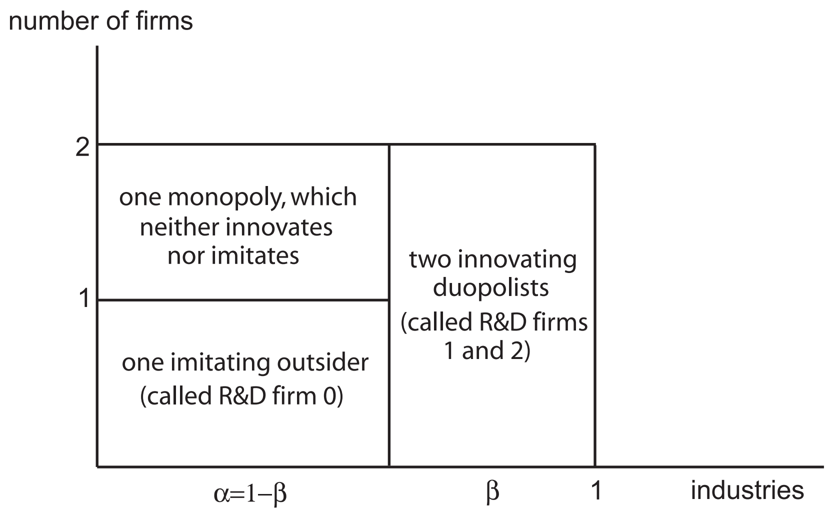

Figure 1: Competition and the number of firms in the economy.

Thus, *the property (ii) of a stationary-state equilibrium is proven.*

3.3 Employment, output and growth

Noting (10) and (11) and summing up throughout industries $j \in [0,1]$ yield

$$x_j = x \doteq \int_{k\in[0,1]} x_k dj = \frac{1}{\mu w}. \tag{12}$$

Given this and (1), the wage w becomes a function of total labor in R&D, l:

$$w(l) = \frac{1}{\mu x} = \frac{1}{\mu}\frac{1}{N-l}, \quad w' > 0. \tag{13}$$

Higher demand for labor in R&D (i.e. a bigger l) raises the wage w. This *proves the property (iv) of a stationary-state equilibrium.*

According to the properties (i) and (ii) of a stationary-state equilibrium, duopolists labeled 1 and 2 innovate and none imitates in duopoly industries $j \notin \Theta$, while outsiders imitate and none innovates in monopoly industries $j \in \Theta$. Because according to technology (8) imitation yields constant returns to scale, all outsiders in monopoly industry $j \in \Theta$ behave as if there were a single outsider firm labeled 0. The structure of industries is given by Fig. 1.

Total employment in R&D is given by

$$l \doteq \int_{j\notin\Theta}(l_{j1}+l_{j2})dj+\int_{j\in\Theta}l_j dj. \tag{14}$$

Given (6), the average productivity in the economy, $B(\{t_k\})$, is a function of the technologies τ_j of all industries $j \in [0,1]$ as follows:

$$\log\, B(\{t_k\}) \doteq \int_0^1 \log\, B_j dj = (\log\mu)\int_0^1 \tau_j dj. \tag{15}$$

The arrival rate of innovations in industry $j \notin \Theta$ is the sum of the arrival rates of both duopolists in that industry, $\Lambda_{j1}+\Lambda_{j2}$ [Cf., (7)]. From (1), (5), (12) and (15) it follows that aggregate consumption y is given by

$$y = xB(\{t_k\}) = (N-l)B(\{t_k\}). \tag{16}$$

Because only duopoly industries $j \notin \Theta$ innovate, then the average growth rate of the average productivity $B(\{t_k\})$ in the stationary state is given by

$$g \doteq (\log\mu)\int_0^1 \Pr(\tau_j \text{ increases by one})dj = (\log\mu)\int_{j\notin\Theta}(\Lambda_{j1}+\Lambda_{j2})dj, \tag{17}$$

where $\Pr(\cdot)$ denotes the probability.

3.4 Innovation and imitation

In monopoly industry $j \in \Theta$ outsider 0 and in industry $j \notin \Theta$ duopolists 1 and 2 issue shares to finance their labor expenditure in R&D. Because the households $\iota \in \{1,...,N\}$ invest in these shares, one obtains

$$\sum_{\iota=1}^{N} S_{\iota j0} = wl_{j0} \text{ for } j \in \Theta, \quad \sum_{\iota=1}^{N} S_{\iota j\kappa} = wl_{j\kappa} \text{ for } \kappa \in \{1,2\} \text{ and } j \notin \Theta, \tag{18}$$

where wl_{j0} is the imitative expenditure of outsider 0 in monopoly industry $j \in \Theta$, $wl_{j\kappa}$ the innovative expenditure of duopolist $\kappa \in \{1,2\}$ in industry $j \notin \Theta$, $S_{\iota j0}$ household ι's investment in outsider firm 0 in monopoly industry $j \in \Theta$, $S_{\iota j\kappa}$ household ι's investment in duopolist κ in industry $j \notin \Theta$, $\sum_{\iota=1}^{N} S_{\iota j0}$ aggregate investment in outsider firm 0 in monopoly industry $j \in \Theta$, and

$\sum_{\iota=1}^{N} S_{\iota j\kappa}$ aggregate investment in duopolist κ in industry $j \notin \Theta$. Household ι's relative investment shares in outsiders 0 and duopolists $\kappa \in \{1, 2\}$ are

$$i_{\iota j0} \doteq \frac{S_{\iota j0}}{wl_{j0}} \text{ for } j \in \Theta; \quad i_{\iota j\kappa} \doteq \frac{S_{\iota j\kappa}}{wl_{j\kappa}} \text{ for } j \notin \Theta. \tag{19}$$

When household ι has financed a successful R&D firm, it acquires the right to the firm's profit in proportion to its relative investment share. The profit sharing in the economy can then be characterized as follows:

$s_{\iota j\kappa}$ household ι's profit from duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ when the uncertainty in R&D is taken into account;

$i_{\iota j\kappa}$ household ι's investment share in duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ [Cf. (19)];

Π_{j1} the profit of duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ after innovation have changed it into a monopoly;

$\Pi_{j1} i_{\iota j\kappa}$ household ι's profit from duopolist $\kappa \in \{1, 2\}$ in industry $j \notin \Theta$ after innovation have changed this into a monopoly;

$s_{\iota j0}$ household ι's profit from outsider 0 in industry $j \in \Theta$ when the uncertainty in R&D is taken into account;

$i_{\iota j\kappa}$ household ι's investment share in outsider 0 in industry $j \in \Theta$ [Cf. (19)];

Π_{j2} the profit of outsider 0 in industry $j \in \Theta$ after imitation have changed it as the second duopolist;

$\Pi_{j2} i_{\iota j0}$ household ι's profit from outsider 0 in industry $j \in \Theta$ after imitation have changed it into the second duopolist.

The changes in the profits of firms in industry j are functions of the increments $(dq_{j1}, dq_{j2}, dQ_{j0})$ of Poisson processes (q_{j1}, q_{j2}, Q_{j0}) as follows:[8]

$$\begin{aligned} ds_{\iota j\kappa} &= (\Pi_{j1} i_{\iota j\kappa} - s_{\iota j\kappa}) dq_{j\kappa} - s_{\iota j\kappa} dq_{j(\zeta \neq \kappa)} \text{ when } j \notin \Theta; \\ ds_{\iota j0} &= (\Pi_{j2} i_{\iota j0} - s_{\iota j0}) dQ_{j0} \text{ when } j \in \Theta. \end{aligned} \tag{20}$$

[8] This extends the idea of Wälde (1999a, 1999b).

These functions can be explained as follows. If a household invests in innovating duopolist κ in industry $j \notin \Theta$, then, in the advent of a success for that duopolist, $dq_{j\kappa} = 1$, the amount of its share holdings rises up to $\Pi_{j1} i_{\iota j\kappa}$, $ds_{\iota j\kappa} = \Pi i_{\iota j\kappa} - s_{\iota j\kappa}$, but in the advent of success for the other duopolist $\zeta \neq \kappa$, its share holdings in duopolist κ fall down to zero, $ds_{\iota j\kappa} = -s_{\iota j\kappa}$. If a household invests in imitating outsider 0 in monopoly industry $j \in \Theta$, then, in the advent of a success for firm 0, $dQ_{j0} = 1$, the amount of its share holdings rises up to $\Pi_{j2} i_{\iota j0}$, $ds_{\iota j0} = \pi_2 i_{\iota j0} - s_{\iota j0}$.

3.5 Households

Because investment in shares in R&D firms is the only form of saving in the model, the budget constraint of household ι is given by

$$A_\iota = PC_\iota + \int_{j\in\Theta} S_{\iota j0} dj + \int_{j\notin\Theta} (S_{\iota j1} + S_{\iota j2}) dj, \tag{21}$$

where A_ι is the household's total income, C_ι its consumption, P the consumption price, $S_{\iota j0}$ the household's investment in outsider firm 0 in monopoly industry $j \in \Theta$, and $S_{\iota j\kappa}$ the household's investment in duopolist κ in industry $j \notin \Theta$. Household ι's total income A_ι consists of its wage income w (the household supplies one labor unit), its profits $s_{\iota j1}$ from the monopoly in each industry $j \in \Theta$ and its profits $s_{\iota j1}$ and $s_{\iota j2}$ from the duopolists 1 and 2 in each industry $j \notin \Theta$. This yields

$$A_\iota = w + \int_{j\in\Theta} s_{\iota j1} dj + \int_{j\notin\Theta} (s_{\iota j1} + s_{\iota j2}) dj. \tag{22}$$

Household ι maximizes its utility (4) by its investment, $\{S_{\iota j0}\}$ for $j \in \Theta$ and $\{S_{\iota j1}, S_{\iota j2}\}$ for $j \notin \Theta$, subject to its budget constraint (21), the stochastic changes (20) in its profits, the composition of its income, (22), and the determination of its relative investment shares, (19), given the arrival rates $\{\Lambda_{j\kappa}, \Gamma_{j0}\}$, the wage w and the consumption price P. In the households' stationary equilibrium in which the allocation of resources is invariable across technologies, this maximization yields (see the Appendix):

$$\begin{matrix} l_{j\kappa} = \ell_\beta = \xi l & \text{for } j \notin \Theta, \\ l_{j0} = \ell_\alpha = \frac{1-2\beta\xi}{1-\beta} l & \text{for } j \in \Theta, \end{matrix} \quad \xi \doteq \mu^{\sigma/\nu} \left(\frac{\lambda}{\varepsilon}\right)^{1/\nu}, \quad \varepsilon \doteq \gamma\epsilon_2, \tag{23}$$

$$\rho + \frac{1-\mu^\sigma}{\log\,\mu} g = \Delta(l)\varepsilon, \quad \Delta' < 0, \tag{24}$$

$$g = (2\lambda \log \mu)\beta\xi^{1-\nu} l = (2\lambda \log \mu)\beta\mu^{(1/\nu-1)\sigma}(\lambda/\varepsilon)^{1/\nu-1} l, \tag{25}$$

where ℓ_α (ℓ_β) is employment per firm in imitative (innovative) R&D and ε the extent of property rights (= the "width" ϵ_2 times the "length" γ for an innovation). The results (23)-(25) can be explained as follows:

(23) With "wider" and "longer" innovations (i.e. a smaller $\varepsilon \doteq \gamma\epsilon_2$), households invest more in innovative R&D to escape the competition and ℓ_β/l rises. Because of the spillover effect,[9] inputs in innovative R&D, ℓ_β, and imitative R&D, ℓ_α, are proportional to total labor in R&D, l.

(24) A household's subjective discount factor $\rho + \frac{1-\mu^\sigma}{\log\,\mu} g$ is equal to the marginal rate of return to savings, $\Delta\varepsilon$, which is proportional to the extent of property rights, ε, and decreases with an increase in labor input l to R&D.

(25) The growth rate g is in fixed proportion to aggregate labor devoted to R&D, l. A higher number of innovating industries (i.e. a bigger β) increases this proportion and thereby promotes growth.

Given (23), *the property (v) of a stationary-state equilibrium is proven.* The equation (24) defines the function

$$l(g,\varepsilon), \quad \frac{\partial l}{\partial \varepsilon} = -\frac{\Delta}{\varepsilon\Delta'} > 0, \quad \frac{\varepsilon}{l}\frac{\partial l}{\partial \varepsilon} = -\frac{\Delta(l)}{l\Delta'(l)} \doteq \eta(l) > 0,$$

$$\frac{g}{l}\frac{\partial l}{\partial g} = \frac{1-\mu^\sigma}{\varepsilon \log \mu}\frac{g}{l\Delta'} = \frac{\mu^\sigma - 1}{\varepsilon \log \mu}\frac{\eta g}{\Delta} = \frac{\mu^\sigma - 1}{\log \mu} g \left[\rho + \frac{1-\mu^\sigma}{\log\,\mu} g\right]^{-1} \eta(l) > 0. \tag{26}$$

4 The product cycle

Given the property (ii) of the stationary-state equilibrium, the rate at which industries leave the group of duopoly industries $k \notin \Theta$ in a small interval dt, $\beta(\Lambda_{j1} + \Lambda_{j2})dt$, is then equal to the rate at which the industries leave the

[9] See assumption 7 in section 2.

group of monopoly industries $j \in \Theta$, $\alpha\Gamma_{j0}dt$ in that interval dt. This implies the equilibrium condition

$$\beta(\Lambda_{k1} + \Lambda_{k2}) = \alpha\Gamma_{j0} \text{ for } k \notin \Theta \text{ and } j \in \Theta. \tag{27}$$

From equations (11), (7), (8), (23) and (27) it follows that

$$1 = \frac{\Lambda_{k1} + \Lambda_{k2}}{\alpha\Gamma_{j0}/\beta} = \frac{\lambda(l_{k1}^{1-\nu} + l_{k2}^{1-\nu})l^\nu}{(1-\beta)\gamma l_{j0}/\beta} = \frac{2\lambda\ell_\beta^{1-\nu}l^\nu}{(1-\beta)\gamma\ell_\alpha/\beta} = \frac{2\lambda\xi^{1-\nu}/\gamma}{1/\beta - 2\,\xi}.$$

Solving for β from this equation yields

$$\beta \doteq \frac{1}{2}\Big[\xi + \frac{\lambda}{\gamma}\xi^{1-\nu}\Big]^{-1}. \tag{28}$$

From (23), (25) and (28) it follows that

$$\frac{g}{l(g,\varepsilon)} = (2\lambda\log\mu)\beta\xi^{1-\nu} = \frac{\lambda\log\mu}{\xi^\nu + \lambda/\gamma} = \frac{\log\mu}{\mu^\sigma/\varepsilon + 1/\gamma} \doteq \theta(\gamma,\varepsilon),$$
$$\partial\theta/\partial\gamma > 0, \quad \partial\theta/\partial\varepsilon > 0. \tag{29}$$

Unfortunately, the variable g appears in both sides of the equation (25), which makes the outcome ambiguous. For this reason, I assume the following stability property for the equation (25).[10] After a small perturbation $\theta l - g$, the actual growth rate of the economy, g, adjusts to its equilibrium level θl according to an increasing function δ of the perturbation $\theta l - g$:

$$dg/dt = \delta\Big(\theta(\gamma,\varepsilon)l(g,\varepsilon) - g\Big) \doteq \Upsilon(g,\gamma,\varepsilon), \quad \delta' > 0, \quad \delta(0) = 0, \tag{30}$$

where the stability of the system requires $\partial\Upsilon/\partial g < 0$. Noting (23), (26), (28), (29) and (30), we obtain

$$\frac{\partial\Upsilon}{\partial\gamma} = \underbrace{\delta'}_{+}\, l\underbrace{\frac{\partial\theta}{\partial\gamma}}_{+} > 0, \quad \frac{\partial\Upsilon}{\partial\varepsilon} = \underbrace{\delta'}_{+}\Big[l\underbrace{\frac{\partial\theta}{\partial\varepsilon}}_{+} + \theta\underbrace{\frac{\partial l}{\partial\varepsilon}}_{+}\Big] > 0.$$

Given this, (28) and (30), the equilibrium with $dg/dt = 0$ defines the function

$$g = G(\gamma,\varepsilon), \quad \frac{\partial G}{\partial\gamma} = -\frac{\partial\Upsilon}{\partial\gamma}\Big/\frac{\partial\Upsilon}{\partial g} > 0, \quad \frac{\partial G}{\partial\varepsilon} = -\frac{\partial\Upsilon}{\partial\varepsilon}\Big/\frac{\partial\Upsilon}{\partial g} > 0. \tag{31}$$

This result can be rephrased as follows:

[10] Cf. Dixit (1986), for the use of stability properties in refining comparative statics.

Proposition 1 *The more extensive property rights (i.e. the higher ε), or the more these favor "short" innovations (i.e. a big γ), the faster growth.*

Property rights increase rewards for innovations and the growth rate. With "shorter" innovations, firms spend shorter periods as a monopoly and longer periods as a duopolist, which increases the proportion β of duopoly industries. Because only duopolists innovate, this boosts innovation and growth.

5 Public policy

Noting (23), the equation (25) can be written also as follows:

$$\beta \doteq \frac{g\xi^{\nu-1}}{(2\lambda\log\mu)l(g,\varepsilon)} = \frac{g\mu^{(1-1/\nu)\sigma}\varepsilon^{1/\nu-1}}{(2\log\mu)\lambda^{1/\nu}l(g,\varepsilon)}. \tag{32}$$

The symmetry across the households $\iota = 1,...,n$ yields $C_\iota = y/N$. Noting $C_\iota = y/N$, (11), (16), (26) and (32), a single household's consumption relative to the level of productivity, $c \doteq C_\iota/B(\{t_k\})$, can be written as follows:

$$c(g,\varepsilon) \doteq \frac{C_\iota}{B(\{t_k\})} = \frac{y/N}{B(\{t_k\})} = \left[\alpha J(1) + \beta J(2)\right]\left(1 - \frac{l}{N}\right) = \beta J(2)\left(1 - \frac{l}{N}\right)$$
$$= \frac{J(2)g\mu^{(1-1/\nu)\sigma}\varepsilon^{1/\nu-1}}{(2\log\mu)\lambda^{1/\nu}}\left[\frac{1}{l(g,\varepsilon)} - \frac{1}{N}\right], \quad \frac{g}{c}\frac{\partial c}{\partial g} = 1 - \frac{N}{N-l}\frac{g}{l}\frac{\partial l}{\partial g}. \tag{33}$$

Noting this and (31), a single household's utility function (4) takes the form

$$U(C_\iota, T) = E\int_T^\infty c(g,\varepsilon)^\sigma B(\{t_k\})^\sigma e^{-\rho(\nu-T)}d\nu. \tag{34}$$

Given $\varepsilon \doteq \epsilon\gamma$ and (31), the government controls the growth rate g and the extent of property rights, ε, by the "width" and "length" of the innovations, ϵ_2 and γ. It chooses ε and g to maximize a household's welfare (34) subject to stochastic technological change (7). I denote by $\Upsilon(\{t_k\})$ the value of any industry using current technology t_k, and by $\Upsilon\big(t_j + 1, \{t_{k\neq j}\}\big)$ the value of industry j using technology $t_j + 1$, when other industries $k \neq j$ use current technology t_k. In each duopoly industry $j \notin \Theta$, the arrival rate of innovations that change technology from t_j to t_{j+1} is equal to $\Lambda_{j1}+\Lambda_{j2}$, while there are no innovations in monopoly industries $j \in \Theta$. Noting this, the Bellman equation corresponding to the government's maximization problem is given by[11]

$$\rho\Upsilon(\{t_k\}) = \max_{g,,\varepsilon,\lambda} \mathcal{F}(g,\varepsilon), \quad \text{where}$$
$$\mathcal{F}(g,\varepsilon,\lambda) \doteq \frac{c(g,\varepsilon)^\sigma}{B(\{t_k\})^{-\sigma}} + \int_{j\notin\Theta} (\Lambda_{j1}+\Lambda_{j2})\big[\Upsilon\big(t_j+1,\{t_{k\neq j}\}\big) - \Upsilon\big(\{t_k\}\big)\big]dj.$$

Because in equilibrium technological change is symmetric throughout all innovating industries, $\Upsilon\big(t_j+1,\{t_{k\neq j}\}\big)-\Upsilon\big(\{t_k\}\big) = \Upsilon\big(t_\iota+1,\{t_{k\neq\iota}\}\big)-\Upsilon\big(\{t_k\}\big)$ for $j \notin \Theta$, then, noting (17), this Bellman equation changes into

$$\rho\Upsilon(\{t_k\}) = \max_{g,\varepsilon} \mathcal{F}(g,\varepsilon,\lambda), \quad \text{where}$$
$$\mathcal{F}(g,\varepsilon,\lambda) = \frac{c(g,\varepsilon)^\sigma}{B(\{t_k\})^{-\sigma}} + \big[\Upsilon\big(t_\iota+1,\{t_{k\neq\iota}\}\big) - \Upsilon\big(\{t_k\}\big)\big]\int_{j\notin\Theta}(\Lambda_{j1}+\Lambda_{j2})dj$$
$$= \frac{c(g,\varepsilon)^\sigma}{B(\{t_k\})^{-\sigma}} + \big[\Upsilon\big(t_\iota+1,\{t_{k\neq\iota}\}\big) - \Upsilon\big(\{t_k\}\big)\big]\frac{g}{\log\mu}. \tag{35}$$

5.1 The optimal extent of property rights

Noting (33) and (35), one obtains

$$\varepsilon = \arg\max_{\varepsilon} \mathcal{F}(g,\varepsilon,\lambda) = \arg\max_{\varepsilon} c(g,\varepsilon,\lambda) = \arg\max_{\varepsilon}\big\{\varepsilon^{1/\nu-1}\big[1/l - 1/N\big]\big\}$$
$$= \arg\max_{\varepsilon}\big\{(1/\nu-1)\log\varepsilon + \log\big[1/l(g,\varepsilon) - 1/N\big]\big\}. \tag{36}$$

This result can be rephrased as follows:

Proposition 2 *The extent of property rights, ε, should be chosen to maximize current consumption c holding the level of productivity, B, constant.*

More extensive property rights (i.e. a higher ε) (i) decrease the variety of products and the index of consumption, but (ii) diminishes imitation, which releases resources from R&D to consumption. The opposite effects (i) and (ii) are in balance when the extent of property rights maximizes consumption.

Noting (26), the first-order condition for the maximization in (36) is

$$0 = \frac{\partial\{\}}{\partial\varepsilon} = \Big(\frac{1}{\nu}-1\Big)\frac{1}{\varepsilon} - \frac{N/l}{N-l}\frac{\partial l}{\partial\varepsilon} = \frac{1}{\nu}\left[\frac{1}{\nu} - 1 - \frac{N\eta(l)}{N-l}\right]$$

[11] Cf. Dixit and Pindyck (1994).

and

$$N\eta(l)/(N-l) = 1/\nu - 1. \tag{37}$$

This defines total labor in R&D as a function of the spillover parameter ν:

$$l(\nu). \tag{38}$$

5.2 The optimal growth rate

I try the solution that the value function is of the form

$$\Upsilon(\{t_k\}) = c^\sigma B(\{t_k\})^\sigma/\vartheta \tag{39}$$

where ϑ is independent of the endogenous variables of the system. From (6), (15) and (39) it then follows that

$$\frac{\Upsilon\big(t_j+1,\{t_{k\neq j}\}\big)}{\Upsilon(\{t_k\})} = \left(\frac{B\big(t_j+1,\{t_{k\neq j}\}\big)}{B(\{t_k\})}\right)^\sigma = \left(\frac{B_j(t_j+1)}{B_j(t_j)}\right)^\sigma = \mu^\sigma. \tag{40}$$

Inserting (39) and (40) into the Bellman equation (35), we obtain

$$\rho = \vartheta + (\mu^\sigma - 1)\int_{j\notin\Theta}(\Lambda_{j1}+\Lambda_{j2})dj = \vartheta + (\mu^\sigma-1)\frac{g}{\log\mu}$$

and

$$\vartheta = \rho - \frac{\mu^\sigma - 1}{\log\mu}g. \tag{41}$$

Noting (26), (33), (37), (39), (40) and (41), the first-order condition corresponding to g in the maximization (35) is given by

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial g} &= \sigma c^{\sigma-1}B^\sigma\frac{\partial c}{\partial g} + \frac{\mu^\sigma-1}{\log\mu}\Upsilon(\{t_k\}) = \left[\vartheta\frac{g}{c}\frac{\partial c}{\partial g} + \frac{\mu^\sigma-1}{\sigma\log\mu}g\right]\frac{\sigma}{g}\Upsilon(\{t_k\})\\ &= \left[\vartheta - \frac{\vartheta N}{N-l}\frac{g}{l}\frac{\partial l}{\partial g} + \frac{\mu^\sigma-1}{\sigma\log\mu}g\right]\frac{\sigma}{g}\Upsilon(\{t_k\})\\ &= \left[\rho + \left(\frac{1}{\sigma}-1\right)\frac{\mu^\sigma-1}{\log\mu}g - \frac{\vartheta N}{N-l}\frac{g}{l}\frac{\partial l}{\partial g}\right]\frac{\sigma}{g}\Upsilon(\{t_k\})\\ &= \left[\frac{\rho}{g} + \left(\frac{1}{\sigma}-\frac{1}{\nu}\right)\frac{\mu^\sigma-1}{\log\mu}\right]\sigma\Upsilon(\{t_k\}) = 0.\end{aligned}$$

This yields the optimal growth rate

$$g(\rho,\sigma,\nu) \doteq \frac{(\log\mu)\rho}{(\mu^\sigma-1)(1/\nu-1/\sigma)}, \quad \frac{\partial g}{\partial\sigma} < 0, \tag{42}$$

for which $g > 0$ if and only if $\sigma > \nu$. These results can be rephrased as:

Proposition 3 *The optimal growth rate is the higher, the more risk averse the households (i.e. the smaller σ). The economy has persistent growth only if the spillover effect is not too large (i.e. $\nu < \sigma$ holds true).*

When households are more risk averse (i.e. a bigger RRA $1-\sigma$ and a smaller σ), they claim a higher rate of return, $\rho + \frac{1-\mu^\sigma}{\log \mu} g$, on their investment in R&D. Because the true rate of return on R&D cannot raise that much, the optimal growth rate g must increase. When spillover of knowledge is too extensive, $\nu \geq \sigma$, growth accelerates, i.e. there is no stable growth path.

5.3 The optimal "width" and "length" of innovations

Inserting (38) and (42) into (24) and solving for ε yield

$$\varepsilon(\rho, \sigma, \nu) \doteq \left(1 - \frac{1}{1/\nu - 1/\sigma}\right) \frac{\rho}{\Delta(l(\nu))}, \quad \frac{\partial \varepsilon}{\partial \sigma} > 0. \tag{43}$$

Using (31) and (43), one obtains an equation that determines the optimal "length" of innovations, γ: $g\big(\rho, \sigma, \nu\big) = G(\gamma, \varepsilon(\rho, \sigma, \nu))$. Differentiating this equation totally and noting (23), (31), (42) and (43) yield

$$\gamma(\rho, \sigma, \nu), \quad \frac{\partial \gamma}{\partial \sigma} \doteq \left(\frac{\partial g}{\partial \sigma} - \frac{\partial G}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \sigma}\right) \bigg/ \frac{\partial G}{\partial \gamma} < 0,$$

$$\epsilon_2(\rho, \sigma, \nu) \doteq \frac{\varepsilon(\rho, \sigma, \nu)}{\gamma(\rho, \sigma, \nu)}, \quad \frac{\partial \epsilon_2}{\partial \sigma} \doteq \frac{1}{\gamma}\left(\frac{\partial \varepsilon}{\partial \sigma} - \frac{\varepsilon}{\gamma}\frac{\partial \gamma}{\partial \sigma}\right) > 0.$$

These results can be rephrased as follows:

Proposition 4 *The less risk averse the households (i.e. the greater σ), the "longer" (i.e. the smaller γ) and the "narrower" (i.e. the greater ϵ_2) the welfare-maximizing innovations.*

Where households are only a little risk averse, the welfare-maximizing growth rate is low [cf. proposition 3] and innovations must be taylored "long" and "narrow" to maintain this growth rate [cf. proposition 1].

6 Conclusions

This study examines a multi-industry economy in which growth is generated by creative destruction. In each industry, a firm that creates the newest technology by a successful innovation crowds out the other firms with older technologies from the market and becomes the first producer of the industry. A firm creating a copy of the newest technology starts producing a close substitute for the innovator's product and establishes an innovation race with the first producer. Because systematic investment risk cannot be eliminated by diversification, the households hold the shares of all firms in their portfolios. The main findings of this study are as follows.

Because monopolies have no incentives to innovate, the growth rate increases with the proportion of duopoly industries. With non-diversifiable risk, the "width" and "shortness" of innovations boost economic growth. In that case, firms spend shorter periods as a monopoly and the proportion of innovating duopoly industries increases. Property rights decrease the variety of products and the index of consumption, but decrease also imitation, which releases resources from R&D to consumption. When these opposite effects are in balance, there is an optimal extent of property rights.

The optimal growth rate is the higher, the more risk averse the households. More risk averse households claim a higher rate of return on their investment in R&D. Because the true rate of return on R&D cannot raise that much, the growth rate must increase to keep households' claims down. Moderate spillover of knowledge boosts growth. With too large spillover growth accelerates, i.e. there is no stable growth path.

The less risk averse households, the "longer" and the "narrower" the welfare-maximizing innovations. Where households are only a little risk averse, the welfare-maximizing growth rate is low. To maintain that low growth rate, innovations must be taylored "long" and "narrow" .

Appendix

I denote:

$\Omega\big(\{s_{\iota k\upsilon}\},\{\tau_k\}\big)$ the value of receiving profits $s_{\iota k\upsilon}$ from all firms υ in all industries k using current technology τ_k.

$\Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)\upsilon}\}, \tau_j+1, \{\tau_{k\neq j}\}\big)$ the value of receiving the profit $\Pi i_{\iota j\kappa}$ from firm κ in industry $j \notin \Theta$ using technology $\tau_j + 1$, but receiving no profits from the other firm which was a producer in that industry when technology τ_j was used, and receiving profits $s_{\iota(k\neq j)\upsilon}$ from all firms υ in other industries $k \neq j$ with current technology τ_k.

$\Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j2}, \{s_{\iota(k\neq j)\upsilon}\}, \{\tau_k\}\big)$ the value of receiving profits $\pi_2 i_{\iota j\kappa}$ from firms $\kappa \in \{1,2\}$ in industry $j \in \Theta$, but receiving profits $s_{\iota(k\neq j)\upsilon}$ from all firms υ in the other industries $k \neq j$ with current technology τ_k.

The Bellman equation associated with the household's maximization is[12]

$$\rho\Omega\big(\{s_{\iota k\upsilon}\}, \{\tau_k\}\big) = \max_{S_{\iota j}\,\geq\, 0 \text{ for all } j} \Xi_\iota \tag{44}$$

with

$$\begin{aligned}\Xi_\iota \doteq C_\iota^\sigma &+ \int_{j\in\Theta} \Gamma_{j0}\Big[\Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j1}, \{s_{\iota(k\neq j)\upsilon}\}, \{\tau_k\}\big) - \Omega\big(\{s_{\iota k\upsilon}\}, \{\tau_k\}\big)\Big]dj \\ &+ \int_{j\notin\Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa}\Big[\Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)\upsilon}\}, \tau_j+1, \{\tau_{k\neq j}\}\big) - \Omega\big(\{s_{\iota k\upsilon}\}, \{\tau_k\}\big)\Big]dj,\end{aligned} \tag{45}$$

where $\Lambda_{j\kappa}$ is the arrival rate of innovations for duopolist κ in industry $j \notin \Theta$ and Γ_{j0} the arrival rate of imitations for outsider 0 in industry $j \in \Theta$. Because $\partial C_\iota/\partial S_{\iota j\kappa} = -1/P$ by (21), the first-order conditions are given by

$$\Lambda_{j\kappa}\frac{d}{dS_{\iota j\kappa}}\Big[\Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)\upsilon}\}, \tau_j+1, \{\tau_{k\neq j}\}\big) - \Omega\big(\{s_{\iota k\upsilon}\}, \{\tau_k\}\big)\Big] = \frac{\sigma}{P}C_\iota^{\sigma-1} \quad \text{for } j \notin \Theta \text{ and } \kappa \in \{1,2\}, \tag{46}$$

$$\Gamma_{j0}\frac{d}{dS_{\iota j0}}\Big[\Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j2}, \{s_{\iota(k\neq j)\upsilon}\}, \{\tau_k\}\big) - \Omega\big(\{s_{\iota k\upsilon}\}, \{\tau_k\}\big)\Big] = \frac{\sigma}{P}C_\iota^{\sigma-1} \quad \text{for } j \in \Theta. \tag{47}$$

I try the solution that for each household ι the propensity to consume, h_ι, and the subjective interest rate r_ι are independent of income A_ι, i.e. $PC_\iota = h_\iota A_\iota$ and $\Omega = C_\iota^\sigma/r_\iota$. I denote variables depending on technology τ_j by superscript τ_j and a vector that consists of t_k for all k by $\{t_k\}$. Since

[12] Cf. Dixit and Pindyck (1994).

according to (22) income $A_\iota^{\{\tau_k\}}$ depends directly on variables $\{s_{\iota k}^{\tau_k}\}$, I denote $A_\iota^{\{\tau_k\}}(\{s_{\iota k}^{\tau_k}\})$. Assuming that h_ι is invariant across technologies yields

$$P^{\{\tau_k\}}C_\iota^{\{\tau_k\}} = h_\iota A_\iota^{\{\tau_k\}}(\{s_{\iota k}^{\tau_k}\}). \tag{48}$$

The share in the next innovator τ_j+1 is determined by investment under the present technology τ_j, $s_{\iota j\kappa}^{\tau_j+1} = \Pi i_{\iota j\kappa}^{\tau_j}$ for $j \notin \Theta$. The share in the next imitator is determined by investment under the same technology τ_j, $s_{\iota j\kappa}^{\tau_j} = \pi_2 i_{\iota j\kappa}^{\tau_j}$ for $j \in \Theta$. The value functions are then given by

$$\begin{aligned}
&\Omega\big(\{s_{\iota kv}\}, \{\tau_k\}\big) = \Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j2}, \{s_{\iota(k\neq j)v}\}, \{\tau_k\}\big) = \frac{1}{r_\iota}\big(C_\iota^{\{\tau_k\}}\big)^\sigma,\\
&\Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)v}\}, \tau_j+1, \{\tau_{k\neq j}\}\big) = \frac{1}{r_\iota}\big(C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}\big)^\sigma.
\end{aligned} \tag{49}$$

Given this, one obtains

$$\frac{\partial\Omega\big(\{s_{\iota kv}\}, \{\tau_k\}\big)}{\partial S_{\iota j}^{\tau_j}} = 0. \tag{50}$$

From (19), (22), (48), (49), $s_{\iota j\kappa}^{\tau_j+1} = \Pi i_{\iota j\kappa}^{\tau_j}$ for $j \notin \Theta$, and $s_{\iota j\kappa}^{\tau_j} = \pi_2 i_{\iota j\kappa}^{\tau_j}$ for $j \in \Theta$ it follows that

$$\begin{aligned}
&\frac{\partial s_{\iota j\kappa}^{\tau_j+1}}{\partial i_{\iota j\kappa}^{\tau_j}} = \Pi \text{ for } j\notin\Theta, \quad \frac{\partial s_{\iota j0}^{\tau_j}}{\partial i_{\iota j0}^{\tau_j}} = \pi_2 \text{ for } j\in\Theta, \quad \frac{\partial A_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}{\partial s_{\iota j\kappa}^{\tau_j+1}} = \frac{\partial A_\iota^{\{\tau_k\}}}{\partial s_{\iota j\kappa}^{\tau_j}} = 1,\\
&\frac{\partial i_{\iota j0}^{\tau_j}}{\partial S_{\iota j0}^{\tau_j}} = \frac{1}{w^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \text{ for } j\in\Theta, \quad \frac{\partial i_{\iota j\kappa}^{\tau_j}}{\partial S_{\iota j\kappa}^{\tau_j}} = \frac{1}{w^{\{\tau_k\}} l_{j\kappa}^{\{\tau_k\}}} \text{ for } j\notin\Theta,\\
&\frac{d\Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)v}\}, \tau_j+1, \{\tau_{k\neq j}\}\big)}{dS_{\iota j\kappa}^{\tau_j}}\\
&= \frac{\sigma}{r_\iota}\big(C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}\big)^{\sigma-1} \underbrace{\frac{\partial C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}{\partial A_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}}_{h_\iota/P^{\tau_j+1,\{\tau_{k\neq j}\}}} \underbrace{\frac{\partial A_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}{\partial s_{\iota j\kappa}^{\tau_j+1}}}_{=1} \underbrace{\frac{\partial s_{\iota j\kappa}^{\tau_j+1}}{\partial i_{\iota j\kappa}^{\tau_j}}}_{=\pi_2} \frac{\partial i_{\iota j\kappa}^{\tau_j}}{\partial S_{\iota j\kappa}^{\tau_j}}\\
&= \frac{\Pi\sigma h_\iota\big(C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}\big)^{\sigma-1}}{r_\iota P^{\tau_j+1,\{\tau_{k\neq j}\}}} \frac{\partial i_{\iota j\kappa}^{\tau_j}}{\partial S_{\iota j\kappa}^{\tau_j}} = \frac{\Pi h_\iota\sigma\big(C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}\big)^{\sigma-1}}{r_\iota w^{\{\tau_k\}} P^{\tau_j+1,\{\tau_{k\neq j}\}} l_{j\kappa}^{\{\tau_k\}}} \quad \text{for } j\notin\Theta, \qquad (51)\\
&\frac{d\Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j2}, \{s_{\iota(k\neq j)v}\}, \{\tau_k\}\big)}{dS_{\iota j0}^{\tau_j}}\\
&= \frac{\sigma}{r_\iota}\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1} \underbrace{\frac{\partial C_\iota^{\{\tau_k\}}}{\partial A_\iota^{\{\tau_k\}}}}_{=h_\iota/P^{\{\tau_k\}}} \underbrace{\frac{\partial A_\iota^{\{\tau_k\}}}{s_{\iota j0}^{\tau_j}}}_{=1} \underbrace{\frac{s_{\iota j0}^{\tau_j}}{\partial i_{\iota j0}^{\tau}}}_{=\pi_2} \frac{\partial i_{\iota j0}^{\tau}}{\partial S_{\iota j0}^{t}}
\end{aligned}$$

$$= \frac{\pi_2 \sigma h_\iota}{r_\iota P^{\{\tau_k\}}} \big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1} \frac{\partial i^\tau_{\iota j0}}{\partial S^t_{\iota j0}} = \frac{\pi_2 h_\iota \sigma \big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}}{r_\iota w^{\{\tau_k\}} P^{\{\tau_k\}} l_{j0}^{\{\tau_k\}}} \quad \text{for } j \in \Theta. \tag{52}$$

I focus on a stationary equilibrium where the growth rate g and the allocation of labor, $(l_{j\kappa}, x)$, are invariant across technologies. Given (9), (13), (15) and (1), this implies

$$l_{j\kappa}^{\{\tau_k\}} = l_{j\kappa}, \quad x^{\{\tau_k\}} = x = N - l, \quad w^{\{\tau_k\}} = w = x/\varphi,$$
$$\frac{P^{\{\tau_k\}}}{P^{\tau_j+1,\{\tau_{k\neq j}\}}} = \frac{C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}{C_\iota^{\{\tau_k\}}} = \frac{A_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}}{A_\iota^{\{\tau_k\}}} = \frac{y^{\tau_j+1,\{\tau_{k\neq j}\}}}{y^{\{\tau_k\}}} = \frac{B^{\tau_j+1,\{\tau_{k\neq j}\}}}{B^{\{\tau_k\}}} = \mu. \tag{53}$$

Inserting (17), (45), (48), (49), (53) and $g \doteq \int_{j\notin\Theta} l_j dj$ into (44) yields

$$\begin{aligned}
0 &= \Big[\rho + \int_{j\notin\Theta} (\Lambda_{j1} + \Lambda_{j2}) dj + \int_{j\in\Theta} \Gamma_{j0} dj\Big] \Omega\big(\{s_{\iota k v}\}, \{\tau_k\}\big) - \big(C_\iota^{\{\tau_k\}}\big)^\sigma \\
&\quad - \int_{j\notin\Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \Omega\big(\Pi i_{\iota j\kappa}, 0, \{s_{\iota(k\neq j)v}\}, \tau_j + 1, \{\tau_{k\neq j}\}\big) dj \\
&\quad - \int_{j\in\Theta} \Gamma_{j0} \Omega\big(\pi_2 i_{\iota j1}, \pi_2 i_{\iota j2}, \{s_{\iota(k\neq j)v}\}, \{\tau_k\}\big) dj \\
&= \Big[\rho + \int_{j\notin\Theta} (\Lambda_{j1} + \Lambda_{j2}) dj\Big] \frac{\big(C_\iota^{\{\tau_k\}}\big)^\sigma}{r_\iota} - \big(C_\iota^{\{\tau_k\}}\big)^\sigma - \int_{j\notin\Theta} \sum_{\kappa=1,2} \Lambda_{j\kappa} \frac{\mu^\sigma}{r_\iota} \big(C_\iota^{\{\tau_k\}}\big)^\sigma dj \\
&= \frac{1}{r_\iota} \big(C_\iota^{\{\tau_k\}}\big)^\sigma \Big[\rho - r_\iota + \frac{1-\mu^\sigma}{\log \mu} g\Big].
\end{aligned}$$

This equation is equivalent to

$$r_\iota = \rho + \frac{1-\mu^\sigma}{\log \mu} g. \tag{54}$$

Because there is symmetry throughout all households ι, their propensity to consume is equal, $h_\iota = h$. This, (18), (3), (21), (22) and (48) yield

$$\begin{aligned}
wl &= w \int_{j\in\Theta} l_{j0} dj + w \int_{j\notin\Theta} (l_{j1} + l_{j2}) dj = w \int_{j\in\Theta} l_{j0} dj + \int_{j\notin\Theta} (l_{j1} + l_{j2}) dj \\
&= \sum_{\iota=1}^N \Big[\int_{j\in\Theta} S_{\iota j0} dj + \int_{j\notin\Theta} (S_{\iota j1} + S_{\iota j2}) dj\Big] = \sum_{\iota=1}^N (A_\iota - PC_\iota) = (1-h) \sum_{\iota=1}^N A_\iota \\
&= (1-h)(1+wl)
\end{aligned}$$

and

$$h_\iota = h = (1+wl)^{-1}. \tag{55}$$

Inserting (7), (8), (50), (51), (52), (53), (54), (55) and $\varepsilon \doteq \gamma\epsilon$ into (46) and (47), one obtains

$$\begin{aligned}
&\frac{\Pi h\sigma\mu^\sigma\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}\lambda(l/l_{j\kappa})^\nu}{\Big(\rho+\frac{1-\mu^\sigma}{\log\,\mu}g\Big)wP^{\{\tau_k\}}} = \frac{\sigma\Pi h_\iota\mu^\sigma\Lambda_{j\kappa}\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}}{r_\iota wl_{j\kappa}P^{\{\tau_k\}}} \\
&= \frac{\sigma\Pi h_\iota\Lambda_{j\kappa}\big(C_\iota^{\tau_j+1,\{\tau_{k\neq j}\}}\big)^{\sigma-1}}{r_\iota wl_{j\kappa}P^{\tau_{\tau_j+1},\{\tau_{k\neq j}\}}} = \Lambda_{j\kappa}\frac{d}{dS_{\iota j\kappa}}\Omega\big(\Pi i_{\iota j},\{s_{\iota(k\neq j)}\},\tau_j+1,\{\tau_{k\neq j}\}\big) \\
&= \frac{\sigma}{P^{\{\tau_k\}}}\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1} \text{ for } j\notin\Theta \text{ and } \kappa\in\{1,2\},
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\frac{\varepsilon\Pi h\sigma\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}}{\Big(\rho+\frac{1-\mu^\sigma}{\log\,\mu}g\Big)wP^{\{\tau_k\}}} = \frac{\sigma\epsilon\Pi h_\iota\gamma\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}}{r_\iota wP^{\{\tau_k\}}} = \frac{\sigma\pi_2h_\iota\Gamma_{j0}\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1}}{r_\iota wl_{j0}P^{\{\tau_k\}}} \\
&= \Gamma_{j0}\frac{d}{dS_{\iota j0}}\Omega\big(\{\pi_2 i_{\iota j1},\pi_2 i_{\iota j2},\{s_{\iota m(k\neq j)}\},\{\tau_k\}\big) = \frac{\sigma}{P^{\{\tau_k\}}}\big(C_\iota^{\{\tau_k\}}\big)^{\sigma-1} \text{ for } j\in\Theta.
\end{aligned} \tag{57}$$

Given $\mu>1$, $\pi_2\le\Pi/2$, (56) and (57), one obtains

$$\begin{matrix} l_{j\kappa}=\ell_\beta & \text{for } j\notin\Theta, \\ l_{j0}=\ell_\alpha & \text{for } j\in\Theta, \end{matrix} \quad \frac{\ell_\beta}{l}=\xi(\varepsilon)\doteq\left(\frac{\mu^\sigma\lambda}{\varepsilon}\right)^{1/\nu}. \tag{58}$$

Equations (11), (13), (7), (14), (17), (55), (57) and (58) yield

$$\begin{aligned}
&l = \int_{j\notin\Theta}(l_{j1}+l_{j2})dj + \int_{j\in\Theta}l_jdj = 2\ell_\beta\int_{j\notin\Theta}dj + \ell_\alpha\int_{j\in\Theta}dj \\
&\quad = (1-\beta)\ell_\alpha + 2\beta\ell_\beta = (1-\beta)\ell_\alpha + 2\beta\xi l, \\
&\ell_\alpha = [1-2\beta\xi(\varepsilon)]l/(1-\beta),
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\Lambda_{j\kappa} = \lambda l_{j\kappa}^{1-\nu}l^\nu = \lambda\ell_\beta^{1-\nu}l^\nu = \lambda\xi(\varepsilon)^{1-\nu}l \text{ for } j\notin\Theta \text{ and } \kappa\in\{1,2\}, \\
&g = (\log\mu)\int_{j\notin\Theta}(\Lambda_{j1}+\Lambda_{j2})dj = (2\log\mu)\beta\Lambda_{j\kappa} = (2\lambda\log\mu)\beta\xi(\varepsilon)^{1-\nu}l,
\end{aligned} \tag{60}$$

$$\rho + \frac{1-\mu^\sigma}{\log\,\mu}g = \frac{h\varepsilon}{w}\Pi = \frac{\varepsilon\Pi}{[1+w(l)l]w(l)} \doteq \Delta(l)\varepsilon, \quad \Delta'<0. \tag{61}$$

Equations (58)-(61) define (23)-(25).

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